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AN ERROR ESTIMATE FOR VISCOUS APPROXIMATE SOLUTIONS TO DEGENERATE ANISOTROPIC CONVECTION-DIFFUSION EQUATIONS

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Abstract. We consider a viscous approximation for a nonlinear degenerate convection-diffusion equations in two space dimensions, and prove an L^1 error estimate. Precisely, we show that the L^1_{loc} difference between the approximate solution and the unique entropy solution converges at a rate $\mathcal{O}(\varepsilon^{1/2})$, where ε is the viscous parameter.

1. **Introduction.** In this paper, we are interested in certain "viscous" approximations of entropy solutions of the following Cauchy problem

$$\begin{cases} u_t + f(u)_y = u_{xx}, & (x,y,t) \in \Pi_T, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $\Pi_T = \mathbb{R} \times \mathbb{R} \times (0,T)$ with $T > 0$ fixed, $u : \Pi_T \to \mathbb{R}$ is the unknown function and $f : \mathbb{R} \to \mathbb{R}$ is the convective flux function. The main characteristics of this type of equations is that it has mixed parabolic-hyperbolic type, due to the directional separation of the diffusion and convection effects: while matter is convected along the y axis, it is simultaneously diffused along orthogonal direction.

The existence of solutions of equation (1) can be obtained by the classical method of adding a vanishing viscosity, in other terms a diffusion, in the missing direction (along the y axis). Since the most characteristics of Equation (1) is that it has mixed parabolic-hyperbolic type, or in other words, it has strong degeneracy due to the lack of diffusion in the x-direction, it is difficult to establish the uniqueness of solutions of (1). In [1], authors have proved the existence as well as uniqueness of such problems.

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In this paper we are interested in certain approximate solutions of (1) coming from solving the uniformly parabolic problem

$$u_t^\varepsilon + f(u^\varepsilon)_y = u_{xx}^\varepsilon + \varepsilon u_{yy}^\varepsilon. \tag{2}$$

We refer to u^ε as a "viscous" approximate solution of (1). Since the convergence of u^ε to the unique entropy solution u of (1) as $\varepsilon \downarrow 0$ is well known, so our interest here is to give an explicit rate of convergence for u^ε as $\varepsilon \downarrow 0$, i.e., an L^1 error estimate for viscous approximate solutions. There are several ways to prove such an error estimate. One way is to view it as a consequence of a continuous dependence estimate. Combining the ideas of [1] with a variant of Kruzkov's "doubling of variables" device for (1), we prove that $\|u^\varepsilon - u\|_{L^1(\Pi_T)} = \mathcal{O}(\varepsilon^{1/2})$. Although our proof is of independent interest, it may also shed some light on how to obtain an error estimates for numerical methods.

2. **Preliminaries.** Independently of the smoothness of the initial data, due to the lack of diffusion in the x-direction, jumps may form in the solution u. Therefore we consider solutions in the weak sense, i.e.,

Definition 2.1. Set $\Pi_T = \mathbb{R} \times \mathbb{R} \times (0, T)$. A function

$$u(t, x) \in C\left([0, T]; L^1(\mathbb{R}^2)\right) \cap L^\infty(\Pi_T)$$

is a weak solution of the initial value problem (1) if it satisfies:

(a) For all test functions $\varphi \in \mathcal{D}(\mathbb{R}^2 \times [0, T))$

$$\iiint_{\Pi_T} \left(u\varphi_t + f(u)\varphi_y + u\varphi_{xx}\right) \, dx\, dy\, dt + \iint_{\mathbb{R}^2} u_0(x, y)\varphi(x, y, 0)\, dx\, dy = 0. \tag{3}$$

(b) $u = u(x, y, t)$ is continuous at $t = 0$ as a function: $[0, T) \to L^1(\mathbb{R}^2)$.

Since weak solutions are not uniquely determined by their initial data, one must impose an additional entropy condition to single out the physically relevant solution.

