Global weak solutions for a nonlinear hyperbolic system

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Abstract

In this paper, we study the global existence of weak solutions for the Cauchy problem of the nonlinear hyperbolic system of three equations (1.1) with bounded initial data (1.2). When we fix the third variable s, the system about the variables ρ and u is the classical isentropic gas dynamics in Eulerian coordinates with the pressure function $P(\rho, s) = e^s e^{-\frac{1}{\rho}}$, which , in general, does not form a bounded invariant region. We introduce a variant of the viscosity argument, and construct the approximate solutions of (1.1) and (1.2) by adding the artificial viscosity to the Riemann invariants system ((2.1)). When the amplitude of the first two Riemann invariants $(w_1(x,0), w_2(x,0))$ of system (1.1) is small, $(w_1(x,0), w_2(x,0)$ are nondecreasing and the third Riemann invariant $s(x,0)$ is of the bounded total variation, we obtained the necessary estimates and the pointwise convergence of the viscosity solutions by the compensated compactness theory. This is an extension of the results in [1].

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1 Introduction

In this paper, we study the global solutions of the nonlinearly conservation system of three equations

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho, s))_x = 0 \\ ((\rho s)_t + (\rho u s)_x = 0 \end{cases} \tag{1.1}$$

with bounded initial data

$$(\rho, u, s)|_{t=0} = (\rho_0(x), u_0(x), s_0(x)), \quad \rho_0(x) \geq 0, \;\; s_0(x) \geq 0, \tag{1.2}$$

where $P(\rho, s)$ is fixed as $e^s e^{-\frac{1}{\rho}}$.

Substituting the first equation in (1.1) into the second and the third, we have, for the smooth solution, the following equivelent system about the variables (ρ, u, s),

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + \frac{1}{\rho^3} e^{s-\frac{1}{\rho}} \rho_x + u u_x + \frac{1}{\rho} e^{s-\frac{1}{\rho}} s_x = 0 \\ s_t + u s_x = 0. \end{cases} \tag{1.3}$$

Let the matrix $dF(U)$ be

$$dF(U) = \begin{pmatrix} u & \rho & 0 \\ \frac{1}{\rho^3} e^{s-\frac{1}{\rho}} & u & \frac{1}{\rho} e^{s-\frac{1}{\rho}} \\ 0 & 0 & u \end{pmatrix}. \tag{1.4}$$

Then three eigenvalues of (1.1) are

$$\lambda_1 = u - \frac{1}{\rho} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}}, \quad \lambda_2 = u + \frac{1}{\rho} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} \quad \lambda_3 = u \tag{1.5}$$

with corresponding three Riemann invariants

$$w_1 = u - 2e^{\frac{s}{2}} e^{-\frac{1}{2\rho}}, \quad w_2 = u + 2e^{\frac{s}{2}} e^{-\frac{1}{2\rho}}, \quad w_3 = s. \tag{1.6}$$

Based on the following condition (H), the existence of global smooth solution for the Cauchy problem (1.3) and (1.2) was first studied in [1]:

(H): $\rho_0(x), u_0(x), s_0(x)$ are bounded in $C^1(R)$, and there exists a positive constant M such that

$$\begin{cases} 0 \leq \frac{d}{dx}(u_0(x) - 2e^{\frac{s_0(x)}{2}} e^{-\frac{1}{2\rho_0(x)}}) \leq M, \\ 0 \leq \frac{d}{dx}(u_0(x) + 2e^{\frac{s_0(x)}{2}} e^{-\frac{1}{2\rho_0(x)}}) \leq M, \\ |\frac{ds_0(x)}{dx}| \leq M. \end{cases} \tag{1.7}$$

However, there is a gap in the proof in [1] because the same estimates, like (1.7), on the solution $(\rho(x,t), u(x,t), s(x,t))$, do not ensure the boundedness of $(\rho(x,t), u(x,t), s(x,t))$.

In this paper, we shall repair this gap by assuming that the amplitude of the first two Riemann invariants $(w_1(x,0), w_2(x,0))$ of system (1.1) is suitable small (see the condition (1.9) below), and weaken the condition (H).

Mainly we have the following result.

