Hyperbolic Approximation on System of Elasticity in Lagrangian Coordinates

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Abstract

In this paper, we construct a sequence of hyperbolic systems (1.13) to approximate the general system of one-dimensional nonlinear elasticity in Lagrangian coordinates (1.14). For each fixed approximation parameter δ, we establish the existence of entropy solutions for the Cauchy problem (1.13) with bounded initial data (1.24).

1 Introduction

Three most classical, hyperbolic systems of two equations in one-dimension are the system of isentropic gas dynamics in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{1.1}$$

where ρ is the density of gas, u the velocity and $P = P(\rho)$ the pressure; the nonlinear hyperbolic system of elasticity

$$\begin{cases} u_t + f(v)_x = 0 \\ v_t + u_x = 0, \end{cases} \tag{1.2}$$

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where v denotes the strain, $f(v)$ is the stress and u the velocity, which describes the balance of mass and linear momentum, and is equivalent to the nonlinear wave equation

$$v_{tt} = f(v)_{xx}; \tag{1.3}$$

and the system of compressible fluid flow

$$\begin{cases} u_t + (\frac{1}{2}u^2 + F(\rho))_x = 0 \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.4}$$

To obtain the global existence of weak solutions for nonstrictly hyperbolic systems (two eigenvalues are real, but coincide at some points or lines), the compensated compactness theory (cf. [22, 26] or the books [15, 23, 24]) is still a powerful and unique method until now.

For the polytropic gas $P(\rho) = c\rho^\gamma$, where $\gamma \geq 1$ and c is an arbitrary positive constant, the Cauchy problem (1.1) with bounded initial data was completely resolved by many authors (cf. [1, 4, 6, 9, 13, 14]). When $P(\rho)$ has the same principal singularity as the γ-law in the neighborhood of vacuum ($\rho = 0$), a compact framework was first provided in [2, 3] and later, the necessary H^{-1} compactness of weak entropy-entropy flux pairs for general pressure function was completed in [19].

Under the strictly hyperbolic condition $f'(v) \geq c > 0$ and some linearly degenerate conditions $v\dot{f}''(v) > 0$ or $v\dot{f}''(v) < 0$ as $v \neq 0$, the global existence of weak bounded solutions, or L^p solutions, $1 < p < \infty$ was obtained by Diperna [5] and Lin [12], Shearer [25] respectively.

Without the strictly hyperbolic restriction, a preliminary existence result of the nonlinear wave equation (1.3) was proved in [17] for the special case $f(v) = v|v|^{\gamma-1}, \gamma > 1$ under the assumption $v \geq 0$ or $v \leq 0$.

Using the Glimm's scheme method (cf. [8]), Diperna [7] first studied the system (1.4) in a strictly hyperbolic region. Roughly speaking, for the polytropic case $F(\rho) = c\rho^{\gamma-1}$, Diperna's results cover the case $1 < \gamma < 3$.

Since the solutions for the case of $\gamma > 3$ always touch the vacuum, its existence was obtained in [18] by using the compensated compactness method coupled with some basic ideas of the kinetic formulations (cf. [13, 14]). The existence of the Cauchy problem (1.4) for more general function $F(\rho)$ was given in [16] under some conditions to ensure the H^{-1} compactness for all smooth entropy-entropy flux pairs.

If all smooth entropy-entropy flux pairs satisfy the H^{-1} compactness, an ideal compactness framework to prove the global existence was provided by Diperna in [5]. For the above three systems (1.1)-(1.2) and (1.4), we can prove the H^{-1} compactness only for half of the entropies (weak or strong entropy).

In [19] (see also [20] for inhomogeneous system), the author constructed a sequence of regular hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = 0, \end{cases} \tag{1.5}$$

to approximate system (1.1), where $\delta > 0$ in (1.5) denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t)dt = P(\rho) - 2\delta \int_{2\delta}^{\rho} \frac{1}{t} P'(t)dt. \tag{1.6}$$

The most interesting point of this kind approximation is that both systems (1.5) and (1.1) have the same entropies (or the same entropy equation). In [19], the H^{-1} compactness of weak entropy-entropy flux pairs was also proved for general pressure function $P(\rho)$.

