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On Global Lipshitz-continuous Solutions of Isentropic Gas Dynamics

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Building on work in [Yunguang Lu (1993). The global Hölder continuous solution of isentropic gas dynamics. *Proc. Roy. Soc. Edinburgh Sect.*, **123A**, 231–238.] we consider the Cauchy problem for the isentropic equations of gas dynamics in Eulerian coordinates. For a certain class of initial data we prove existence of Lipshitz-continuous solution for a wide class of the pressure functions.

Keywords: Hyperbolic conservation laws; Existence; Smooth solutions; Isentropic gas dynamics

AMS Subject Classifications: 35B50; 35B65; 35L65

1. INTRODUCTION

In this article we consider the existence of a global Lipshitz-continuous solution for the isentropic equations of gas dynamics in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{1.1}$$

with initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)). \tag{1.2}$$

The global Hölder continuous solution of (1.1) in a non-strictly hyperbolic domain ($\rho = 0$) was first studied in [4]. It is well-known in the study of global smooth solutions for hyperbolic systems that tracking for the development or the formation of singularities of solutions for these systems is quite helpful. This study was started by Lax [1] in some strictly hyperbolic systems (See also [3]) and later was extended by Li [2] to some linearly degenerate systems. All these articles used the method of the characteristic lines to prove the existence of smooth solutions, which seems impossible to apply to

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non-strictly hyperbolic systems where characteristic lines coincide at some points or lines. In [4], the second author introduced a variant of the viscosity argument to study the global smooth solutions for the Cauchy problem (1.1), (1.2). To our knowledge this is the only non-strictly hyperbolic system for which we can prove the existence of global smooth solution directly in the non-strictly hyperbolic domain. The main trick used in [4] is to add a small perturbation δ to the system which is different from the traditional method where the perturbation is added to the initial data (See [5], for example). The new or perturbated system is still non-strictly hyperbolic on the vacuum line $\rho = 0$. Fortunately, the viscosity solutions of the perturbated system have an *a priori* boundness estimates, in C^1 space, which are independent of the viscosity and the perturbation parameter δ. Then these estimates are used to get the convergence of the sequence of viscosity solutions and the existence of the global solution for the Cauchy problem (1.1), (1.2).

In [5], a general technique different from [4] was also introduced to study some non-strictly hyperbolic systems. This technique is based on the following two steps. The first step is to add a small perturbation δ to the intial data (not to the equations) such that the system we consider is strictly hyperbolic in the domain of the perturbated initial data, and then to get some necessary estimates in the same way as with strictly hyperbolic systems. The second step is to let the perturbation δ vanish and then get a global smooth solution in the given non-strictly hyperbolic domain of the original initial data. Since the solution is smooth, from the mathematical point of view, the method introduced in [5] is almost the same to that given in the strictly hyperbolic case.

In the article [4], the results are based on the following basic assumptions on a function f:

$$(\bar{A}) \qquad f(\rho) > 0,\ f'(\rho) \le 0,\ 2f(\rho) + \rho f'(\rho) \ge 0, \qquad \int_0^{\infty} f(s)\,ds = +\infty$$

for $\rho \ge 0$, where $f(\rho)$ given by the relation $P(\rho) = \int_0^{\rho} s^2 f^2(s)\,ds$. For the special case of a polytropic gas, $P(\rho) = c\rho^{\gamma}$ $(c > 0, \gamma \ge 1)$, the assumption $(\bar{A})$ is satisfied only for $\gamma \in [1, 3]$.

In this article, we study the Lipshitz-continuous solution of the Cauchy problem (1.1), (1.2) in a more general case, where $P(\rho)$ satisfies

$$(A) \qquad f(\rho) > 0,\ f'(\rho) \ge 0 \quad \text{or} \quad f'(\rho) \le 0,\ 2f(\rho) + \rho f'(\rho) \ge 0, \quad \int_0^{\infty} f(s)ds = +\infty$$

for $\rho \ge 0$.

For the polytropic gas, the assumption (A) includes the exponent $\gamma > 3$.

Following the method introduced in [4], we consider the perturbated system of (1.1)

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P_1(\rho))_x = 0, \end{cases} \tag{1.3}$$

where $P_1(\rho) = \int_0^{\rho} s^2 f_1^2(s)\,ds$, $f_1(\rho) = f(\rho + 2\delta)$ and $\delta > 0$ is the perturbation parameter.

