A general framework for the construction of high order well-balanced finite volume schemes for balance laws

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Structure preserving numerical methods for hyperbolic equations.
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Outline

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Introduction

- We consider 1d balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \quad (1)$$

where $U(x,t)$ takes values in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$).

- **Stationary solutions:**

$$F(U)_x = S(U)H_x.$$

Main objective

To design high-order finite volume methods that preserve all or a representative set of the stationary solutions of system (1): high order Well-balanced finite volume schemes.

High-order finite volume methods

- For systems of conservation laws:

$$U_t + F(U)_x = 0,$$

we consider numerical methods of the form:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right), \tag{2}$$

where

- $U_i(t)$ is the approximation given by the numerical method of the average of the exact solution at the ith cell, $[x_{i-1/2}, x_{i+1/2}]$ at time t;
- $F_{i+1/2}(t) = \mathbb{F}(U^{t,-}_{i+1/2}, U^{t,+}_{i+1/2})$;
- $\mathbb{F}$ is a consistent first order numerical flux;
- $U^{t,-}_{i+1/2} = P^t_i(x_{i+1/2}), \quad U^{t,+}_{i+1/2} = P^t_{i+1}(x_{i+1/2})$;
- $P^t_i(x)$ is the approximation of the solution at the ith cell obtained by applying a high-order reconstruction operator to $\{U_i(t)\}$.
- **E**xamples: ENO, WENO, CWENO, or hyperbolic reconstructions: see [HEOC97], [Mar94], [Shu97], [SO89],[DBTM08, DK07, DKTT07], [LPR00], [CS16].

High-order finite volume methods

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$$U_t + F(U)_x = 0,$$

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Time discretization

(2) is an ODE system which is solved by using a high order numerical solver with good properties, as the TVD Runge-Kutta schemes (see [GS98], [SO88]) or ADER schemes (see [DK07], [DKTT07]).

Outline

Well-balanced high-order finite volume scheme

- Let us consider the system

$$U_t + F(U)_x = S(U)H_x, \tag{3}$$

where $H(x)$ is supposed to be a known continuous function.

- We consider high-order finite volume numerical methods of the form:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right) + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx. \tag{4}$$

Well-balanced

The numerical scheme (4) is well balanced for the stationary solution U if the vector of the cell-averages of U, $\{U_i\}$, is an equilibrium of the ODE system (4)

Remarks

1. Typically,

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x)\,dx \approx U(x_i) = U_i,$$

for low order schemes (first and second order).

Well-balanced high-order finite volume scheme

- Let us consider the system

$$U_t + F(U)_x = S(U)H_x, \tag{3}$$

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- We consider high-order finite volume numerical methods of the form:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right) + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx. \tag{4}$$

Well-balanced

The numerical scheme (4) is well balanced for the stationary solution U if the vector of the cell-averages of U, $\{U_i\}$, is an equilibrium of the ODE system (4)

Remarks

1. In general, we use a high-order quadrature formula to approximate the cell averages

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x)\,dx \approx \sum_{j=0}^{M} \alpha_j U(x_i^j) = U_i,$$

Well-balanced high-order finite volume scheme

- Given a continuous stationary solution U of (3), the reconstruction operator is said to be well-balanced for U if

$$P_i(x) = U(x), \quad \forall x \in [x_{i-1/2}, x_{i+1/2}], \ \forall i, \tag{5}$$

where P_i is the approximation of U obtained by applying the reconstruction operator to the sequence of cell averages of U, $\{U_i\}$

Theorem

If the reconstruction operator is well-balanced for a stationary solution U, then the numerical method (4) *is also well-balanced for U, in the sense that the vector of the cell-averages of U is an equilibrium of the ODE system* (4).

Well-balanced high order reconstruction operator

But, a standard reconstruction operator is not expected in general to be well-balanced.

Well-balanced high order reconstruction operator (see [CLP08, CP20])

Given a family of cell averages $\{U_i\}$, at every cell $[x_{i-1/2}, x_{i+1/2}]$:

1. Look for the stationary solution $U_i^*(x)$ such that

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = U_i. \tag{6}$$

2. Apply the reconstruction operator to the cell values $\{V_j\}_{j\in\mathcal{S}_i}$ given by

$$V_j = U_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx,$$

to obtain

$$Q_i(x) = Q_i(x; \{V_j\}_{j\in\mathcal{S}_i}).$$

3. Define

$$P_i(x) = U_i^*(x) + Q_i(x).$$

Well-balanced high order reconstruction operator: comments

- **Remarks:**
  - If Q_i is exact for the zero function, conservative, and high order accurate, then P_i is well-balanced for every stationary solution; conservative, i.e.

    $$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} P_i(x)\,dx = U_i, \quad \forall i,$$

    and it is high-oder accurate provided that the stationary solutions are smooth (see [CLP08]).
  - If the problem (6) has no solution, then $U_i^* \equiv 0$ is chosen so that the reconstruction operator reduces to the standard one.
  - If the problem (6) has more than one solution a criterion to select one of them is needed (see for example [CDLGP13]).
  - This strategy can be easily adapted to obtain an operator which is well-balanced for a prescribed set $\mathcal{C}$ of stationary solutions.
  - If there is only a stationary solution U^* to preserve, then step 1 is skipped and $U_i^* = U^*$ in steps 2 and 3.

Well-balanced high-order finite volume scheme: quadrature formula

- The well-balancing property can be lost if a quadrature formula is used to compute the integral:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right) + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx.$$

- To avoid this problem the numerical scheme is written in the equivalent form:

$$\begin{aligned} U_i'(t) &= -\frac{1}{\Delta x}\Big(F_{i+1/2}(t) - F_{i-1/2}(t) - F(U_i^{t,*}(x_{i+1/2})) + F(U_i^{t,*}(x_{i-1/2}))\Big) \\ &+ \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \Big(S(P_i^t(x)) - S(U_i^{t,*}(x))\Big)\,H_x(x)\,dx, \end{aligned} \tag{7}$$

where $U_i^{t,*}$ is the stationary solution found at the first step of the reconstruction procedure at the ith cell and time t.

Well-balanced high-order finite volume scheme: quadrature formula

- The well-balancing property can be lost if a quadrature formula is used to compute the integral:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right) + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx.$$

- Now, a high-order quadrature formula could be used to approximate the source term integral.

$$\begin{aligned} U_i'(t) &= -\frac{1}{\Delta x}\Big(F_{i+1/2}(t) - F_{i-1/2}(t) - F(U_i^{t,*}(x_{i+1/2})) + F(U_i^{t,*}(x_{i-1/2}))\Big) \\ &+ \sum_{j=0}^{M} \alpha_j^i \Big(S(P_i^t(x_j^i)) - S(U_i^{t,*}(x_j^i))\Big) H_x(x_j^i), \end{aligned} \tag{7}$$

where $\alpha_0^i, \dots, \alpha_M^i$ and $x_0^i, \dots, x_M^i$ are respectively the weights and the nodes of the quadrature rule used in the ith cell.

Well-balanced high-order finite volume scheme: quadrature formula

- The well-balancing property can be lost if a quadrature formula is used to compute the integral:

$$U_i'(t) = -\frac{1}{\Delta x}\left(F_{i+1/2}(t) - F_{i-1/2}(t)\right) + \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx.$$

- First and second order schemes reduce to

$$U_i'(t) = -\frac{1}{\Delta x}\Big(F_{i+1/2}(t) - F_{i-1/2}(t) - F(U_i^{t,*}(x_{i+1/2})) + F(U_i^{t,*}(x_{i-1/2}))\Big),$$

supposing that (6) is satisfied.