Let the entropy-entropy flux pairs of systems (1.1) and (1.5) be $(\eta(\rho, u), q(\rho, u))$ and $(\eta(\rho, u), q(\rho, u) + \delta q_a(\rho, u))$ respectively. Then by using Murat-Tartar theorem, we have

$$\begin{cases} \overline{\eta_1 q_2 - \eta_2 q_1} + \delta(\overline{\eta_1 q_{a_2} - \eta_2 q_{a_1}}) \\ \\ = \overline{\eta_1} \quad \overline{q_2} - \overline{\eta_2} \quad \overline{q_1} + \delta(\overline{\eta_1} \quad \overline{q_{a_2}} - \overline{\eta_2} \quad \overline{q_{a_1}}), \end{cases} \tag{1.7}$$

for any fixed $\delta \geq 0$, where the weak-star limit is denoted by $w^\star\!-\!\lim \eta(u^\varepsilon) = \overline{\eta(u^\varepsilon)}$ as ε goes to zero.

Paying attention to the approximation function (1.6), we know that

$$(\eta(\rho, u), \delta q_a(\rho, u)) \quad \text{or} \quad (\eta(\rho, u), q_a(\rho, u)) \tag{1.8}$$

are the entropy-entropy flux pairs of system

$$\begin{cases} \rho_t - 2\delta u_x = 0 \\ (\rho u)_t - (\delta u^2 + 2\delta \int_{2\delta}^{\rho} \frac{1}{t} P'(t)dt)_x = 0, \end{cases} \tag{1.9}$$

or system

$$\begin{cases} \rho_t - 2u_x = 0 \\ (\rho u)_t - (u^2 + 2\int_{2\delta}^{\rho} \frac{1}{t} P'(t)dt)_x = 0 \end{cases} \tag{1.10}$$

respectively.

If we could prove from the arbitrary of δ in (1.7) that

$$< \eta_1 q_2 - \eta_2 q_1 >=< \eta_1 >< q_2 > - < \eta_2 >< q_1 > \tag{1.11}$$

and

$$< \eta_1 q_{a_2} - \eta_2 q_{a_1} >=< \eta_1 > q_{a_2} > - < \eta_2 >< q_{a_1} >, \tag{1.12}$$

where $< h >$ denotes the weak-star limit $w^\star - \lim h(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ as ε, δ tend to zero, then we would have more function equations (1.12) to reduce the strong convergence of $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ as ε, δ tend to zero.

Between systems (1.2) and (1.4), we have the following approximation

$$\begin{cases} v_t - u_x + \delta(vu)_x = 0 \\ u_t + p(v)_x + \delta(\frac{1}{2}u^2 - vp(v) + \int^v p(s)ds)_x = 0 \end{cases} \tag{1.13}$$

which has also the same entropy equation like system (1.2). If we could prove (1.11) and (1.12) from (1.7), then similarly we could prove the equivalence of systems (1.2) and (1.4). Moreover, we have much more information from system (1.13) to prove the existence of solutions for system (1.2) or (1.4).

Systems (1.13) and

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \tag{1.14}$$

have many common basic behaviors, such as the nonstrict hyperbolicity, the same entropy equation, same Riemann invariants and so on.

By simple calculations, two eigenvalues of system (1.13) are

$$\lambda_1 = \delta u - (\delta v - 1)\sqrt{-p'(v)}, \quad \lambda_2 = \delta u + (\delta v - 1)\sqrt{-p'(v)} \tag{1.15}$$

with corresponding right eigenvectors

$$r_1 = (1, -\sqrt{-p'(v)})^T, \quad r_2 = (1, \sqrt{-p'(v)})^T \tag{1.16}$$

and Riemann invariants

$$z(u,v) = u - \int^v \sqrt{-p'(s)}ds, \quad w(u,v) = u + \int^v \sqrt{-p'(s)}ds. \tag{1.17}$$

Moreover

$$\nabla\lambda_1 \cdot r_1 = \frac{4\delta p'(v) + (\delta v - 1)p''(v)}{2\sqrt{-p'(v)}} \tag{1.18}$$

and

$$\nabla\lambda_2 \cdot r_2 = -\frac{4\delta p'(v) + (\delta v - 1)p''(v)}{2\sqrt{-p'(v)}}. \tag{1.19}$$