Then $f_1(\rho)$ satisfies

$$(A_1) \qquad f_1(\rho) > 0, f'_1(\rho) \geq 0 \quad \text{or} \quad f'_1(\rho) \leq 0,\ 2f_1(\rho) + \rho f'_1(\rho) \geq 0,\ \int_0^\infty f_1(s)\,ds = +\infty$$

for $\rho \geq -\delta$.

Two eigenvalues of (1.3) are

$$\mu_1 = u - \rho f_1(\rho), \quad \mu_2 = u + \rho f_1(\rho), \tag{1.4}$$

which coincide when $\rho = 0$ and so in which (1.3) is non-strictly hyperbolic; two Riemann invariants of (1.3) are

$$z = u - \int_0^\rho f_1(s)\,ds, \quad w = u + \int_0^\rho f_1(s)\,ds. \tag{1.5}$$

In discussing the smooth solution we consider the viscosity solutions for the following Cauchy problem

$$\begin{cases} w_t + \mu_2 w_x = \epsilon w_{xx} \\ z_t + \mu_1 z_x = \epsilon z_{xx} \end{cases} \tag{1.6}$$

with initial data

$$(w(x,0), z(x,0)) = (w_0(x), z_0(x)), \tag{1.7}$$

where

$$z_0(x) = u_0(x) - \int_0^{\rho_0(x)} f(s)\,ds, \quad w_0(x) = u_0(x) + \int_0^{\rho_0(x)} f(s)\,ds \tag{1.8}$$

and $u_0(x)$, $\rho_0(x)$ are given by (1.2). Then the existence of the Lipshitz-continuous solution of the Cauchy problem (1.1), (1.2) is reduced to obtain the necessary C^1 estimates of (w^ϵ, z^ϵ) of the Cauchy problem (1.6), (1.7). These estimates are given in Section 2 and the existence theorem of the Lipshitz-continuous solution of (1.1), (1.2) is given in Section 3.

2. VISCOSITY SOLUTIONS

In this section we consider the existence of the global solutions (w, z) of the Cauchy problem (1.6), (1.7) and the necessary boundness estimates indepedent of ϵ and δ.

We differentiate (1.6) with respect to x and let $w_x = r$, $z_x = s$; then

$$\begin{cases} r_t + \mu_2 r_x + (\mu_{2w} r + \mu_{2z} s) r = \epsilon r_{xx}, \\ s_t + \mu_1 s_x + (\mu_{1w} r + \mu_{1z} s) s = \epsilon s_{xx}. \end{cases} \tag{2.1}$$

A simple calculation and the assumption (A_1) yield

$$u_w = \frac{1}{2}, \quad \rho_w = \frac{1}{2f_1(\rho)}, \quad u_z = \frac{1}{2}, \quad \rho_z = -\frac{1}{2f_1(\rho)} \tag{2.2}$$

and

$$\mu_{1w} = \mu_{2z} = -\frac{\rho f_1'(\rho)}{2f_1(\rho)}, \quad \mu_{1z} = \mu_{2w} = \frac{2f_1(\rho) + \rho f_1'(\rho)}{2f_1(\rho)}. \tag{2.3}$$

Similarly to the article [4], the existence of the global solutions (w^ϵ, z^ϵ) of the Cauchy problem (1.6), (1.7) can be obtained based on the *a priori* estimates given in the following Lemma 1.

LEMMA 1 *Let $f_1(\rho)$ satisfy (A_1) and let $w_0(x)$, $z_0(x)$ be bounded in C^1 space and satisfy*

$$c_1 \le z_0(x) \le c_2, \;\; c_2 \le w_0(x) \le c_3, \;\; 0 \le z_{0x}(x) \le M, \;\; 0 \le w_{0x}(x) \le M. \tag{2.4}$$

Moreover, suppose that $(w(x,t), z(x,t))$ is a smooth solution of (1.6), (1.7) *defined in a strip $(-\infty, \infty) \times [0, T]$ with $0 < T < \infty$. Then*

$$c_1 \le z(x,t) \le c_2, \quad c_2 \le w(x,t) \le c_3, \tag{2.5}$$

and

$$0 \le z_x(x,t) \le M, \quad 0 \le w_x(x,t) \le M. \tag{2.6}$$

Proof The estimates in (2.5) are analogous to these of the linear case. The estimates $z_x(x,t) \ge 0$ and $w_x(x,t) \ge 0$, the estimates $z_x \le M$, $w_x \le M$ in (2.6) for the case of $f_1'(\rho) \le 0$ are proved in [4], Lemma 2.2. Here we only prove the upper bound estimates in (2.6) for the case of $f_1'(\rho) \ge 0$.