Burgers equation with source term: Example 1

- Let us consider the Burgers equation with a non-linear source term

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = u^2 H_x, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x). \end{cases}$$

- The system can be written in the form

$$U_t + F(U)_x = S(U)H_x,$$

with

$$U = u, \quad F(U) = \frac{U^2}{2}, \quad S(U) = U^2.$$

- Stationary solutions are given by

$$u'(x) = u(x)H_x,$$

that is

$$u^*(x) = C_0 e^{H(x)}.$$

H continuous: stationary solution

Setting of the experiment

- $x \in (-1, 1)$, $t \in (0, 8]$. $CFL = 0.9$, $H(x) = x$.
- We consider 4 uniform meshes with 100, 200, 400, and 800 cells.
- Inflow boundary condition is set at $x = -1$ and free boundary conditions are set at $x = 1$.
- Initial condition: $U_0(x) = e^x$.
- Rusanov flux.
- First, second (MUSCL) and third order (CWENO) schemes are considered.
- First, second and third order Runge-Kutta TVD are considered.

H continuous: stationary solution

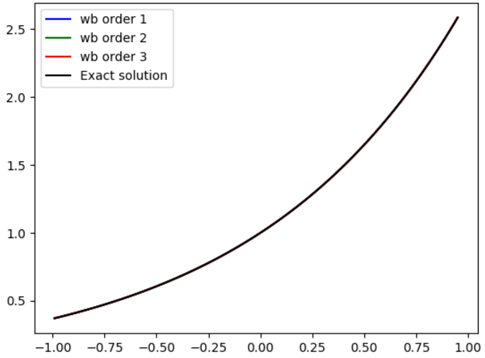

Figure: Solution at $t = 8$s: first, second and third order well-balanced schemes. Number of cells: 100

H continuous: stationary solution

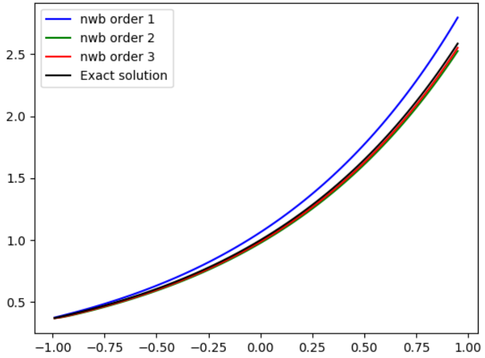

Figure: Solution at $t = 8$s: first, second and third order non well-balanced schemes. Number of cells: 100

H continuous: stationary solution

Cells	Error (1^{st})	Order (1^{st})	Error (2^{nd})	Order (2^{nd})	Error (3^{rd})	Order (3^{rd})
100	4.21E-15	-	8.87E-16	-	3.20E-16	-
200	2.90E-15	-	4.42E-16	-	2.54E-16	-
400	1.84E-14	-	1.82E-15	-	7.40E-14	-
800	4.45E-16	-	1.83E-16	-	2.61E-15	-

Table: Well-balanced schemes: L^1 errors and convergence rates for the stationary solution.

Cells	Error (1^{st})	Order (1^{st})	Error (2^{nd})	Order (2^{nd})	Error (3^{rd})	Order (3^{rd})
100	1.58E-01	-	5.42E-02	-	3.12E-02	-
200	7.51E-02	1.08	1.48E-02	1.87	5.41E-03	2.54
400	3.66E-02	1.04	3.91E-03	1.92	7.33E-04	2.87
800	1.81E-02	1.04	1.01E-03	1.98	9.49E-05	2.95

Table: Non well-balanced schemes: L^1 errors and convergence rates for the stationary solution.

H continuous: Perturbation of a stationary solution

Setting of the experiment

- $x \in (-1, 1)$, $t \in (0, 8]$. $CFL = 0.9$, $g = 1.0$.
- We consider 4 uniform meshes with 100, 200, 400, and 800 cells.
- Inflow boundary condition is set at $x = -1$ and free boundary conditions are set at $x = 1$.
- Initial condition: $U_0(x) = e^x + 0.3e^{-200(x+0.5)^2}$.
- Rusanov flux.
- First, second (MUSCL) and third order (CWENO reconstruction) schemes are considered.
- First, second and third order Runge-Kutta TVD are considered.
- A reference solution computed with a first order well-balanced scheme on a fine mesh (12800 cells) is considered.

H continuous: Perturbation of a stationary solution

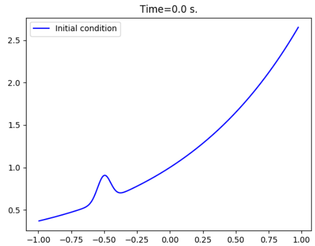

Figure: Initial condition. Number of cells: 200.

H continuous: Perturbation of a stationary solution

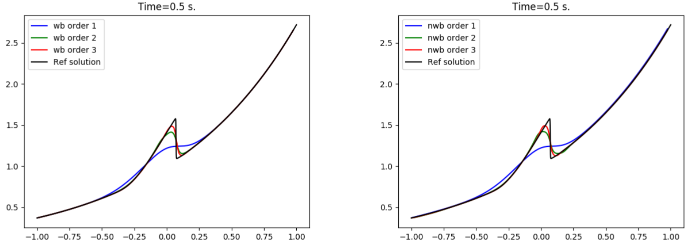

Figure: Solution at time $t = 0.5$ s. Well-balanced schemes (left) (100 cells) and non well-balanced schemes (right) (200 cells) .

H continuous: Perturbation of a stationary solution

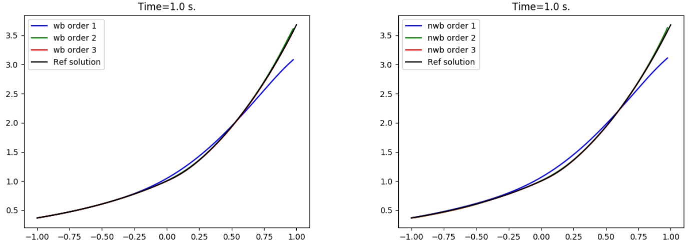

Figure: Solution at time $t = 1.0$ s. Well-balanced schemes (left) (100 cells) and non well-balanced schemes (right) (200 cells).

H continuous: Perturbation of a stationary solution

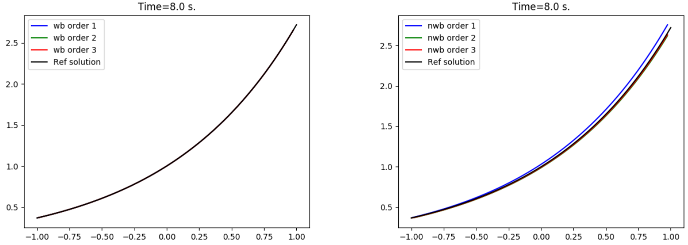

Figure: Solution at time $t = 8.0$ s. Well-balanced schemes (left) (100 cells) and non well-balanced schemes (right) (200 cells).

1D Shallow-water equations

$$U_t + F(U)_x = S(U)H_x$$

with

$$U = \begin{bmatrix} h \\ q \end{bmatrix}, \quad F(U) = \begin{bmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2}gh^2 \end{bmatrix}, \quad S(U) = \begin{bmatrix} 0 \\ gh \end{bmatrix}.$$

The variable x makes reference to the axis of the channel and t is the time; $q(x,t)$ and $h(x,t)$ are the discharge and the thickness, respectively; g is the gravity and $H(x)$ is the depth function measured from a fixed reference level.

Stationary solutions:

$$q = \bar{q}, \qquad g(h - H) + \frac{\bar{q}^2}{2h^2} = C,$$

where $\bar{q}$ and C are two real constants.
In [CPMP07] we propose a first order fully well-balanced scheme for the shallow-water system with variable cross-section, and later on, in [CDLGP13], we extend it to high-order. More recently in [CCMdL18] and [MdLCC19] we propose a first and high-order L-P scheme for the shallow-water system as well as a first order semi-implicit version of the scheme.