Any entropy-entropy flux pair $(\eta(v,u), q(v,u))$ of system (1.13) satisfies the additional system

$$q_v = \delta u\eta_v + (1-\delta v)p'(v)\eta_u, \quad q_u = (\delta v - 1)\eta_v + \delta u\eta_u. \tag{1.20}$$

Eliminating the q from (1.20), we have

$$\eta_{vv} = -p'(v)\eta_{uu}. \tag{1.21}$$

Therefore systems (1.13) and (1.14) have the same entropies. From these calculations, we know that system (1.13) is strictly hyperbolic in the domain $\{(x,t) : 0 < v < \frac{1}{\delta}\}$ or $\{(x,t) : v > \frac{1}{\delta}\}$, while it is nonstrictly hyperbolic on the domain $\{(x,t) : v = \frac{1}{\delta}\}$ since $\lambda_1 = \lambda_2$ when $v = \frac{1}{\delta}$.

However, from (1.18) and (1.19), for each fixed δ, both characteristic fields of system (1.13) are genuinely nonlinear in the domain $\{(x,t) : 0 < v \leq \frac{1}{\delta}\}$ if $p'(v) < 0, p''(v) > 0$ or in the domain $\{(x,t) : v \geq \frac{1}{\delta}\}$ if $p'(v) < 0, p''(v) < 0$. In the first case $(p'(v) < 0, p''(v) > 0)$, we have an a-priori L^∞ estimate for the solutions of system (1.13)

$$c_1 \leq v \leq \frac{1}{\delta}, \quad |u| \leq M_1 \tag{1.22}$$

because the region

$$R_\delta = \{(v,u) : w(v,u) \geq -M, \ z(v,u) \leq M, v \leq \frac{1}{\delta}\}$$

is an invariant region, where $c_1 \leq c_0$,(c_0 is given in Theorem 1), M and M_1 are positive constants depending on the initial date, but being independent of δ. In the second case $(p'(v) < 0, p''(v) < 0)$, we have the L^∞ estimate

$$\frac{1}{\delta} \leq v \leq M_1, \quad |u| \leq M_1 \tag{1.23}$$

because the region

$$R_\delta = \{(v,u) : w(v,u) \leq M, \ z(v,u) \geq -M, v \geq \frac{1}{\delta}\}$$

is an invariant region.

In this paper, for fixed $\delta > 0$, we first establish the existence of entropy solutions for the Cauchy problem (1.13) with bounded measurable initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)). \tag{1.24}$$

In a further coming paper, we will study the relation between the functions equations (1.11) and (1.12), and the convergence of approximated solutions of system (1.13) as δ goes to zero.

Theorem 1 *Suppose the initial data $(v_0(x), u_0(x))$ be bounded measurable. Let (I): $p'(v) < 0, p''(v) > 0, c_0 \le v_0(x) \le \frac{1}{\delta}$, where $c_0 > 0$ is a positive constant, or (II): $p'(v) < 0, p''(v) < 0, v_0(x) \ge \frac{1}{\delta}$. Then the Cauchy problem (1.13) with the bounded measurable initial data (1.24) has a global bounded entropy solution.*

Note 1. The idea to use the flux perturbation coupled with the vanishing viscosity was well applied by the author in [21] to control the super-line, source terms and to obtain the L^∞ estimate for the nonhomogeneous system of isentropic gas dynamics.

Note 2. It is well known that system (1.14) is equivalent to system (1.1), but (1.1) is different from system (1.4) although the latter can be derived by substituting the first equation in (1.1) into the second. However, (1.4) can be considered as the approximation of (1.14). In fact, let $\rho = \frac{1}{\delta} - v, x = \delta y$ in (1.13). Then (1.13) is rewritten to the form

$$\begin{cases} \rho_t + (\rho u)_y = 0 \\ u_t + (\frac{u^2}{2} + g(\rho, \delta))_y = 0 \end{cases} \tag{1.25}$$

for some nonlinear function $g(\rho, \delta)$.