Theorem 1 *Let $(\rho_0, u_0(x), s_0(x))$ be bounded, $\rho_0 \geq 0, s_0 \geq 0$. $w_1(x,0), w_2(x,0)$ are nondecreasing, and there exist constants c_0, c_1, c_2, M such that*

$$c_1 \leq w_1(x,0) \leq c_0, \quad c_0 \leq w_2(x,0) \leq c_2, \quad \int_R |s_x(x,0)| dx \leq M, \tag{1.8}$$

where

$$0 < c_2 - c_1 < 4 \tag{1.9}$$

and

$$w_1(x,0) = u_0(x) - 2e^{\frac{s_0(x)}{2}} e^{-\frac{1}{2\rho_0(x)}}, \quad w_2(x,0) = u_0(x) + 2e^{\frac{s_0(x)}{2}} e^{-\frac{1}{2\rho_0(x)}}. \tag{1.10}$$

Then the Cauchy problem (1.1) and (1.2) has a generalized bounded solution $(\rho(x,t), u(x,t), s(x,t))$ satisfying that the first and the second Riemann invariants $w_1(x,t), w_2(x,t)$ are nondecreasing and $\int_R |s_x(x,t)| dx \leq M$.

It is worthwhile to point out that a similar result on nonlinear system of three equations was obtained in [2], where the authors studied the global weak solution for the following system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho, s))_x = 0 \\ ((\rho s)_t + (\rho u s)_x = (\frac{1}{\rho} s_x)_x, \end{cases} \tag{1.11}$$

where $P(\rho, s) = e^{(\gamma-1)s}\rho^\gamma$.

This paper is organized as follows: In Section 2, we introduce a variant of the viscosity argument, and construct the approximated solutions of the Cauchy problem (1.1) and (1.2) by using the solutions of the parabolic system (2.1) with the initial data (2.2). Under the conditions in Theorem 1, we can easily obtain the necessary boundedness estimates (2.8), (2.9) and (2.10) on the approximated solutions $(w_1^{\varepsilon,\delta}(x,t), w_2^{\varepsilon,\delta}(x,t), w_3^{\varepsilon,\delta}(x,t))$, where the bound $M(\delta, T)$ in (2.9) could tend to infinity as δ goes to zero or T goes to infinity. In Section 3, based on the estimates (2.9) and (2.10) , we obtained the pointwise convergence of the viscosity solutions $(\rho^{\varepsilon,\delta}(x,t), u^{\varepsilon,\delta}(x,t), s^{\varepsilon,\delta}(x,t))$ by using the compensated compactness theory [3, 4, 5, 6, 7, 8, 9].

2 Viscosity Solutions

In this section we construct the approximated solutions of the Cauchy problem (1.1) and (1.2) by using the following parabolic systems

$$\begin{cases} w_{1t} + \lambda_1 w_{1x} = \varepsilon w_{1xx} \\ w_{2t} + \lambda_2 w_{2x} = \varepsilon w_{2xx} \\ w_{3t} + \lambda_3 w_{1x} = \varepsilon w_{3xx} \end{cases} \tag{2.1}$$

with initial data

$$(w_1(x,0), w_2(x,0), w_3(x,0)) = (w_{10}(x) * G^\delta - \delta, w_{20}(x) * G^\delta + \delta, s_0(x) * G^\delta), \tag{2.2}$$

where ε, δ are small positive constants, G^δ is a mollifier. and $(\rho_0, u_0(x), s_0(x))$ are given by (1.2). Thus $w_i(x,0), i = 1,2,3$ are smooth functions, and satisfy

$$c_1 - \delta \le w_1(x,0) \le c_0 - \delta, \quad c_0 + \delta \le w_2(x,0) \le c_2 + \delta, \quad 0 \le w_3(x,0) \le c_3, \tag{2.3}$$

$$0 \le w_{1x}(x,0) \le M(\delta), \quad 0 \le w_{2x}(x,0) \le M(\delta), \quad |w_{3x}(x,0)| \le M(\delta) \tag{2.4}$$

and

$$\int_R |w_{3x}(x,0)| dx \le M, \tag{2.5}$$

where M is a positive constant being independent of δ and $M(\delta)$ is a constant, which could tend to infinity as δ tends to zero.