We rewrite (2.1) by

$$\begin{cases} r_t + \mu_2 r_x + (\mu_{2w} r) r + (\mu_{2z} r) s = \epsilon r_{xx}, \\ s_t + \mu_1 s_x + (\mu_{1w} s) r + (\mu_{1z} s) s = \epsilon s_{xx}. \end{cases} \tag{2.7}$$

Make a transformation

$$r = \bar{r} + M + \frac{N(x^2 + cLe^t)}{L^2}, \quad s = \bar{s} + M + \frac{N(x^2 + cLe^t)}{L^2}, \tag{2.8}$$

where L, c, N are positive constants and N is the upper bound of r, s on $R \times [0, T]$ (N can be obtained by the local existence). The function $\bar{r}, \bar{s}$, as are easily seen, satisfy the equations

$$\begin{cases} \bar{r}_t + \mu_2 \bar{r}_x + (\mu_{2w} r + \mu_{2z} r)\left(M + \dfrac{N(x^2 + cLe^t)}{L^2} \right) \\ \quad + (cLe^t + 2\mu_2 x - 2\epsilon)\dfrac{N}{L^2} + (\mu_{2w} r)\bar{r} + (\mu_{2z} r)\bar{s} = \epsilon \bar{r}_{xx}, \\ \bar{s}_t + \mu_1 \bar{s}_x + (\mu_{1w} s + \mu_{1z} s)\left(M + \dfrac{N(x^2 + cLe^t)}{L^2} \right) \\ \quad + (cLe^t + 2\mu_1 x - 2\epsilon)\dfrac{N}{L^2} + (\mu_{1w} s)\bar{r} + (\mu_{1z} s)\bar{s} = \epsilon \bar{s}_{xx} \end{cases} \tag{2.9}$$

resulting from (2.7). Moreover

$$\begin{aligned} \bar{r}(x,0) &= r(x,0) - M - \frac{N(x^2 + cL)}{L^2} < 0, \\ \bar{s}(x,0) &= s(x,0) - M - \frac{N(x^2 + cL)}{L^2} < 0, \end{aligned} \tag{2.10}$$

$$\bar{r}(+L,t) < 0, \;\; \bar{r}(-L,t) < 0, \;\; \bar{s}(+L,t) < 0, \;\; \bar{s}(-L,t) < 0. \tag{2.11}$$

From (2.9)–(2.11), we have

$$\bar{r}(x,t) < 0, \quad \bar{s}(x,t) < 0, \quad \text{on } (-L,L) \times (0,T). \tag{2.12}$$

If (2.12) is violated at a point $(x,t) \in (-L,L) \times (0,T)$, let $\bar{t}$ be the least upper bound of values of t at which $\bar{r} < 0$ (or $\bar{s} < 0$); then by the continuity we see that $\bar{r} = 0,\ \bar{s} \le 0$ at some points $(\bar{x},\bar{t}) \in (-L,L) \times (0,T)$. So

$$\bar{r}_t \ge 0, \quad \bar{r}_x = 0, \quad \epsilon \bar{r}_{xx} \ge 0, \quad \text{at } (\bar{x},\bar{t}). \tag{2.13}$$

Since $r \ge 0$, then we have from (2.3),

$$\mu_{2z} r \bar{s} \ge 0, \quad (\mu_{2w} + \mu_{2z}) r = r \ge 0.$$

If we choose sufficiently large constant c (which may depend on δ in the case of $f_1(\rho) \le 0$) such that

$$cLe^t + 2\mu_2 x - 2\epsilon > 0 \quad \text{on } (-L,L) \times (0,T), \tag{2.14}$$

the first equation in (2.9) gives a conclusion contradicting (2.13). So (2.12) is proved. Therefore, for any point $(x_0,t_0) \in (-L,L) \times (0,T)$,

$$r(x_0,t_0) < M + \frac{N(x_0^2 + cLe_0^t)}{L^2}, \quad s(x_0,t_0) < M + \frac{N(x_0^2 + cLe_0^t)}{L^2}, \tag{2.15}$$

which gives the desired estimates $r_x \le M,\ s_x \le M$ if we let L go to infinity. So Lemma 1 is proved.

Based on the *a priori* estimates in Lemma 1 and the existence of the local solution given in Lemma 2.1 of [4], we have the following global existence result:

THEOREM 2 *Let $f_1(\rho)$ satisfy the assumption (A_1) and let $(z_0(x),\ w_0(x)) \in C^1$ satisfy*

$$c_1 \leq z_0(x) \leq c_2,\ \ c_2 \leq w_0(x) \leq c_3,\ \ 0 \leq w_{0x} \leq M,\ \ 0 \leq z_{0x} \leq M;$$

then the Cauchy problem (1.6), (1.7) *has a unique global smooth solution satisfying* (2.5), (2.6).