Stationary transcritical smooth solution

We consider a stationary solution with a transition at $x_{crit} = 1.5$ characterized by the constants:

$$\bar{q} = 2.5, \quad C = 17.56957396120237, \quad g = 9.812.$$

The depth function is given by

$$H(x) = \begin{cases} 0.5 - 0.25(1 + \cos(5\pi(x + 0.5))) & \text{if} \quad 1.3 \leq x \leq 1.7, \\ 0.5 & \text{otherwise.} \end{cases}$$

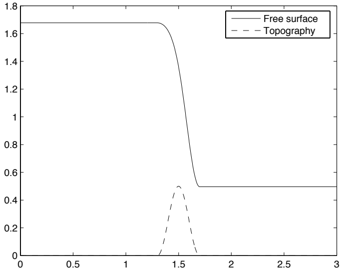

Figure: Stationary solution: Free surface and topography.

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N. cells	L^1 error h	L^1 error q
50	9.99e-17	5.32e-17
100	1.04e-16	1.27e-15
200	1.03e-15	7.95e-15
400	3.36e-15	2.91e-14

Table: Transcritical stationary solution: Errors in L^1 norm for the third order exactly well-balanced for all stationary states method.

Stationary transcritical smooth solution

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N. cells	L^1 error h	L^1 error q	order h	order q
100	1.39e-3	1.06e-2	-	-
200	1.55e-4	1.12e-3	3.16	3.24
400	1.89e-5	1.25e-4	3.03	3.16
800	2.22e-6	1.42e-5	3.08	3.13

Table: Transcritical stationary solution: Errors in L^1 norm for the third order well-balanced for water-at-rest method.

Perturbation around a transcritical solution

A perturbation of size $\Delta h = 0.02$ is imposed to the thickness h in the interval $[1.1, 1.2]$.

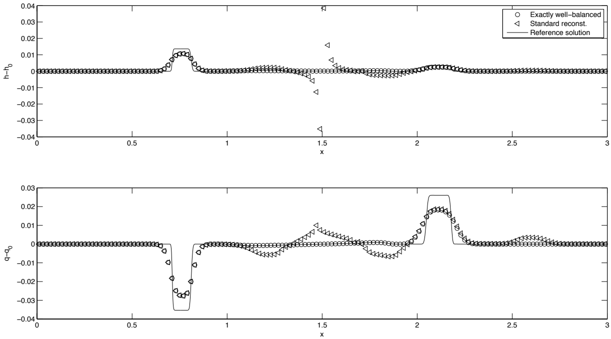


Figure: Evolution of the perturbation in a mesh of 150 cells at time $t = 0.15$ (the reference solution has been computed using a mesh composed by 2000 cells).

1D Euler equations

- System:

$$U_t + F(U)_x = S(U)H(x)_x$$

with

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ u(E + P) \end{bmatrix}, \quad S(U) = \begin{bmatrix} 0 \\ -\rho \\ -\rho u \end{bmatrix}.$$

$$P = (\gamma - 1)\rho e;$$

$$E = \rho e + \frac{1}{2}\rho u^2.$$

$g = 1$, $\gamma = 5/3$. and $H(x) = gx$.

- **A family of stationary solutions with $u = 0$:**

$$\rho^*(x) = C_1 e^{-H(x)} \geq 0 \quad P^*(x) = C_1 e^{-H(x)} + C_2 \geq 0, \quad E^*(x) = \frac{P^*(x)}{\gamma - 1}. \tag{7}$$

where C_1 and C_2 are arbitrary constants.

1D Euler equations: stationary solution

Setting of the experiment

- $x \in (-1, 1)$, $t \in (0, 10]$, $CFL = 0.9$.
- We consider a uniform mesh with 100 cells.
- Wall boundary conditions at $x = -1$ and $x = 1$.
- Initial condition:

$$\rho_0(x) = e^{-x}, \quad P_0(x) = e^{-x}, \quad u_0(x) = 0.$$

- First, second and third order (CWENO reconstruction) schemes are considered.
- First, second and third order Runge-Kutta TVD are considered.

1D Euler equations: stationary solution

Reconstruction procedure:

- Look for $C^t_{1,i}$ such that $\rho_i^{t,*}(x) = \alpha(x, C^t_{1,i})$ satisfies:

$$\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i^{t,*}(x)\, dx = \rho_i.$$

- Look for $C^t_{2,i}$ such that

$$E_i^{t,*}(x) = \frac{\beta(x, C^t_{1,i}, C^t_{2,i})}{\gamma - 1}.$$

  satisfies

$$\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} E_i^{t,*}(x)\, dx = E_i.$$

Notice that the EOS has been used in the second step. In practice, the integrals in the above steps have to be replaced by the corresponding quadrature formula. Momentum is reconstructed using a standard reconstruction operator.

1D Euler equations: stationary solution

Cells	Error (1^{st})	Order (1^{st})	Error (2^{nd})	Order (2^{nd})	Error (3^{rd})	Order (3^{rd})
100	4.28E-16	-	4.03E-16	-	5.93E-15	-
200	5.86E-16	-	3.63E-16	-	3.05E-15	-
400	4.44E-16	-	4.23E-16	-	5.33E-14	-
800	9.59E-16	-	5.90E-16	-	2.21E-14	-

Table: Well-balanced schemes: L^1 errors and convergence rates for the stationary solution.

Cells	Error (1^{st})	Order (1^{st})	Error (2^{nd})	Order (2^{nd})	Error (3^{rd})	Order (3^{rd})
100	2.35E-02	-	2.27E-05	-	2.15E-06	-
200	1.18E-02	1.00	4.87E-06	2.22	7.51E-08	4.84
400	5.85E-03	1.00	1.21E-06	2.00	5.93E-09	3.66
800	2.91E-03	1.00	3.02E-07	2.00	7.38E-10	3.01

Table: Non well-balanced schemes: L^1 errors and convergence rates for the stationary solution.

Shock tube problem

We consider the shock tube problem given by the initial condition with wall boundary conditions:

$$(\rho(x,0),\ u(x,0),\ p(x,0)) = \begin{cases} (1,0,1) & \text{if } x \leq 0.5, \\ (0.125,0,0.1) & \text{if } x > 0.5, \end{cases} \qquad x \in [0,1],$$

Shock tube problem

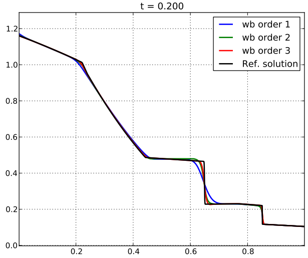
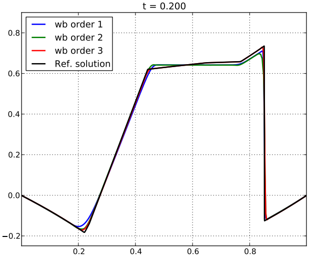


Figure: Shock tube problem: density at $t = 0.2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (400 cells) .

Shock tube problem



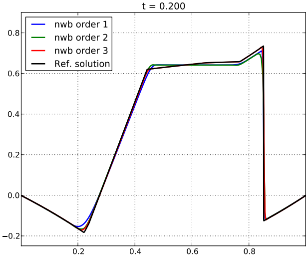


Figure: Shock tube problem: velocity at $t = 0.2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (400 cells) .

Shock tube problem

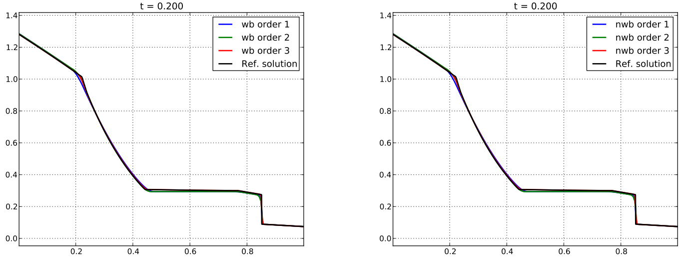


Figure: Shock tube problem: pressure at $t = 0.2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (400 cells) .

Computing the stationary solutions

Main difficulty

At each time step, a local stationary solution defined at the stencil of the reconstruction operator should be computed satisfying:

- $$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = U_i.$$
- In some cases, the stationary solutions are known and this problem can be solved, but in general it may be difficult and costly. Different strategies could be used.