First, following the standard theory of semilinear parabolic systems, the local existence result of the Cauchy problem (2.1), (2.2) can be easily obtained by applying the contraction mapping principle to an integral representation for a solution.

Lemma 2 *Let $w_i(x,0), i = 1,2,3$ be bounded in C^1 space and satisfy (2.3) and (2.4). Then for any fixed $\varepsilon > 0, \delta > 0$, the Cauchy problem (2.1) and (2.2) always has a local smooth solution $w_i^{\varepsilon,\delta}(x,t) \in C^\infty(R \times (0,\tau))$ for a small time τ, which depends only on the L^∞ norm of the initial data $w_i(x,0), i = 1,2,3,$ and satisfies*

$$c_1 - \frac{3\delta}{2} \le w_1^{\varepsilon,\delta}(x,t) \le c_0 - \frac{\delta}{2}, \quad c_0 + \frac{\delta}{2} \le w_2^{\varepsilon,\delta}(x,t) \le c_2 + \frac{3\delta}{2}, \quad |w_3^{\varepsilon,\delta}(x,t)| \le 2c_3, \tag{2.6}$$

and

$$|w_{1x}^{\varepsilon,\delta}(x,t)| \le 2M(\delta), \quad |w_{2x}^{\varepsilon,\delta}(x,t)| \le 2M(\delta), \quad |w_{3x}^{\varepsilon,\delta}(x,t)| \le 2M(\delta). \tag{2.7}$$

Second, by using the maximum principle given in [10], we have the following a priori estimates on the solutions of the Cauchy problem (2.1) and (2.2)

Lemma 3 *Let $w_i(x,0), i = 1,2,3$ be bounded in C^1 space and satisfy (2.3) and (2.4). Moreover, suppose that $(w_1^{\varepsilon,\delta}(x,t), w_2^{\varepsilon,\delta}(x,t), w_3^{\varepsilon,\delta}(x,t))$ is a smooth solution of (2.1), (2.2) defined in a strip $(-\infty,\infty) \times [0,T]$ with $0 < T < \infty$. Then*

$$c_1 - \delta \le w_1^{\varepsilon,\delta}(x,t) \le c_0 - \delta, \; c_0 + \delta \le w_2^{\varepsilon,\delta}(x,t) \le c_2 + \delta, \; 0 \le w_3^{\varepsilon,\delta}(x,t) \le c_3, \tag{2.8}$$

$$0 \le w_{1x}^{\varepsilon,\delta}(x,t) \le M(\delta,T), \quad 0 \le w_{2x}^{\varepsilon,\delta}(x,t) \le M(\delta,T), \quad |w_{3x}^{\varepsilon,\delta}(x,t)| \le M(\delta) \tag{2.9}$$

and

$$\int_R |w_{3x}^{\varepsilon,\delta}(x,t)| dx \le M, \tag{2.10}$$

where the bound $M(\delta,T)$ could go to infinity as δ goes to zero or T goes to infinity.

Proof of Lemma 3. The estimates in (2.8) can be obtained by using the maximum principle to (2.1), (2.2) and the condition (2.3) directly.

We differentiate (2.1) with respect to x and let $w_{ix} = \phi_i, i = 1,2,3$; then we have the following parabolic system

$$\phi_{it} + \lambda_i \phi_{ix} + (\sum_{j=1}^{3} \lambda_{iw_j} \phi_j) \phi_i = \varepsilon \phi_{ixx}, \;\; i = 1,2,3. \tag{2.11}$$

The nonnegativity $w_{1x}^{\varepsilon,\delta}(x,t) \geq 0, w_{2x}^{\varepsilon,\delta}(x,t) \geq 0$ in (2.9) can be obtained by using the maximum principle to the first and second equations in (2.11) and the condition $w_{1x}(x,0) \geq 0, w_{2x}(x,0) \geq 0$ in (2.4).

Since $\lambda_3 = u, w_3 = s$, the third equation in (2.11) is

$$\phi_{3t} + u\phi_{3x} + u_x\phi_3 = \varepsilon\phi_{3xx}. \tag{2.12}$$

Since $u_x = \phi_1 + \phi_2 \geq 0$, by using the maximum principle to (2.12) and the condition $|w_{3x}(x,0)| \leq M(\delta)$ in (2.4), we can easily prove the estimate $|w_{3x}^{\varepsilon,\delta}(x,t)| \leq M(\delta)$ in (2.9).