Definition 2.2. A weak solution u of the initial value problem (1) is called an entropy solution, if the following entropy inequality holds for all test functions $0 \le \varphi \in \mathcal{D}(\mathbb{R}^2 \times (0, T))$:

$$\begin{aligned} &\iiint_{\Pi_T} |u - \psi(x)|\, \varphi_t + \operatorname{sign}(u - \psi(x))(f(u) - f(\psi(x)))\varphi_y \, dxdydt \\ &\qquad \ge \iiint_{\Pi_T} -|u - \psi(x)|\, \varphi_{xx} - \operatorname{sign}(u - \psi(x))\psi_{xx}\varphi \, dxdydt. \end{aligned} \tag{4}$$

Remark 1. Note that the above entropy condition is inspired by Kružkov's entropy condition for scalar conservation laws. However, due to the presence of diffusion term u_{xx} in (1), Kružkov's entropy condition has to be modified in this case. This is done by introducing as entropy test functions all functions of the form $|u - \psi(x)|$ and ψ smooth, while Kružkov's definition asks for ψ to be a constant.

Regarding the regularized problem (2), it is well known [1] that for all $\varepsilon > 0$ this problem has a unique solution

$$u^\varepsilon \in C([0, T]; L^1(\mathbb{R}^2)) \cap L^\infty(\Pi_T).$$

In fact,

$$u^\varepsilon \in C([0, T]; W^{2,p}(\mathbb{R}^2)) \cap C^1([0, T]; L^p(\mathbb{R}^2)),$$

for all $1 < p < \infty$. In addition, u^ε satisfies the following properties

(a) $\iint_{\mathbb{R}^2} u^\varepsilon(x,y,t)\,dxdy = \iint_{\mathbb{R}^2} u_0(x,y)\,dxdy$,
(b) $\mathrm{TV}\,(u^\varepsilon(\cdot,\cdot,t)) \le \mathrm{TV}\,(u_0)$,
(c) $\|\partial_t u^\varepsilon(t)\|_1 \le C[\mathrm{TV}\,(u_0) + \mathrm{TV}\,((u_0)_x)]$.

Now we are in a position to state our main result, which is the following

Main Theorem. *Let u be the unique entropy solution to (1) and u^ε be as defined by (2). Assume that $u_0 \in BV$, f is Lipschitz continuous. Choose a constant*

$$M > \max_{|u|<\|u_0\|_{L^\infty(\mathbb{R}^2)}} |f'(u)|,$$

and another constant $L > MT$, where $T > 0$. Then there exists a constant C, independent of ε, but depending on f, L, T and u_0, such that

$$\int_{\mathbb{R}} \int_{-L+Mt}^{L-Mt} |u(t,x,y) - u^\varepsilon(t,x,y)|\,dxdy \le C\varepsilon^{1/2} \quad \text{for } t \le T.$$

3. **Proof of the main theorem.** The theorem will be proved by a "doubling of the variables" argument, which was introduced by Kruzhkov [3, 4] in the context of hyperbolic conservation laws.

First, observe that from (4) we have for any test function φ with compact support in $\mathbb{R} \times \mathbb{R} \times (0,T)$ and $u = u(x,z,s)$,

$$\begin{aligned}\iiint_{\Pi_T} &|u - \psi(x)|\,\varphi_s + \mathrm{sign}(u - \psi(x))(f(u) - f(\psi(x)))\varphi_z\,dxdzds \\ &\ge \iiint_{\Pi_T} -|u - \psi(x)|\,\varphi_{xx} - \mathrm{sign}(u - \psi(x))\psi_{xx}\varphi\,dxdzds.\end{aligned} \tag{5}$$

For the regularized equation (2), we start not with the entropy condition, but in the argument leading up to this condition. To do that, first define the regularized counterpart of the signum function as

$$\mathrm{sign}_\eta(\sigma) = \begin{cases} \mathrm{sign}(\sigma) & |\sigma| > \eta, \\ \sin\left(\frac{\pi\sigma}{2\eta}\right) & \text{otherwise,} \end{cases}$$

where $\eta > 0$ and the signum function is defined as

$$\mathrm{sign}(\sigma) = \begin{cases} -1 & \sigma < 0, \\ 0 & \sigma = 0, \\ 1 & \sigma > 0, \end{cases}$$