Computing the stationary solutions: the control problem

- We reformulate the previous problem as a control one (see [GCP20]):

$$\min_{U_0 \in \mathbb{R}^N} \left| \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U_i^*(x)\, dx - U_i^n \right|^2 \tag{8}$$

subject to :

$$\begin{cases} F(U_i^*)_x = S(U_i^*) H_x, \\ U_i^*(x_{i-\frac{1}{2}}) = U_0, \end{cases} \tag{9}$$

- A numerical method is used to approximate the Cauchy problems.
- Newton method is used based on the adjoint state computation with the initial guess $U_0 = U_i^n$.

Computing the stationary solutions

Approximated WB reconstruction operators

To implement the well-balanced reconstruction operator, the following ingredients have to be chosen:

- Quadrature rules at the cells

$$\int_{x_{i-1/2}}^{x_{i+1/2}} g(x)\,dx \cong \Delta x \sum_{l=0}^{M} \alpha_i^l g(x_i^l). \tag{10}$$

- A numerical method for solving the Cauchy problems that is reversible or symmetric, that is,

$$\Phi_h \circ \Phi_{-h} = Id, \quad \text{or equivalently} \quad \Phi_h = \Phi_{-h}^{-1}, \tag{11}$$

  where Φ_h is the iteration funcion that characterizes the method.
- Meshes of maximum step h at the cells $[x_{i-1/2}, x_{i+1/2}]$ whose set of nodes include the quadrature points x_i^l and $x_{i\pm 1/2}$.

Computing the stationary solutions

With the previous ingredients, it can be proved that the numerical method is well-balanced in the following sense: given the vector of cell averages $\{U_i\}$ of a stationray solution U that has been computed with the cuadrature formula (10) and with a numerical method that satisfies (11), then $\{U_i\}$ is an equilibrium of the ODE system (4).
In the previous result, we consider that the numerical procedure to solve the control problem converges up to machine precision.

Some particular situations

Let us consider the smooth stationary solutions for the Euler equations of gas dynamics with $u = 0$. Those could be determined, if for example, the stationay density profile ρ is given, and then pressure P should be a primitive of the density and E is determined using EOS. For example $E(x) = \frac{P(x)}{\gamma - 1}$.

We could proceed as follows:

- Given a stencil S_i around the cell I_i, compute some high-order polynomial reconstruction of the density ρ, on the complete stencil: $\rho_i^*(x)$, satisfying that
$$\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \rho^*(x)\, dx = \rho_i.$$
- Define $P_i^*(x)$ a primitive of $-\rho_i^*(x) H(x)_x$ satisfying
$$\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} P_i^*(x)\, dx = P_i$$
- Define $E_i^*(x)$ using EOS.
- Next, we apply the general procedure to the approximated stationary solution given by
$$U_i^*(x) = \begin{bmatrix} \rho_i^*(x) \\ 0 \\ E_i^*(x) \end{bmatrix}.$$
to construct a well-balanced numerical scheme that preserves the previous discrete approximation of the stationary solutions.

Euler equations

Smooth stationary solutions of the Euler equations of gas dynamics are given by

$$\begin{cases} q_x = 0, \\ \dfrac{d\hat{U}}{dx} = G(x, \hat{U}), \end{cases} \tag{12}$$

where

$$\hat{U} = \begin{pmatrix} \rho \\ E \end{pmatrix}, \quad G(x, \hat{U}) = -\begin{pmatrix} \dfrac{\rho}{c^2 - u^2} \\ \dfrac{\rho}{\gamma - 1}\left(1 + \dfrac{3-\gamma}{2}\dfrac{u^2}{c^2 - u^2}\right) \end{pmatrix} H_x,$$

where

$$c = \sqrt{\gamma \frac{p}{\rho}}$$

is the wave speed.

Euler equations

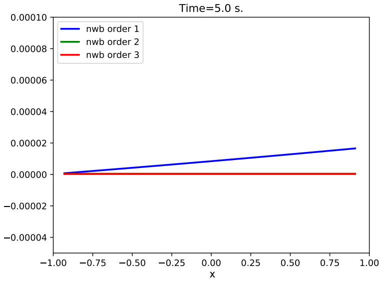
As initial condition, we compute the supersonic stationary solution which solves the Cauchy problem:

$$\begin{cases} q_x = 0, \\ \dfrac{d\hat{U}}{dx} = G(x, \hat{U}), \\ \rho(-1) = 1, \ q(-1) = 10, \ E(-1) = 52. \end{cases} \tag{13}$$

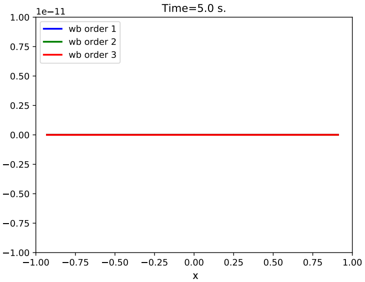
with $x \in [-1, 1]$, being $H(x) = x$.

Now, we consider first, second and third order well-balanced and not well-balanced schemes with $CFL = 0.9$. Boundary conditions are imposed upstream and open boundary conditions are set downstream.

Euler equations

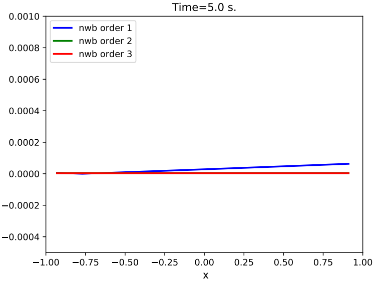


(a) NWB schemes: Density

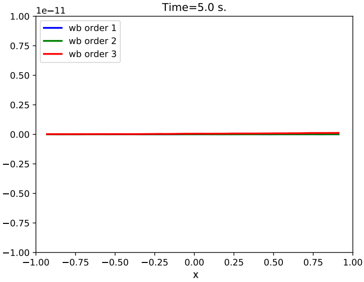


(b) WB schemes: Density

Figure: Differences between the stationary solution and the numerical solutions at time $t = 5s$. Number of cells: 100.



(a) NWB schemes: Momentum

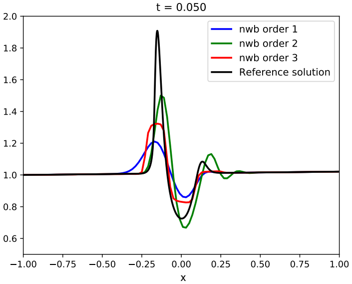


(b) WB schemes: Momentum

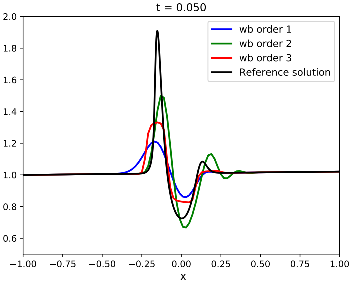
Figure: Differences between the stationary solution and the numerical solutions at time $t = 5s$. Number of cells: 100.

Now we consider a small perturbation of the previous stationary solution $U^*(x)$

$$U_0(x) = U^*(x) + \begin{pmatrix} 0.3e^{-200(x+0.5)^2} \\ 0.0 \\ 0.0 \end{pmatrix},$$



(a) NWB schemes. $t = 0.05$.



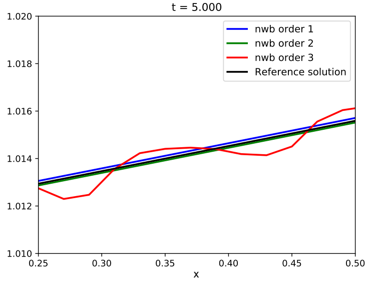
(b) WB schemes. $t = 0.05$.

Figure: Density at time $t = 0.05s$. Number of cells: 100.