To prove the left estimates in (2.9), we first calculate $\lambda_{iw_j}, i = 1,2, j = 1,2,3$. By simple calculations, we have from the Riemann invariants given in (1.6)

$$\begin{cases} u_{w_1} = \frac{1}{2}, \ \ \rho_{w_1} = -\frac{\rho^2}{2}e^{\frac{-s}{2}}e^{\frac{1}{2\rho}}, \ \ s_{w_1} = 0, \\ u_{w_2} = \frac{1}{2}, \ \ \rho_{w_2} = \frac{\rho^2}{2}e^{\frac{-s}{2}}e^{\frac{1}{2\rho}}, \ \ s_{w_2} = 0, \\ u_{w_3} = 0, \ \ \rho_{w_3} = 0, \ \ s_{w_3} = -\rho^2 \end{cases} \tag{2.13}$$

and so

$$\begin{cases} \lambda_{1w_1} = \lambda_{1u}u_{w_1} + \lambda_{1\rho}\rho_{w_1} + \lambda_{1s}s_{w_1} = \frac{1}{4\rho}, \ \ \lambda_{1w_2} = 1 - \frac{1}{4\rho}, \ \ \lambda_{1w_3} = -\frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}, \\ \lambda_{2w_1} = 1 - \frac{1}{4\rho}, \ \ \lambda_{2w_2} = \frac{1}{4\rho}, \ \ \lambda_{2w_3} = \frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}. \end{cases} \tag{2.14}$$

Then the first and second equations in (2.11) are

$$\begin{cases} \phi_{1t} + \lambda_1\phi_{1x} + (\frac{1}{4\rho}\phi_1 + (1 - \frac{1}{4\rho})\phi_2)\phi_1 - \frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\phi_3\phi_1 = \varepsilon\phi_{1xx}, \\ \phi_{2t} + \lambda_2\phi_{2x} + ((1 - \frac{1}{4\rho})\phi_1 + \frac{1}{4\rho}\phi_2)\phi_2 + \frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\phi_3\phi_2 = \varepsilon\phi_{1xx}. \end{cases} \tag{2.15}$$

By using the nonnegativity of ϕ_1, ϕ_2, we have from (2.15) that

$$\begin{cases} \phi_{1t} + \lambda_1\phi_{1x} + (\frac{1}{4\rho}\phi_1 - \frac{1}{4\rho})\phi_2)\phi_1 - \frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\phi_3\phi_1 \leq \varepsilon\phi_{1xx}, \\ \phi_{2t} + \lambda_2\phi_{2x} + (-\frac{1}{4\rho}\phi_1 + \frac{1}{4\rho}\phi_2)\phi_2 + \frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\phi_3\phi_2 \leq \varepsilon\phi_{1xx}. \end{cases} \tag{2.16}$$

Let the bound of $|\frac{1}{2\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\phi_3|$ be $M_1(\delta)$, and

$$\phi_1 = Xe^{M_1(\delta)t}, \ \ \phi_2 = Ye^{M_1(\delta)t}. \tag{2.17}$$

Then we have from (2.16) that

$$\begin{cases} X_t + \lambda_1 X_x + \frac{1}{4\rho} e^{M_1(\delta)t}(X-Y)X + (M_1(\delta) - \frac{1}{2\rho} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} \phi_3)X \le \varepsilon X_{xx}, \\ \\ Y_t + \lambda_2 Y_x + \frac{1}{4\rho} e^{M_1(\delta)t}(Y-X)Y + (M_1(\delta) + \frac{1}{2\rho} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} \phi_3)Y \le \varepsilon Y_{xx}. \end{cases} \tag{2.18}$$

By using the nonnegativity of X, Y again, we have from (2.18) that

$$\begin{cases} X_t + \lambda_1 X_x + \frac{1}{4\rho} e^{M_1(\delta)t}(X-Y)X \le \varepsilon X_{xx}, \\ \\ Y_t + \lambda_2 Y_x + \frac{1}{4\rho} e^{M_1(\delta)t}(Y-X)Y \le \varepsilon Y_{xx}. \end{cases} \tag{2.19}$$