Set

$$\psi_\eta(u,\psi) = \int_\psi^u \mathrm{sign}_\eta(z - \psi)\,dz.$$

This is a convex entropy for all ψ. Set $u^\varepsilon = u^\varepsilon(x,y,t)$ and rewrite (2) as

$$u_t^\varepsilon + (f(u^\varepsilon) - f(\psi))_y = u_{xx}^\varepsilon + \varepsilon(u^\varepsilon - \psi(x))_{yy},$$

and multiply this with $(\psi_\eta)_u(u,\psi)\,\varphi$ where φ is a test function with compact support in $\mathbb{R} \times \mathbb{R} \times (0,T)$. Observe that the solution u^ε of (2) is smooth. Hence after a partial integration, we arrive at

$$\begin{aligned}\iiint_{\Pi_T} &\psi_\eta(u^\varepsilon,\psi)\varphi_t + Q_\eta(u^\varepsilon,\psi)\varphi_y\,dxdydt \\ &= \iiint_{\Pi_T} -\mathrm{sign}_\eta(u^\varepsilon - \psi)u_{xx}^\varepsilon\varphi - \varepsilon\,\mathrm{sign}_\eta(u^\varepsilon - \psi)(u^\varepsilon - \psi(x))_{yy}\varphi\,dxdydt,\end{aligned}$$

where we have used $Q'_\eta(u,\psi) = \psi'_\eta(u,\psi) f'(u)$. Next, taking limit as $\eta \to 0$, we end up with the parabolic equality

$$\begin{aligned}
&\iiint_{\Pi_T} |u^\varepsilon - \psi(x)|\, \varphi_t + \operatorname{sign}(u^\varepsilon - \psi(x))(f(u^\varepsilon) - f(\psi(x)))\varphi_y \, dxdydt \\
&\quad = \iiint_{\Pi_T} -\operatorname{sign}(u^\varepsilon - \psi(x)) u^\varepsilon_{xx}\varphi - \varepsilon \operatorname{sign}(u^\varepsilon - \psi(x))(u^\varepsilon - \psi(x))_{yy}\varphi \, dxdydt, \\
&\quad = \iiint_{\Pi_T} -\operatorname{sign}(u^\varepsilon - \psi(x)) u^\varepsilon_{xx}\varphi - \varepsilon \, |u^\varepsilon - \psi(x)|_{yy}\, \varphi \, dxdydt.
\end{aligned} \tag{6}$$

At this point we are ready to use "doubling of the variables" device. First, using the entropy inequality (5) for the solution $u = u(x,z,s)$ with $\psi(x) = u^\varepsilon(x,y,t)$, we get for $(y,t) \in \mathbb{R} \times (0,T)$

$$\begin{aligned}
&\iiint_{\Pi_T} |u - u^\varepsilon|\, \varphi_s + \operatorname{sign}(u - u^\varepsilon)(f(u) - f(u^\varepsilon))\varphi_z \, dxdzds \\
&\qquad \ge \iiint_{\Pi_T} - |u - u^\varepsilon|\, \varphi_{xx} - \operatorname{sign}(u - u^\varepsilon) u^\varepsilon_{xx}\varphi \, dxdzds.
\end{aligned} \tag{7}$$

Similarly, from the parabolic equality (6) for the solution $u^\varepsilon = u^\varepsilon(x,y,t)$ with $\psi(x) = u(x,z,s)$, we get for $(z,s) \in \mathbb{R} \times (0,T)$

$$\begin{aligned}
&\iiint_{\Pi_T} |u^\varepsilon - u|\, \varphi_t + \operatorname{sign}(u^\varepsilon - u)(f(u^\varepsilon) - f(u))\varphi_y \, dxdydt \\
&\qquad = \iiint_{\Pi_T} -\operatorname{sign}(u^\varepsilon - u) u^\varepsilon_{xx}\varphi - \varepsilon \, |u^\varepsilon - u|_{yy}\, \varphi \, dxdydt.
\end{aligned} \tag{8}$$