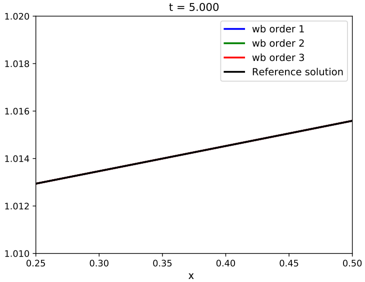
Euler equations

Now we consider a small perturbation of the previous stationary solution $U^*(x)$

$$U_0(x) = U^*(x) + \begin{pmatrix} 0.3e^{-200(x+0.5)^2} \\ 0.0 \\ 0.0 \end{pmatrix},$$



(a) NWB schemes. Zoom at $t = 5$.



(b) WB schemes. Zoom at $t = 5$.

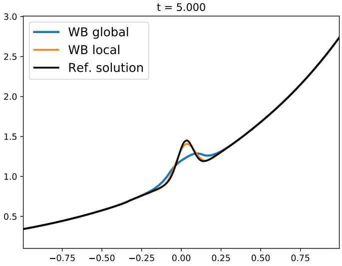
Figure: Density: zoom at time $t = 5s$. Number of cells: 100.

Euler equations

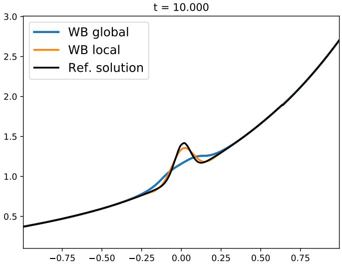
Now we consider a small perturbation of a $u = 0$ smooth stationary solution given by

$$U_0(x) = U^*(x) + \begin{pmatrix} 0.4e^{-200x^2} \\ 0.0 \\ 0.0 \end{pmatrix}, \quad U^*(x) = \begin{pmatrix} e^x \\ 0 \\ \frac{e^x}{\gamma - 1} \end{pmatrix}.$$

We compare two different WB numerical schemes of second order:



(a) Comparison of two different WB schemes. $t = 5s$.



(b) Comparison of two different WB schemes. $t = 10s$.

2D Euler equations with gravity in polar coordinates

We consider the 2D Euler equations with gravity in polar coordinates (see [GCD18]):

$$U_t + F(U)_r + G(U)_\varphi = S(U)H(\varphi)_\varphi .$$

with $H(\varphi) = \varphi$,

$$U = \begin{pmatrix} r\rho \\ r\rho u_r \\ r\rho u_\varphi \\ r\rho E \end{pmatrix}, \quad F(U) = \begin{pmatrix} r\rho u_r \\ r\rho u_r^2 + rP \\ r\rho u_r u_\varphi \\ r u_r(\rho E + P) \end{pmatrix}, \quad G(U) = \begin{pmatrix} \rho u_\varphi \\ \rho u_r u_\varphi \\ \rho u_\varphi^2 + P \\ u_\varphi(\rho E + P) \end{pmatrix}$$

$$S(U) = \begin{pmatrix} 0 \\ -\rho \frac{Gm_s}{r} + P + \rho u_\varphi^2 \\ -\rho u_r u_\varphi \\ -\rho u_r \frac{Gm_s}{r} \end{pmatrix}$$

m_s is the solar mass, G the gravitational constant and the pressure P is given by

$$P = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho\left(u_r^2 + u_\varphi^2\right)\right), \quad \gamma = \frac{c_p}{c_v} > 1,$$

where γ is the ratio between heat at constant pressure and volume, which is taken to be constant.

2D Euler equations with gravity in polar coordinates

We consider the 2D Euler equations with gravity in polar coordinates (see [GCD18]):

$$U_t + F(U)_r + G(U)_\varphi = S(U)H(\varphi)_\varphi.$$

with $H(\varphi) = \varphi$,

$$U = \begin{pmatrix} r\rho \\ r\rho u_r \\ r\rho u_\varphi \\ r\rho E \end{pmatrix}, \; F(U) = \begin{pmatrix} r\rho u_r \\ r\rho u_r^2 + rP \\ r\rho u_r u_\varphi \\ r u_r(\rho E + P) \end{pmatrix}, \; G(U) = \begin{pmatrix} \rho u_\varphi \\ \rho u_r u_\varphi \\ \rho u_\varphi^2 + P \\ u_\varphi(\rho E + P) \end{pmatrix}$$

$$S(U) = \begin{pmatrix} 0 \\ -\rho \frac{Gm_s}{r} + P + \rho u_\varphi^2 \\ -\rho u_r u_\varphi \\ -\rho u_r \frac{Gm_s}{r} \end{pmatrix}$$

Stationary solutions:

$$\rho = \rho(r),\; u_r = 0,\; \frac{\partial u_\varphi}{\partial \varphi} = 0,\; \frac{\partial rP}{\partial r} = -\rho\left(\frac{Gm_s}{r} - u_\varphi^2\right) + P.$$

Keplerian disc with Kelvin-Helmholtz instabilities

Stationary Keplerian disc:

$$\rho^* = \rho_0 + \rho_1 \tanh\left(\frac{r - r_m}{\sigma}\right), \; u_r^* = 0, \; u_\varphi^* = \sqrt{\frac{Gm_s}{r}}, \; P^* = 1,$$

with $G = 1$, $m_s = 1$, $\rho_0 = 1$, $\rho_1 = 0.25$, $r_m = 1.5$ and $\sigma = 0.01$. The computational domain is given by: $[1, 2] \times [0, \pi/2]$. Periodic boundary conditions are consider in $\varphi = 0$ and $\varphi = \pi/2$ and the exact solution is imposed at $r = 1$ and $r = 2$.
Now, a small perturbation over the density, radial velocity and pressure is prescribed:

$$\rho = \rho^* + A\rho_0 \sin(k\varphi) e^{\frac{-(r-r_m)^2}{s}}$$
$$u = u_r^* + A \sin(k\varphi) e^{\frac{-(r-r_m)^2}{s}}$$
$$P = P^* + A \sin(k\varphi) e^{\frac{-(r-r_m)^2}{s}}$$

with $A = 0.1$, $k = 8$ and $s = 0.005$.

Keplerian disc with Kelvin-Helmholtz instabilities

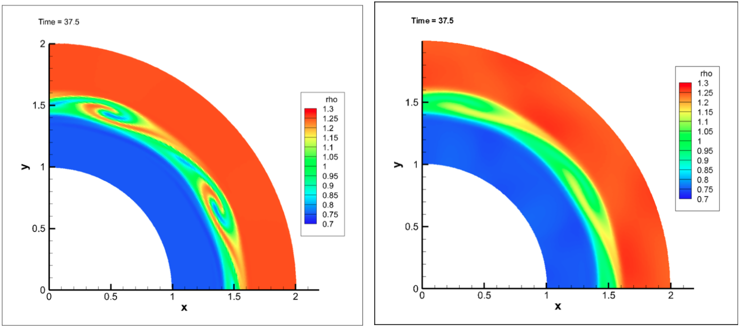


Figure: Keplerian disc with Kelvin-Helmholtz instabilities 100x200 mesh grid. Second order well-balanced scheme (left) and third order non well-balanced scheme (PLUTO solver) (right).

2D shallow-water over the sphere

We consider the 2D shallow-water over the sphere (see [COP17]):

$$
\begin{aligned}
&\partial_t h + \frac{1}{R\cos(\varphi)}\Big(\partial_\theta q_\theta + \partial_\varphi (q_\varphi \cos(\varphi))\Big) = 0,\\
&\partial_t q_\theta + \frac{1}{R\cos(\varphi)}\partial_\theta \left(\frac{q_\theta^2}{h}\right) + \frac{1}{R}\partial_\varphi \left(\frac{q_\theta q_\varphi}{h}\right) - 2\frac{q_\theta q_\varphi}{Rh}\tan(\varphi) + \frac{gh}{R\cos(\varphi)}\partial_\theta h = \frac{gh\,\partial_\theta H}{R\cos(\varphi)},\\
&\partial_t q_\varphi + \frac{1}{R\cos(\varphi)}\partial_\theta \left(\frac{q_\varphi q_\theta}{h}\right) + \frac{1}{R}\partial_\varphi \left(\frac{q_\varphi^2}{h}\right) + \frac{(q_\theta^2 - q_\varphi^2)}{hR}\tan(\varphi) + \frac{gh}{R}\partial_\varphi h = \frac{gh}{R}\partial_\varphi H,
\end{aligned}
$$

where (θ, φ) is the longitude and latitude, R is the radius, g is the gravity constant and

$$q_\theta = hu_\theta, \quad q_\varphi = hu_\varphi.$$

Stationary solution (water at rest):

$$u_\theta = u_\varphi = 0,\ (h - H)\cos(\varphi) = \bar{\eta}\cos(\varphi), \bar{\eta} = \text{constant}.$$

In [COP17] a third order well-balanced scheme for the water at rest solutions of the 2D shallow-water system over the sphere has been developed.