Note 3. For any fixed $\delta > 0$, the invariant region R_δ above is bounded, so the vacuum is avoided. However, the limit of R_δ, as δ goes to zero, is the original invariant region of system (1.14) because v could be infinity from the estimates in (1.22).

In the next section, we will use the compensated compactness method coupled with the construction of Lax entropies [11] to prove Theorem 1.

2 Proof of Theorem 1

In this section, we prove Theorem 1.

Consider the Cauchy problem for the related parabolic system

$$\begin{cases} v_t - u_x + \delta(vu)_x = \varepsilon v_{xx} \\ u_t + p(v)_x + \delta(\frac{1}{2}u^2 - vp(v) + \int^v p(s)ds)_x = \varepsilon u_{xx}, \end{cases} \tag{2.1}$$

with the initial data (1.24).

We multiply (2.1) by (w_v, w_u) and (z_v, z_u), respectively, to obtain

$$w_t + \lambda_2 w_x = \varepsilon w_{xx} + \frac{\varepsilon p''(v)}{2\sqrt{-p'(v)}} v_x^2, \tag{2.2}$$

and

$$z_t + \lambda_1 z_x = \varepsilon z_{xx} - \frac{\varepsilon p''(v)}{2\sqrt{-p'(v)}} v_x^2. \tag{2.3}$$

Then the assumptions on $p(v)$ yield

$$w_t + \lambda_2 w_x \geq \varepsilon w_{xx} \tag{2.4}$$

and

$$z_t + \lambda_1 z_x \leq \varepsilon z_{xx} \tag{2.5}$$

if $p'(v) < 0, p''(v) > 0$; or

$$w_t + \lambda_2 w_x \leq \varepsilon w_{xx} \tag{2.6}$$

and

$$z_t + \lambda_1 z_x \geq \varepsilon z_{xx} \tag{2.7}$$

if $p'(v) < 0, p''(v) < 0$.

If we consider (2.4) and (2.5) (or (2.6) and (2.7)) as inequalities about the variables w and z, then we can get the estimates $w(v^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \geq -M, z(v^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \leq M$ by applying the maximum principle to (2.4) and (2.5) (or $w(v^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \leq M, z(v^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \geq -M$ by applying the maximum principle to (2.6) and (2.7)). Then, using the first equation in (2.1), we get $v^{\varepsilon,\delta} \leq \frac{1}{\delta}$ or $v^{\varepsilon,\delta} \geq \frac{1}{\delta}$ depending on the conditions on $v_0(x)$. Therefore, the region

$$R_\delta = \{(v,u) : w(v,u) \geq -M,\ z(v,u) \leq M, v \leq \frac{1}{\delta}\}$$

or

$$R_\delta = \{(v,u) : w(v,u) \le M,\ z(v,u) \ge -M, v \ge \frac{1}{\delta}\}$$

is respectively an invariant region. Thus we obtain the estimates given in (1.22) or (1.23) respectively.

It is easy to check that system (1.13) has a strictly convex entropy when $v \le \frac{1}{\delta}$ or $v \ge \frac{1}{\delta}$

$$\eta^\star = \frac{u^2}{2} - \int^v \int^s p'(\tau) d\tau ds. \tag{2.8}$$

We multiply (2.1) by $(\eta_v^\star, \eta_u^\star)$ to obtain the boundedness of

$$\varepsilon(v_x, u_x) \cdot \nabla^2 \eta^\star(v,u) \cdot (v_x, u_x)^T \tag{2.9}$$

in $L^1_{loc}(R \times R^+)$. Then it follows that

$$-\varepsilon p'(v) v_x^2 + \varepsilon u_x^2 \tag{2.10}$$

is bounded in $L^1_{loc}(R \times R^+)$. Since $0 < C_1(\delta) \le -p'(v) \le C_2(\delta)$ for some bounded constants $C_1(\delta), C_2(\delta)$ when $v \le \frac{1}{\delta}$ or $v \ge \frac{1}{\delta}$, we get the boundedness of

$$\varepsilon v_x^2, \quad \varepsilon u_x^2 \quad \text{in} \quad L^1_{loc}(R \times R^+) \tag{2.11}$$

for any fixed $\delta > 0$.