We now integrate (7) over $(y,t) \in \mathbb{R} \times (0,T)$ and (8) over $(z,s) \in \mathbb{R} \times (0,T)$. Addition of those two results yields

$$\begin{aligned}
&\int \iiiint_{Q_T} |u^\varepsilon - u|\, (\varphi_t + \varphi_s) + \operatorname{sign}(u^\varepsilon - u)(f(u^\varepsilon) - f(u))(\varphi_y + \varphi_z)\, dX \\
&\quad \ge \int \iiiint_{Q_T} - |u^\varepsilon - u|\, \varphi_{xx} - \varepsilon \, |u^\varepsilon - u|_{yy}\, \varphi \, dX, \\
&\quad := \int \iiiint_{Q_T} \mathcal{Q}_1 + \mathcal{Q}_2 \, dX,
\end{aligned} \tag{9}$$

where $dX = dx\, dy\, dz\, dt\, ds$ and $Q_T = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,T) \times (0,T)$.

Following Kruzkov and Kuznetsov [3, 4] we now specify a nonnegative test function $\varphi = \varphi(x,y,t,z,s)$ defined in Q_T. To this end, let $\omega \in C_0^\infty(\mathbb{R})$ be a function satisfying

$$\operatorname{supp}(\omega) \subset [-1,1], \qquad \omega(\sigma) \ge 0, \qquad \int_{\mathbb{R}} \omega(\sigma)\, d\sigma = 1,$$

and define $\omega_r(x) = \omega(x/r)/r$. Next, let us choose $\phi \in C_c^\infty(\mathbb{R})$ such that

$$\phi = \begin{cases} 1, & |x| < 1, \\ 0 & |x| \ge 2, \end{cases}$$

and $0 \leq \phi \leq 1$ when $1 \leq |x| \leq 2$. Then we define $K_\beta(x) = \phi(x/\beta)$. We will let $\beta \to \infty$ later. Furthermore, let $h(z)$ be defined as

$$h(z) = \begin{cases} 0, & z < -1, \\ z+1 & z \in [-1,0], \\ 1 & z > 0. \end{cases}$$

and set $h_\alpha(z) = h(\alpha z)$. Let $\nu < \tau$ be two numbers in $(0,T)$, for any $\alpha > 0$ define

$$H_\alpha(t) = \int_{-\infty}^{t} \omega_\alpha(\xi)\, d\xi,$$

$$\begin{aligned}\Psi(y,t) &= (H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau)) \left(h_\alpha(y - L_l(t)) - h_\alpha(y - L_r(t) - \frac{1}{\alpha}) \right) \\ &=: \chi^{\alpha_0}_{(\nu,\tau)}(t)\, \chi^{\alpha}_{(L_l,L_r)}(y,t)\end{aligned}$$

where the lines $L_{l,r}$ are given by

$$L_l(t) = -L + Mt,\ L_r(t) = L - Mt$$

where M and L are positive numbers, M will be specified below. With $0 < r < \min\{\nu, T-\tau\}$ and $\alpha_0 \in (0, \min\{\nu - r, T - \tau - r\})$ we set

$$\varphi(x,y,t,z,s) = K_\beta(x)\, \Psi(y,t)\, \omega_r(y-z)\, \omega_{r_0}(t-s). \tag{10}$$

We note that φ has compact support and also that we have,

$$\begin{aligned}\varphi_t + \varphi_s &= K_\beta(x)\, \Psi_t(y,t)\, \omega_r(y-z)\, \omega_{r_0}(t-s), \\ \varphi_y + \varphi_z &= K_\beta(x)\, \Psi_y(y,t)\, \omega_r(y-z)\, \omega_{r_0}(t-s).\end{aligned}$$

For the record, we note that

$$\begin{aligned}\Psi_t(y,t) &= -\chi^{\alpha_0}_{(\nu,\tau)}(t) M \left(h'_\alpha(y - L_l(t)) + h'_\alpha(y - L_r(t) - \frac{1}{\alpha}) \right) \\ &\quad + (\omega_{\alpha_0}(t-\nu) - \omega_{\alpha_0}(t-\tau))\, \chi^{\alpha}_{(L_l,L_r)}(y,t), \\ \Psi_y(y,t) &= \chi^{\alpha_0}_{(\nu,\tau)}(t) \left(h'_\alpha(y - L_l(t)) - h'_\alpha(y - L_r(t) - \frac{1}{\alpha}) \right).\end{aligned} \tag{11}$$

We shall let all the "small parameters" α, α_0, r, r_0, ε and Δx be sufficiently small and the "large parameter" β be sufficiently big, but fixed.