Propagation of a hypothetical tsunami in the Mediterranean Sea

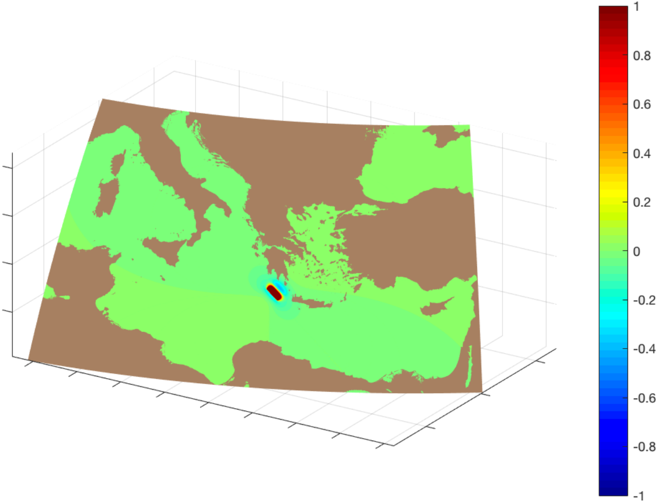


Figure: Propagation of a hypothetical tsunami in the Mediterranean Sea. Initial condition. Third order well-balanced scheme.

Propagation of a hypothetical tsunami in the Mediterranean Sea

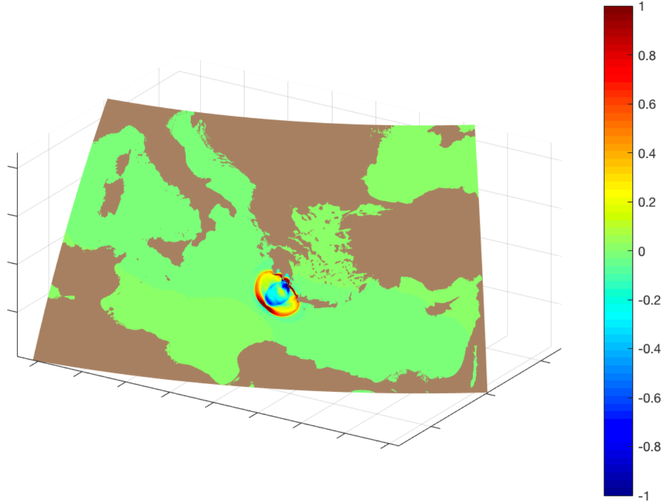


Figure: Propagation of a hypothetical tsunami in the Mediterranean Sea. Initial condition. Third order well-balanced scheme. $T = 10$ minutes

Propagation of a hypothetical tsunami in the Mediterranean Sea

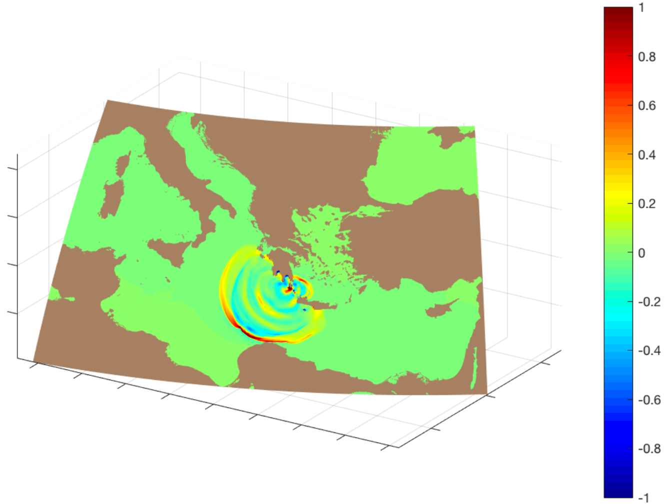


Figure: Propagation of a hypothetical tsunami in the Mediterranean Sea. Initial condition. Third order well-balanced scheme. $T = 30$ minutes

Propagation of a hypothetical tsunami in the Mediterranean Sea

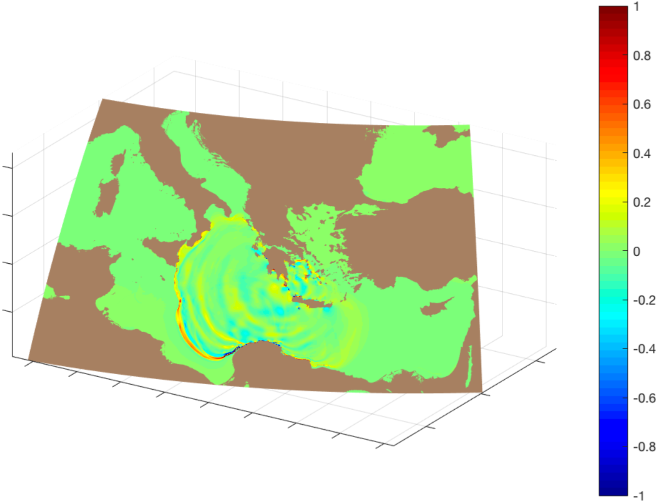


Figure: Propagation of a hypothetical tsunami in the Mediterranean Sea. Initial condition. Third order well-balanced scheme. $T = 60$ minutes

We could apply the same strategy to define a numerical scheme that exactly preserve some representative time dependent solutions like solitons or paekons. Here is one example for the KdV-BBM equation:

$$\partial_t u + \partial_x(\alpha u + \frac{\beta}{2}u^2) - \gamma \partial_{txx} u + \delta \partial_{xxx} u = 0, \quad (14)$$

with $\alpha \geq 0$, $\beta > 0$, $\gamma \geq 0$, and $\delta > 0$. u represents the free surface elevation with respect to water at rest.
System (14) admits exact solitons of the form:

$$u(x,t) = 3\frac{c_s - \alpha}{\beta}\text{sech}^2\Big(\frac{1}{2}\sqrt{\frac{c_s - \alpha}{\gamma c_s + \delta}}(x - c_s t)\Big).$$

Soliton preserving scheme for KdV-BBM equation

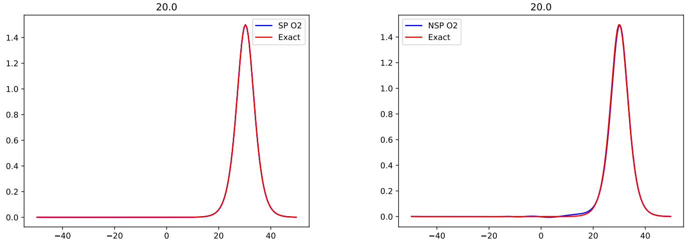


Figure: Soliton evolution ($t = 20$): second order soliton preserving scheme (left) and second order standard scheme (right) (200 cells) .

Soliton preserving scheme for KdV-BBM equation

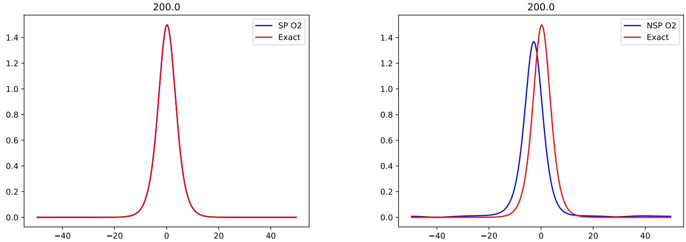


Figure: Soliton evolution ($t = 200$): second order soliton preserving scheme (left) and second order standard scheme (right) (200 cells) .