From the conditions in (2.4), we have

$$X(x,0) \le M(\delta), \quad Y(x,0) \le M(\delta). \tag{2.20}$$

Repeating the proof of Lemma 2.4 in [10], where $\lambda = 0$ in our case, we can obtain by using the maximum principle to (2.19), (2.20) that

$$X(x,t) \le M(\delta), \quad Y(x,t) \le M(\delta) \quad \text{for } 0 < t \le T, \tag{2.21}$$

and so

$$\phi_1 = Xe^{M_1(\delta)t}) \le M(\delta)e^{M_1(\delta)T}, \quad \phi_2 = Ye^{M_1(\delta)t}) \le M(\delta)e^{M_1(\delta)T}. \tag{2.22}$$

Finally using the same technique to estimate (2.8) in [11], we can prove (2.10) from (2.5), and so complete the proof of Lemma 3.

From the estimates (2.8) and (2.9), we have the following estimates about the functions $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, s^{\varepsilon,\delta})$. First, from (2.8), we have

$$\frac{c_0 + c_1}{2} \le u^{\varepsilon,\delta} = \frac{w_1^{\varepsilon,\delta} + w_2^{\varepsilon,\delta}}{2} \le \frac{c_0 + c_2}{2}, \quad 0 \le s^{\varepsilon,\delta} \le c_3 \tag{2.23}$$

and

$$4e^{\frac{s^{\varepsilon,\delta}}{2}} e^{-\frac{1}{2\rho^{\varepsilon,\delta}}} = \frac{w_2^{\varepsilon,\delta} - w_1^{\varepsilon,\delta}}{2} \ge 2\delta$$

or

$$\rho^{\varepsilon,\delta} \ge c(\delta) > 0 \tag{2.24}$$

for a suitable constant $c(\delta)$, which could go to zero as δ goes to zero. Moreover,

$$4e^{\frac{s^{\varepsilon,\delta}}{2}} e^{-\frac{1}{2\rho^{\varepsilon,\delta}}} \le c_2 + \delta - (c_1 - \delta) = c_2 - c_1 + 2\delta$$

or

$$e^{-\frac{1}{2\rho^{\varepsilon,\delta}}} \leq \frac{c_2 - c_1}{4} + \frac{\delta}{2} < c_4 < 1,$$

which implies

$$\rho^{\varepsilon,\delta} < -\frac{1}{2\ln c_4}. \tag{2.25}$$

Second, from (2.9), we have

$$0 \leq u_x^{\varepsilon,\delta} \leq M(\delta, T), \quad |\rho_x^{\varepsilon,\delta}| \leq M(\delta, T), \quad |s_x^{\varepsilon,\delta}| \leq M(\delta). \tag{2.26}$$

Based on the a priori estimates given in (2.8) and (2.9) on the local solution, and the positive lower bound (2.24), we may extend the local time τ in Lemma 2 step by step to arbitrary large time T and obtain the following global existence of solution for the Cauchy problem (2.1) and (2.2).

Theorem 4 *Let $w_i(x, 0), i = 1, 2, 3$ be bounded in C^1 space and satisfy (2.3)-(2.5), then the Cauchy problem (2.1) and (2.2) has a unique global smooth solution satisfying (2.8)-(2.10).*

3 Proof of Theorem 1.

In this section, we shall prove Theorem 1. First, based on the BV estimate (2.10) on the sequence of functions $s^{\varepsilon,\delta}$, we have the following lemmas

Lemma 5 *For any constant c, $c_t + s_x^{\varepsilon,\delta}$ and $s_t^{\varepsilon,\delta} + c_x$ are compact in $H_{loc}^{-1}(R \times R^+)$.*

Proof of Lemma 5. Since $c_t + s_x^{\varepsilon,\delta}$ are bounded in $L_{loc}^1(R \times R^+)$ from (2.10), hence compact in $W_{loc}^{-1,\alpha}$ for $\alpha \in (1, 2)$ by the Sobolev embedding theorem. Noticing that $c_t + s_x^{\varepsilon,\delta}$ are bounded in $W^{-1,\infty}$, we get the proof by Murat's theorem [12] that $c_t + s_x^{\varepsilon,\delta}$ are compact in $H_{loc}^{-1}(R \times R^+)$.