Starting the first term on the left of (9), we write

$$\begin{aligned}\int_{Q_T} |u^\varepsilon - u| (\varphi_s + \varphi_t)\, dX &\leq \underbrace{\int_{\Pi_T} |u^\varepsilon(x,y,t) - u(x,y,t)|\, K_\beta(x) \Psi_t\, dxdydt}_{\delta} \\ &\quad + \underbrace{\int_{\Pi_T} \int_{\mathbb{R}} |u(x,y,t) - u(x,y,s)|\, K_\beta(x)\, |\Psi_t(x,t)|\, \omega_{r_0}(t-s)\, dxdydsdt}_{\beta} \\ &\quad + \underbrace{\int_{Q_T} |u(x,y,s) - u(x,z,s)|\, K_\beta(x)\, |\Psi_t(x,t)|\, \omega_{r_0}(t-s)\, \omega_r(x-y)\, dX}_{\gamma}.\end{aligned}$$

Following [2], it is easy to find that

$$\beta + \gamma \leq C\,(r_0 + r). \tag{12}$$

To continue the estimate with the first term on the left of (9), we split δ as follows

$$\begin{aligned}\delta = &-\iiint_{\Pi_T} \chi^{\alpha_0}_{(\nu,\tau)}(t) M \left(h'_\alpha(y - L_l(t)) + h'_\alpha(y - L_r(t) - \frac{1}{\alpha})\right) \\ &\qquad K_\beta(x)\,|u^\varepsilon(x,y,t) - u(x,y,t)|\; dxdydt \\ &+ \iiint_{\Pi_T} \chi^{\alpha}_{(L_l,L_r)}(y,t)\,|u^\varepsilon(x,y,t) - u(x,y,t)| \\ &\qquad K_\beta(x)\,(\omega_{\alpha_0}(t-\nu) - \omega_{\alpha_0}(t-\tau))\; dxdydt, \\ := &\;\delta_1 + \delta_2.\end{aligned}$$

The term δ_1 will be balanced against the first order derivative term on the left hand side of (9). To estimate δ_2 we set $e(x,y,t) = |u^\varepsilon(x,y,t) - u(x,y,t)|$ and following [2], we find

$$\begin{aligned}\delta_2 \le &\iint \chi^{\alpha}_{(L_l,L_r)}(y,\nu) K_\beta(x)\,|u^\varepsilon(x,y,\nu) - u(x,y,\nu)|\; dxdy \\ &- \iint \chi^{\alpha}_{(L_l,L_r)}(y,\tau) K_\beta(x)\,|u^\varepsilon(x,y,\tau) - u(x,y,\tau)|\; dxdy + C\alpha_0.\end{aligned} \tag{13}$$

Now we rewrite the "first derivative term" on the left hand side of (9). Doing this, we get

$$\begin{aligned}&\int_{Q_T} K_\beta(x)\mathrm{sign}(u^\varepsilon - u)\,(f(u^\varepsilon) - f(u))\,(\varphi_y + \varphi_z)\; dX \\ &\quad= \int_{Q_T} K_\beta(x)\,\mathrm{sg}(x,y,z,t,s)\,(f(u^\varepsilon(x,y,t)) - f(u(x,y,t))) \\ &\qquad\qquad \Psi_y(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\qquad+ \int_{Q_T} K_\beta(x)\,\mathrm{sg}(x,y,z,t,s)\,(f(u(x,y,t)) - f(u(x,z,s))) \\ &\qquad\qquad \Psi_y(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\quad=: \delta_3 + \delta_4,\end{aligned}$$

where we have set $\mathrm{sg}(x,y,z,t,s) = \mathrm{sign}(u^\varepsilon(x,y,t) - u(x,z,s))$. We proceed as follows

$$\begin{aligned}|\delta_4| \le &\int_{Q_T} |f(u(x,y,t)) - f(u(x,z,s))|\,\chi^{\alpha_0}_{(\nu,\tau)}(t) K_\beta(x) \\ &\qquad \omega_{r_0}(t-s)\,\omega_r(y-z)\left(h'_\alpha(y - L_l(t)) + h'_\alpha(y - L_r(t) - \frac{1}{\alpha})\right) dX.\end{aligned}$$