Soliton preserving scheme for KdV-BBM equation: perturbation evolution

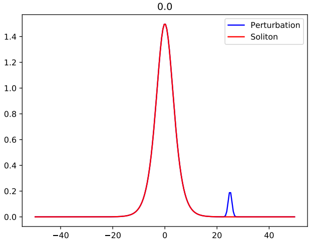


Figure: Initial condition: soliton with a small perturbation.

Soliton preserving scheme for KdV-BBM equation: perturbation evolution

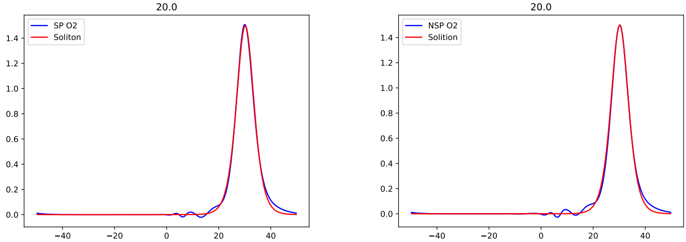
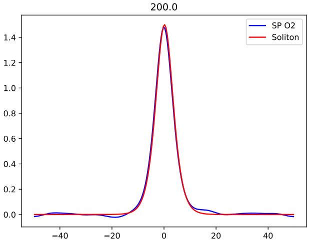


Figure: Soliton+small perturbation ($t = 20$): second order soliton preserving scheme (left) and second order standard scheme (right) (200 cells) .

Soliton preserving scheme for KdV-BBM equation: perturbation evolution



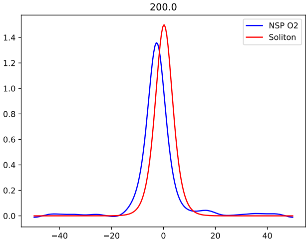


Figure: Soliton+ small perturbation ($t = 200$): second order soliton preserving scheme (left) and second order standard scheme (right) (200 cells) .

Outline

Well-balanced high-order finite volume scheme: H discontinuous

Let us consider the system

$$U_t + F(U)_x = S(U)H_x,$$

where now H has finite isolated jump discontinuities.

- $S(U)H_x$ is in this case a nonconservative product and it can be defined in infinitely many different forms (see [DMLM95]).
- At a discontinuity point of H, x^*, a Dirac measure

  $$\left(\int_0^1 S(\Phi_U(s))\partial_s \Phi_H(s)\, ds \right) \delta_{x=x^*}$$

  is produced, where

  $$s \in [0,1] \mapsto (\Phi_U(s), \Phi_H(s)) \in \Omega \times \mathbb{R}$$

  is a path liking (U^-, H^-) and (U^+, H^+).

Well-balanced high-order finite volume scheme: *H* discontinuous

- A path-conservative approximation [Par06] of these Dirac measures is added to the numerical method:

$$U_i'(t) = -\frac{1}{\Delta x}\Big(F_{i+1/2}-F_{i-1/2}-S^+_{i-1/2}-S^-_{i+1/2}+\int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t(x))H_x(x)\,dx\Big),$$

where $S^{\pm}_{i+1/2}$ are such that:

$$S^-_{i+1/2} + S^+_{i+1/2} = \int_0^1 S(\Phi_U(s))\partial_s\Phi_H(s)\,ds;$$

$$S^{\pm}_{i+1/2} = 0 \text{ if } H^-_{i+1/2} = H^+_{i+1/2}.$$

- The Generalized Hydrostatic reconstruction is used (see [CPMP07]).

Well-balanced high-order finite volume scheme: H discontinuous

- The jumps of the source term are then computed by:

$$\begin{aligned} S^+_{i+1/2} &= F(U^+_{i+1/2}) - F(V^+_{i+1/2}(H^0_{i+1/2})), \\ S^-_{i+1/2} &= F(V^-_{i+1/2}(H^0_{i+1/2})) - F(U^-_{i+1/2}). \end{aligned}$$

where

- $H^0_{i+1/2}$ is an intermediate value between $H^-_{i+1/2}$ and $H^+_{i+1/2}$;
- $V^\pm_{i+1/2}(\sigma)$ are the solution of the equation

$$F(V)_\sigma = S(V)$$

with initial conditions:

$$V(H^\pm_{i+1/2}) = U^\pm_{i+1/2}.$$

Well-balanced high-order finite volume scheme: *H* discontinuous

If a quadrature formula is used, the previous numerical scheme could be written as follows:

$$\begin{aligned} U_i'(t) &= -\frac{1}{\Delta x}\Big(F_{i+1/2} - F_{i-1/2} - F(U_i^{t,*}(x_{i+1/2}^+)) + F(U_i^{t,*}(x_{i-1/2}^-)) \\ &\quad -S_{i-1/2}^+ - S_{i+1/2}^-\Big) \\ &\quad + \sum_{j=0}^{M} \alpha_j^i \left(S(P_i^t(x_j^i)) - S(U_i^{t,*}(x_j^i)) \right) H_x(x_j^i), \end{aligned}$$

with

$$\begin{aligned} S_{i+1/2}^+ &= F(U_{i+1/2}^{t,+}) - F(V_{i+1/2}^{t,+}(H_{i+1/2}^0)), \\ S_{i+1/2}^- &= F(V_{i+1/2}^{t,-}(H_{i+1/2}^0)) - F(U_{i+1/2}^{t,-}). \end{aligned}$$

H discontinuous

$$H(x) = \begin{cases} 0.1 + 0.1x & \text{if } x < 0.0 \\ 1 + x & \text{if } x \geq 0.0 \end{cases}$$

Stationary solutions:

$$U(x) = \begin{cases} C_0 e^{0.1x} & \text{if } x < 0.0 \\ C_0 e^{0.9+x} & \text{if } x \geq 0.0 \end{cases}$$

Setting of the experiment

- $x \in (-1, 1)$, $t \in (0, 1]$. $CFL = 0.9$.
- We consider 5 uniform meshes with 100, 200, 400, 800, and 1600 cells.
- Inflow boundary condition is set at $x = -1$ and free boundary conditions are set at $x = 1$.
- Initial condition: we use the previous expression with $C_0 = 1.0$.
- Rusanov flux.
- First, second (MUSCL) and third order (CWENO) schemes are considered.
- First, second and third order Runge-Kutta TVD are considered.

H discontinuous: stationary solution

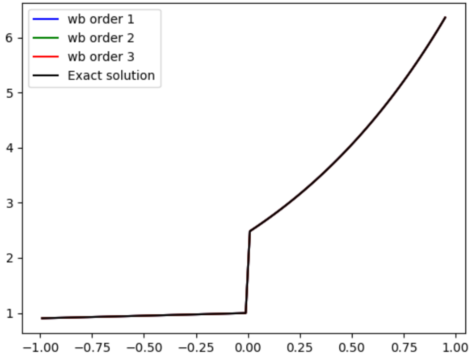


Figure: Solution at $t = 1$s: first, second and third order well-balanced schemes. Number of cells: 400

H discontinuous: stationary solution

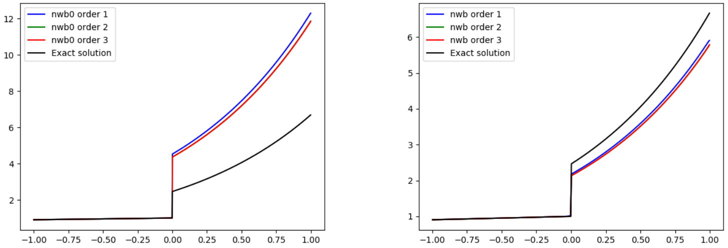


Figure: Solution at time $t = 1.0$ s. Non well-balanced schemes: left (line segment path on U^2). Right (line segment path on U). Blue (first order scheme), green (second order scheme), red (third order scheme), black (exact solution).