Now we multiply (2.1) by $(\eta_v(v,u), \eta_u(v,u))$, where $\eta(v,u)$ is any smooth entropy of system (1.13), to obtain

$$\eta(v,u)_t + q(v,u)_x = \varepsilon \eta(v,u)_{xx} + \varepsilon(v_x, u_x) \cdot \nabla^2 \eta(v,u) \cdot (v_x, u_x)^T, \tag{2.12}$$

where $q(v,u)$ is the entropy-flux corresponding to $\eta(v,u)$. Then using the estimate given in (2.11), we know that the first term in the right-hand side of (2.12) is compact in $W^{-1,\infty}_{loc}(R \times R^+)$, and the second is bounded in $L^1_{loc}(R \times R^+)$. Thus the term in the left-hand side of (2.12) is compact in $H^{-1}_{loc}(R \times R^+)$.

Then for smooth entropy-entropy flux pairs $(\eta_i(\delta, v, u), q_i(\delta, v, u)), i = 1, 2,$ of system (1.13), the following measure equations or the communicate relations are satisfied

$$\begin{aligned} &< \nu^\delta_{(x,t)}, \eta_1(\delta) q_2(\delta) - \eta_2(\delta) q_1(\delta) > \\ &=< \nu^\delta_{(x,t)}, \eta_1(\delta) >< \nu^\delta_{(x,t)}, q_2(\delta) > - < \nu^\delta_{(x,t)}, \eta_2(\delta) >< \nu^\delta_{(x,t)}, q_1(\delta) >, \end{aligned} \tag{2.13}$$

where $\nu^{\delta}_{(x,t)}$ is the family of positive probability measures with respect to the viscosity solutions $(v^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ of the Cauchy problem (2.1) and (1.24).

To finish the proof of Theorem 1, it is enough to prove that Young measures given in (2.13) are Dirac measures.

For applying for the framework given by DiPerna in [5] to prove that Young measures are Dirac ones, we construct four families of entropy-entropy flux pairs of Lax's type in the following special form:

$$\eta^1_k = e^{kw}(a_1(v) + \frac{b_1(v,k)}{k}),\ q^1_k = \eta^1_k(\lambda_2 + \frac{c_1(v,k)}{k} + \frac{d_1(v,k)}{k^2}); \tag{2.14}$$

$$\eta^2_{-k} = e^{-kw}(a_2(v) + \frac{b_2(v,k)}{k}),\ q^2_{-k} = \eta^2_{-k}(\lambda_2 + \frac{c_2(v,k)}{k} + \frac{d_2(v,k)}{k^2}); \tag{2.15}$$

$$\eta^2_k = e^{kz}(a_3(v) + \frac{b_3(v,k)}{k}),\ q^2_k = \eta^2_k(\lambda_1 + \frac{c_3(v,k)}{k} + \frac{d_3(v,k)}{k^2}); \tag{2.16}$$

$$\eta^1_{-k} = e^{-kz}(a_4(v) + \frac{b_4(v,k)}{k}),\ q^1_{-k} = \eta^1_{-k}(\lambda_1 + \frac{c_4(v,k)}{k} + \frac{d_4(v,k)}{k^2}), \tag{2.17}$$

where w, z are the Riemann invariants of system (1.13) given by (1.17). Notice that all the unknown functions $a_i, b_i (i = 1,2,3,4)$ are only of a single variable v. This special simple construction yields an ordinary differential equation of second order with a singular coefficient $1/k$ before the term of the second order derivative. Then the following necessary estimates for functions $a_i(v), b_i(v,k), c_i(v,k), d_i(v,k)$ are obtained by the use of the singular perturbation theory of ordinary differential equations:

$$0 < a_i(v) \le M_2, \quad |b_i(v,k)| \le M_2, \tag{2.18}$$

$$0 < c_i(v,k) \le M_2, \quad (\text{ or } -M_2 \le c_i(v,k) < 0), \quad |d_i(v,k)| \le M_2 \tag{2.19}$$

uniformly for $0 < c_1 \le v \le \frac{1}{\delta}$ or $\frac{1}{\delta} \le v \le M_1$, where $i = 1,2,3,4$ and M_2 is a positive constant independent of k.