To get the Lipshitz-continuous solution of the Cauchy problem (1.1), (1.2), we are going to give the estimates of w_t and z_t.

Let $X = w_t$, and $Y = z_t$; then

$$\begin{cases} X\,|_{t=0} = w_t|_{t=0} = (\epsilon w_{xx} - \mu_2 w_x)|_{t=0}, \\ Y\,|_{t=0} = z_t|_{t=0} = (\epsilon z_{xx} - \mu_1 z_x)|_{t=0}. \end{cases} \tag{2.16}$$

If the conditions in (2.5), (2.6) are satisfied, we can use a mollifier to smooth $(z_0(x), w_0(x))$ such that the obtained functions $(z_0^\tau(x),\ w_0^\tau(x)) \in C^2$ satisfy

$$c_1 \leq z_0^\tau(x) \leq c_2, c_2 \leq w_0^\tau(x) \leq c_3,\ \ 0 \leq w_{0x}^\tau \leq M,\ \ 0 \leq z_{0x}^\tau \leq M;$$
$$|\tau z_{0xx}^\tau| \leq M_1; \quad |\tau w_{0xx}^\tau| \leq M_1,$$

where M_1 is a constant independing of τ. We omit this process and directly assume that $X|_{t=0}$ and $Y|_{t=0}$ are bounded.

Differentiating (1.6) with respect to t, we have

$$\begin{cases} X_t + \mu_2 X_x + (\mu_{2w} r)X + (\mu_{2z} r)Y = \epsilon X_{xx}, \\ Y_t + \mu_1 Y_x + (\mu_{1w} s)X + (\mu_{1z} s)Y = \epsilon Y_{xx}. \end{cases} \tag{2.17}$$

With the same proof of Lemma 1, we have

LEMMA 3. *Let all the conditions in Theorem 2 be satisfied. Moreover $(w_0(x), z_0(x)) \in C^2$ and $|X_0(x)| \leq M_2, |Y_0(x)| \leq M_2$; then*

$$|X(x,t)| \leq M_2, \quad |Y(x,t)| \leq M_2. \tag{2.18}$$

3. LIPSHITZ-CONTINUOUS SOLUTION

In this section we apply the estimates obtained in the last section to give the Lipshitz-continuous solution of the Cauchy problem (1.1), (1.2).

The following estimates for (ρ, u) can be obtained from the representations of w, z and the estimates in (2.5), (2.6) and (2.18).

LEMMA 4 *If the conditions of Lemma* 3 *are satisfied, then for the case of* $f_1'(\rho) \geq 0$, *we have*

$$|u(x,t)| \leq M, \quad 0 \leq \rho(x,t) \leq M; \tag{3.1}$$

$$|u_x(x,t)| \leq M, \quad \left|\left(\int_0^\rho f_1(s)\,ds\right)_x\right| \leq M; \tag{3.2}$$

$$|u_t(x,t)| \leq M, \quad \left|\left(\int_0^\rho f_1(s)\,ds\right)_t\right| \leq M \tag{3.3}$$

and for the case of $f_1'(\rho) \leq 0$, *we have*

$$|u(x,t)| \leq M, \quad 0 \leq \rho(x,t) \leq M; \tag{3.4}$$

$$|u_x(x,t)| \leq M, \quad |\rho_x(x,t)| \leq M; \tag{3.5}$$

$$|u_t(x,t)| \leq M, \quad |\rho_t(x,t)| \leq M, \tag{3.6}$$

where M *is a constant independent of* ϵ *and* δ.

In the case of $f_1'(\rho) \geq 0$, we construct a sequence of the approximate solutions $(\rho^{\epsilon,\delta}, u^{\epsilon,\delta})$ such that $(\int_0^{\rho^{\epsilon,\delta}} f_1(s)\,ds, u^{\epsilon,\delta})$ are uniformly bounded in $W^{1,\infty}$ for $0 < T < \infty$, which by the embedding theorem has a subsequence on any bounded regions G of $R \times R^+$, converging uniformly to a pair of Hölder-continuous functions $(\rho(x,t), u(x,t))$ such that $(\int_0^\rho f(s)\,ds, u)$ are bounded in $W^{1,\infty}(R \times R^+)$.

In the case of $f_1'(\rho) \leq 0$, the above limits $(\rho(x,t), u(x,t))$ are Lipshitz-continuous from the estimates (3.4)–(3.6). Finally, we have our Main Theorem.