H discontinuous: perturbation of a stationary solution

Setting of the experiment

- $x \in (-1, 1)$, $t \in (0, 1]$. $CFL = 0.9$.
- We consider 5 uniform meshes with 100, 200, 400, 800, and 1600 cells.
- Inflow boundary condition is set at $x = -1$ and free boundary conditions are set at $x = 1$.
- Initial condition:

$$U(x) = \begin{cases} e^{0.1x} + 0.3e^{-200(x+0.5)^2} & \text{if } x < 0.0 \\ e^{0.9+x} & \text{if } x \geq 0.0 \end{cases}$$

- Rusanov flux.
- First, second (MUSCL) and third order (CWENO) schemes are considered.
- First, second and third order Runge-Kutta TVD are considered.

H discontinuous: perturbation of a stationary solution

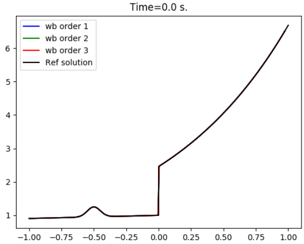


Figure: Initial condition. Number of cells: 400.

H discontinuous: perturbation of a stationary solution

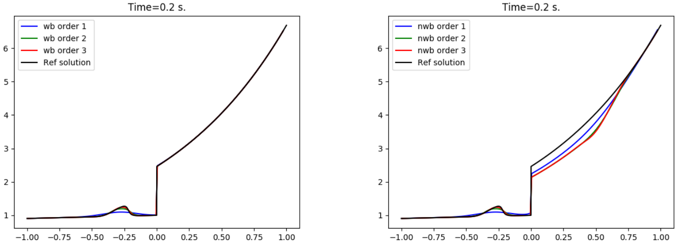


Figure: Solution at time $t = 0.2$ s. Well-balanced schemes (left) and non well-balanced schemes (right). Blue (first order scheme), green (second order scheme), red (third order scheme), black (reference solution)

H discontinuous: perturbation of a stationary solution

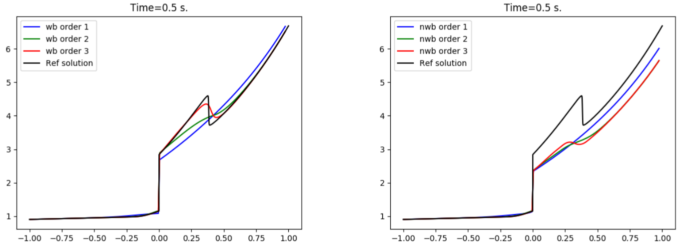


Figure: Solution at time $t = 0.5$ s. Well-balanced schemes (left) and non well-balanced schemes (right). Blue (first order scheme), green (second order scheme), red (third order scheme), black (reference solution).

H discontinuous: perturbation of a stationary solution

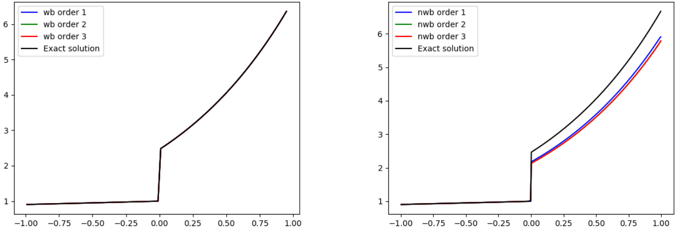


Figure: Solution at time $t = 1.0$ s. Well-balanced schemes (left) and non well-balanced schemes (right). Blue (first order scheme), green (second order scheme), red (third order scheme), black (reference solution).

1D Euler: Non-smooth potential

Let us consider now Euler equations for gas dynamics with a discontinuous potential given by

$$H(x) = \begin{cases} 0.1x & \text{if } x < 0, \\ 1 + x & \text{if } x \geq 0, \end{cases}$$

The isothermal stationary solutions associated to the previous potential are

$$\rho(x,0) = \begin{cases} C_1 e^{-0.1x} & \text{if } x < 0, \\ C_1 e^{-1-x} & \text{if } x \geq 0, \end{cases} \qquad u(x,0) = 0,$$

$$p(x,0) = \begin{cases} C_1 e^{-0.1x} + C_2 & \text{if } x < 0, \\ C_1 e^{-1-x} + C_2 & \text{if } x \geq 0. \end{cases}$$

1D Euler: non-smooth potential

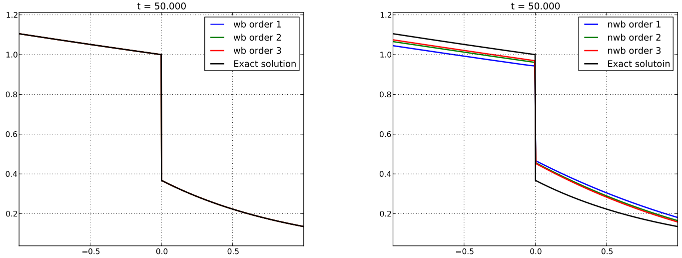


Figure: Non-smooth stationary solution: pressure at $t = 50$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (800 cells) .

1D Euler: Perturbation of a non-smooth stationary solution

We consider a small perturbation on the density:

$$\rho(x,0) = \rho^*(x) + 0.4e^{-200(x+0.5)^2}.$$

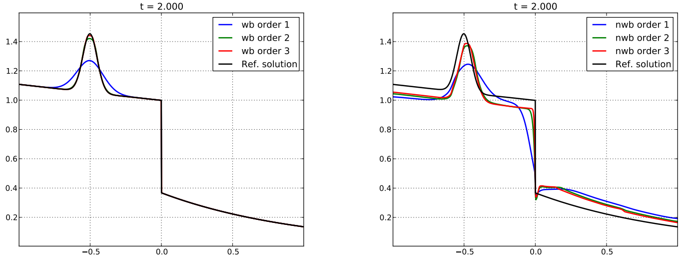


Figure: Perturbation of the stationary a non-smooth stationary solution: density at $t = 2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (800 cells) .

1D Euler: Perturbation of a non-smooth stationary solution

We consider a small perturbation on the density:

$$\rho(x,0) = \rho^*(x) + 0.4e^{-200(x+0.5)^2}.$$

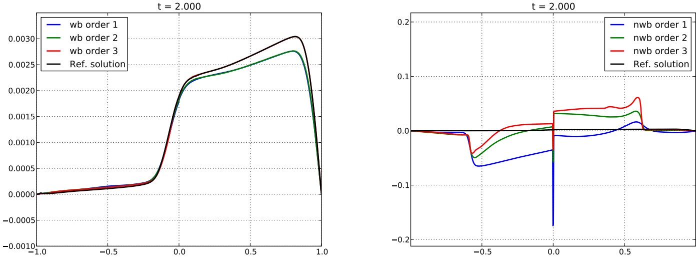


Figure: Perturbation of a non-smooth stationary solution (47): velocity at $t = 2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (800 cells) .

1D Euler: Perturbation of a non-smooth stationary solution

We consider a small perturbation on the density:

$$\rho(x,0) = \rho^*(x) + 0.4e^{-200(x+0.5)^2}.$$

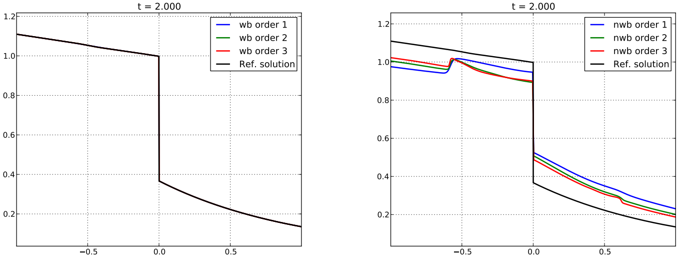


Figure: Perturbation of a non-smooth stationary solution : pressure at $t = 2$ s. Well-balanced schemes (left) and non well-balanced schemes (right) (800 cells) .

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