In fact, substituting entropies $\eta^1_k = e^{kw}(a_1(\rho) + b_1(\rho,k)/k)$ into (1.21), we obtain that

$$k[2\sqrt{-p'(v)}a_1' - \frac{p''(v)}{\sqrt{-2p'(v)}}a_1] + a_1'' + 2\sqrt{-p'(v)}b_1' - \frac{p''(v)}{\sqrt{-2p'(v)}}b_1 + \frac{b_1''}{k} = 0. \tag{2.20}$$

Let

$$2\sqrt{-p'(v)}a_1' - \frac{p''(v)}{\sqrt{-2p'(v)}}a_1 = 0 \tag{2.21}$$

and

$$a_1'' + 2\sqrt{-p'(v)}b_1' - \frac{p''(v)}{\sqrt{-2p'(v)}}b_1 + \frac{b_1''}{k} = 0. \tag{2.22}$$

Then

$$a_1 = \sqrt{-p'(v)} > 0. \tag{2.23}$$

The existence of $b_1(v,k)$ and its uniform bound $|b_1(v,k)| \leq M_2$ on $0 < c_1 \leq v \leq \frac{1}{\delta}$ or $\frac{1}{\delta} \leq v \leq M_1$ with respect to k can be obtained by the following lemma (cf. [10]) (also see Lemma 10.2.1 in [15]):

Lemma 2 *Let $Y(x) \in C^2[0,h]$ be the solution of the equation*

$$F(x, Y, Y') = 0,$$

and functions $f(x,y,z,\lambda), F(x,y,z)$ be continuous on the regions $0 \leq x \leq h, |y - Y(x)| \leq l(x), |z - Y'(x)| \leq m(x)$ for some positive functions $l(x), m(x)$ and $\lambda_0 > \lambda > 0$. In addition,

$$|f(x,y,z,\lambda) - F(x,y,z)| \leq \varepsilon,$$

$$|F(x,y_2,z) - F(x,y_1,z)| \leq M|y_2 - y_1|,$$
$$\frac{F(x,y,z_2) - F(x,y,z_1)}{z_2 - z_1} \geq L$$

for some positive constants ε, M and L.

If $y(x) = y(x,\lambda)$ is a solution of the following ordinary differential equation of second order:

$$\lambda y'' + f(x,y,y',\lambda) = 0,$$

with $y(0) = Y(0)$ and $y'(0)$ being arbitrary, then for sufficiently small $\lambda > 0, \varepsilon > 0$ and $P = |y'(0) - Y'(0)|$, $y(x)$ exists for all $0 \leq x \leq h$ and satisfies

$$|y(x,\lambda) - Y(x)| < \Big[\frac{\varepsilon}{M} + \lambda\big(\frac{P}{L} + \frac{N}{M}\big)\Big] exp\big(\frac{Mx}{L}\big),$$

where $N = \max_{0 \leq x \leq h} |Y(x)|$.

Furthermore, we can use Lemma 2 again to obtain the bound of b_1' with respect to k if we differentiate Equation (2.22) with respect to v.

By the second equation in (1.20), an entropy flux q_k^1 corresponding to η_k^1 is provided by

$$q_k^1 = \lambda_2 \eta_k^1 + e^{kw}\left(\frac{(\delta v - 1)a_1' - \delta a_1}{k} + \frac{(\delta v - 1)b_1' - \delta b_1}{k^2}\right), \tag{2.24}$$

where

$$(\delta v - 1)a_1' - \delta a_1 = -\frac{(\delta v - 1)p''(v)}{2\sqrt{-p'(v)}} - \delta\sqrt{-p'(v)} < 0 \tag{2.25}$$

if $v \leq \frac{1}{\delta}, p''(v) > 0$ or $v \geq \frac{1}{\delta}, p''(v) < 0$, and $(\delta v - 1)b_1' - \delta b_1$ both are bounded uniformly on $v \in [c_1, \frac{1}{\delta}]$ or $v \in [\frac{1}{\delta}, M_1]$.

In a similar way, we can obtain estimates on another three pairs of entropy-entropy flux of Lax type. Hence Theorem 1 is proved when we use these entropy-entropy flux pairs in (2.14)-(2.17) together with the theory of compensated compactness coupled with DiPerna's framework [5].

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