THEOREM 5 (Main Theorem) *Let* $f(\rho)$ *satisfy the assumption* (A), *and let* $(w_0(x), z_0(x))$ *defined by* (1.8) *satisfy* $c_1 \leq z_0(x) \leq c_2,\ c_2 \leq w_0(x) \leq c_3,\ 0 \leq z_{0x}(x) \leq M, 0 \leq w_{0x}(x) \leq M$. *Then the Cauchy problem* (1.1), (1.2) *has a global Lipshitz-continuous solution* (ρ, u) *when* $f'(\rho) \leq 0$; *and a Hölder-continuous solution* (ρ, u) *when* $f'(\rho) \geq 0$ *such that* $(\int_0^\rho f(s)\,ds, u)$ *are Lipshitz-continuous. Moreover*, (ρ, u) *satisfies the estimates given in* (3.1)–(3.3) *or* (3.4)–(3.6).

Proof The case of $f'(\rho) \leq 0$ is similar to the proof of [4]. For the case of $f'(\rho) \geq 0$, we consider the matrix

$$A = \begin{pmatrix} w_\rho & w_m \\ z_\rho & z_m \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{2f_1} & -\dfrac{1}{2f_1} \\ \dfrac{1}{2}\left(\rho + \dfrac{u}{f_1}\right) & \dfrac{1}{2}\left(\rho - \dfrac{u}{f_1}\right) \end{pmatrix},$$

where $m = \rho u$, and multiply it by the two sides in (1.6), then (1.4) can be rewritten as follows:

$$\begin{cases} \rho_t + (\rho u)_x = \dfrac{\epsilon}{2f_1}(w_{xx} - z_{xx}) \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = \dfrac{\epsilon\rho}{2}(w_{xx} + z_{xx}) + \dfrac{\epsilon u}{2f_1}(w_{xx} - z_{xx}). \end{cases} \tag{3.7}$$

For any test function $\phi \in C_0^1(R \times R^+)$, we have from (3.7) that

$$\int_0^\infty \int_R \rho\phi_t + \rho u\phi_x dxdt + \int_{t=0} \rho_0\phi\, dx$$
$$= \epsilon \int_0^\infty \int_R \left(\frac{1}{2f_1}\phi_x - \frac{f'_1\rho_x}{2f_1^2}\phi\right)(w_x - z_x)\, dx\, dt, \tag{3.8}$$

and

$$\begin{cases} \displaystyle\int_0^\infty \int_R \rho u\phi_t + (\rho u^2 + P(\rho))\phi_x dxdt + \int_{t=0} \rho_0 u_0\phi\, dx \\ \displaystyle\quad = \epsilon \int_0^\infty \int_R \frac{1}{2}(\rho_x\phi + \rho\phi_x)(w_x + z_x) \\ \displaystyle\qquad + \left(\frac{u}{2f_1}\phi_x + \frac{u_x}{2f_1}\phi - \frac{uf'_1\rho_x}{2f_1^2}\phi\right)(w_x - z_x)\, dx\, dt. \end{cases} \tag{3.9}$$

Since

$$\frac{1}{2f_1(\rho)} \le \frac{1}{2f(\rho + 2\delta)} \le \frac{1}{2f(2\delta)},$$

$$\left|\frac{f'_1\rho_x}{2f_1^2}\right| \le \left|\frac{f'_1}{2f_1^3}\left(\int_0^\rho f_1(s)\, ds\right)_x\right| \le \frac{M_1}{2f^3(2\delta)},$$

and

$$|\rho_x| = \frac{|(\int_0^\rho f_1(s)\, ds)_x|}{f_1} \le \frac{M_1}{f(2\delta)},$$

and choosing ϵ to be smaller than δ such that $\epsilon/f(2\delta)$ and $\epsilon/f^3(2\delta)$ go to zero as ϵ, δ go to zero, then using the estimates (2.6), (3.4) and (3.5), we obtain immediately by letting ϵ, δ vanish in (3.8) and (3.9),

$$\begin{cases} \displaystyle\int_0^\infty \int_R \rho\phi_t + \rho u\phi_x\, dx\, dt + \int_{t=0} \rho_0\phi\, dx = 0, \\ \displaystyle\int_0^\infty \int_R \rho u\phi_t + (\rho u^2 + P(\rho))\phi_x\, dx\, dt + \int_{t=0} \rho_0 u_0\phi\, dx = 0 \end{cases} \tag{3.10}$$

for all $\phi \in C_0^1(R \times R^+)$. Thus Theorem 5 is proved.

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