From the third equation in (2.1), we have that

$$s_t^{\varepsilon,\delta} + c_x = -u^{\varepsilon,\delta} s_x^{\varepsilon,\delta} + \varepsilon s_{xx}^{\varepsilon,\delta}, \tag{3.1}$$

where $u^{\varepsilon,\delta} s_x^{\varepsilon,\delta}$ are bounded in $L_{loc}^1(R \times R^+)$ from (2.10) and (2.23). If we choose ε to go zero more fast than δ, then $\varepsilon s_{xx}^{\varepsilon,\delta}$ are compact in $H_{loc}^{-1}(R \times R^+)$ from the last estimate in (2.26). Thus $c_t + s_x^{\varepsilon,\delta}$ are compact in $H_{loc}^{-1}(R \times R^+)$. The proof of Lemma 5 is completed.

Second, we prove the pointwise convergence of $s^{\varepsilon,\delta}$.

Lemma 6 *There exists a subsequence (still labelled) $s^{\varepsilon,\delta}$ such that when ε goes to zero more fast than δ,*

$$s^{\varepsilon,\delta}(x,t) \to s(x,t), \ \ a.e. \ \ on \ \Omega, \tag{3.2}$$

where $s(x,t)$ is a bounded function, and $\Omega \subset R \times R^+$ is any bounded open set.

Proof of Lemma 6. Using the results in Lemma 5, we may apply the div-curl lemma to the pairs of functions

$$(c, s^{\varepsilon,\delta}), \quad (s^{\varepsilon,\delta}, c) \tag{3.3}$$

to obtain

$$\overline{s^{\varepsilon,\delta}} \cdot \overline{s^{\varepsilon,\delta}} = \overline{(s^{\varepsilon,\delta})^2}, \tag{3.4}$$

where $\overline{s^{\varepsilon,\delta}}$ denotes the weak-star limit of $s^{\varepsilon,\delta}$, which gives us the pointwise convergence of $s^{\varepsilon,\delta}$. Thus Lemma 6 is proved.

We are going to complete the proof of Theorem 1 by proving the pointwise convergence of $\rho^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$.

Let the matrix A be

$$A = \begin{pmatrix} w_{1\rho} & w_{1m} & w_{1s} \\ w_{2\rho} & w_{2m} & w_{2s} \\ w_{3\rho} & w_{3m} & w_{3s} \end{pmatrix} = \begin{pmatrix} -\frac{m}{\rho^2} - \frac{1}{\rho^2} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} & \frac{1}{\rho} & -e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} \\ -\frac{m}{\rho^2} + \frac{1}{\rho^2} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} & \frac{1}{\rho} & e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

By simple calculations, the inversion of A is

$$A^{-1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} -\frac{\rho^2}{2} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} & \frac{\rho^2}{2} e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} & -\rho^2 \\ \frac{\rho}{2}(1 - me^{-\frac{s}{2}} e^{\frac{1}{2\rho}}) & \frac{\rho}{2}(1 + me^{-\frac{s}{2}} e^{\frac{1}{2\rho}}) & -\rho m \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

We multiply (2.1) by A^{-1} to get

$$\begin{pmatrix} \rho_t \\ (\rho u)_t \\ s_t \end{pmatrix} + \begin{pmatrix} (\rho u)_x \\ (\rho u + P(\rho, s))_x \\ u s_x \end{pmatrix} = \varepsilon \begin{pmatrix} \rho_{xx} \\ (\rho u)_{xx} \\ s_{xx} \end{pmatrix} - \varepsilon (A^{-1})_x \begin{pmatrix} w_{1x} \\ w_{2x} \\ w_{3x} \end{pmatrix}. \tag{3.7}$$

Now we fix s as a constant, and consider the following system about the variables ρ, u:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho, s))_x = 0. \end{cases} \tag{3.8}$$

A pair of smooth functions $(\eta(\rho, u, s), q(\rho, u, s))$ is called a pair of entropy-entropy flux of system (3.8) if $(\eta(\rho, u, s), q(\rho, u, s))$ satisfies the additional system

$$(q_\rho, q_u) = (\eta_\rho, \eta_u) \cdot \begin{pmatrix} u & \rho \\ \frac{P_\rho(\rho,s)}{\rho} & u \end{pmatrix} \tag{3.9}$$

or equivalently

$$q_\rho = u\eta_\rho + \frac{P_\rho(\rho, s)}{\rho}\eta_u, \quad q_u = \rho\eta_\rho + u\eta_u. \tag{3.10}$$

Eliminating the q from (3.10), we have

$$\eta_{\rho\rho} = \frac{P_\rho(\rho, s)}{\rho^2}\eta_{uu}. \tag{3.11}$$

For any smooth pair of entropy-entropy flux $(\eta(\rho, u, s), q(\rho, u, s))$, we multiply $(\eta_\rho(\rho, u, s), \eta_m(\rho, u, s), \eta_s(\rho, u, s))$ to (3.7), where $m = \rho u$, to obtain

$$\eta_t(\rho, u, s) + q_x(\rho, u, s) + (-q_s(\rho, u, s) + \eta_m P_s(\rho, s) + \eta_s u)s_x = R, \tag{3.12}$$

where

$$R = \varepsilon \begin{pmatrix} \eta_\rho & \eta_m & \eta_s \end{pmatrix} \left(\begin{pmatrix} \rho_{xx} \\ (\rho u)_{xx} \\ s_{xx} \end{pmatrix} - (A^{-1})_x \begin{pmatrix} w_{1x} \\ w_{2x} \\ w_{3x} \end{pmatrix} \right). \tag{3.13}$$

By using the estimate (2.10), we have that $(-q_s(\rho, u, s) + \eta_m P_s(\rho, s) + \eta_s u)s_x$ is uniformly bounded in $L^1(R \times R^+)$, and hence compact in $W^{-1,\alpha}_{loc}$ for $\alpha \in (1, 2)$ by the Sobolev embedding theorem. If we choose ε to go zero much fast than δ, by using the estimates in (2.26), we have that R is compact in $H^{-1}_{loc}(R \times R^+)$, Therefore, we have the following Lemma:

Lemma 7 *For any fixed s, let $(\eta(\rho, u, s), q(\rho, u, s))$ be an arbitrary pair of smooth entropy-entropy flux of system (3.8). Then*

$$\eta(\rho^{\varepsilon,\delta}(x, t), u^{\varepsilon,\delta}(x, t), s^{\varepsilon,\delta}(x, t))_t + q(\rho^{\varepsilon,\delta}(x, t), u^{\varepsilon,\delta}(x, t), s^{\varepsilon,\delta}(x, t))_x \tag{3.14}$$

are compact in $H^{-1}_{loc}(R\times R^+)$, where $(\rho^{\varepsilon,\delta}(x,t), u^{\varepsilon,\delta}(x,t), s^{\varepsilon,\delta}(x,t))$ are determinated by the Riemann invariants $(w_1, w_2.w_3)$, which are the solution of the Cauchy problem (2.1) and (2.2).

Thus for fixed s, for smooth entropy-entropy flux pairs $(\eta_i(\rho,u,s), q_i(\rho,u,s)), i = 1,2$, of system (3.8), the following measure equations or the communicate relations are satisfied

$$\begin{aligned}&< \nu_{(x,t)}, \eta_1(\cdot,s)q_2(\cdot,s) - \eta_2(\cdot,s)q_1(\cdot,s) >\\ &\quad =< \nu_{(x,t)}, \eta_1(\cdot,s) >< \nu_{(x,t)}, q_2(\cdot,s) > - < \nu_{(x,t)}, \eta_2(\cdot,s) >< \nu_{(x,t)}, q_1(\cdot,s) >,\end{aligned} \tag{3.15}$$

where $\nu_{(x,t)}$ is the family of positive probability measures with respect to the viscosity solutions $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, s^{\varepsilon,\delta})$ of the Cauchy problem (2.1) and (2.2).

Therefore by using the framework from the compensated compactness theory, given in [3, 4, 5], we may deduce that Young measures given in (3.15) are Dirac measures, and the pointwise convergence of $\rho^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$. **Theorem 1 is proved**.

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