We follow [2] to estimate each of these two terms to conclude

$$|\delta_4| \le C\,(r_0 + r)\,. \tag{14}$$

Again, choosing M larger than the Lipschitz norm of f implies that

$$\delta_1 + \delta_3 \le 0. \tag{15}$$

Collecting all the terms we see that

$$\begin{aligned}\iint K_\beta(x)\chi^{\alpha}_{(L_l,L_r)}(y,\tau)\left|u^\varepsilon(x,y,\tau)-u(x,y,\tau)\right|\,dxdy \\ \le \iint K_\beta(x)\chi^{\alpha}_{(L_l,L_r)}(y,\nu)\left|u^\varepsilon(x,y,\nu)-u(x,y,\nu)\right|\,dxdy \\ + C\left(r_0+r+\alpha_0+\alpha\right)+\left|\int_{Q_T}\mathcal{Q}_1+\mathcal{Q}_2\,dX\right|.\end{aligned} \tag{16}$$

In order to estimate $\mathcal{Q}_1$ and $\mathcal{Q}_2$, we proceed as follows

$$\begin{aligned}\left|\int_{Q_T}\mathcal{Q}_1\right|dX &= \int_{Q_T}\left|u^\varepsilon-u\right|\varphi_{xx}\,dX \\ &= \int_{Q_T}\left|u^\varepsilon-u\right|K''_\beta(x)\Psi(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\le \|u^\varepsilon+u\|_{L^\infty}\int_{Q_T}\frac{1}{\beta^2}\phi''(\frac{x}{\beta})\Psi(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\le \frac{C}{\beta^2}\iiint\phi''(\frac{x}{\beta})\chi^{\alpha_0}_{(\nu,\tau)}(t)\,\chi^{\alpha}_{(L_l,L_r)}(y,t)\,dxdydt \\ &\le \frac{CK}{\beta\alpha},\end{aligned}$$

where $K := \int_{\mathbb{R}}|\phi''(y)|\,dy$.

Next,

$$\begin{aligned}&\left|\int_{Q_T}\mathcal{Q}_2\right|dX = \varepsilon\int_{Q_T}\left|u^\varepsilon-u\right|_y\varphi_y\,dX \\ &= \varepsilon\int_{Q_T}\operatorname{sign}(u^\varepsilon-u)(u^\varepsilon)_yK_\beta(x)\,\Psi_y(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s) \\ &+\varepsilon\int_{Q_T}\operatorname{sign}(u^\varepsilon-u)(u^\varepsilon)_yK_\beta(x)\,\Psi(y,t)\,\omega'_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &:= \mathcal{Q}_{2,1}+\mathcal{Q}_{2,2}.\end{aligned}$$

Each of the above terms can be approximated as follows

$$\begin{aligned}&\left|\int_{Q_T}\mathcal{Q}_{2,1}\right|dX = \varepsilon\int_{Q_T}\operatorname{sign}(u^\varepsilon-u)(u^\varepsilon)_yK_\beta(x)\,\Psi_y(y,t)\,\omega_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\le \varepsilon K\int_{\Pi_T}(u^\varepsilon)_y(x,y,t)\chi^{\alpha_0}_{(\nu,\tau)}(t)\left(h'_\alpha(y-L_l(t))-h'_\alpha(y-L_r(t)-\frac{1}{\alpha})\right)dxdydt \\ &\le K\varepsilon\alpha\iint\left|u^\varepsilon_x\right|\,dxdy \\ &\le CK\alpha\varepsilon,\end{aligned}$$

where $K=\|K_\beta\|_\infty$ and $C=\mathrm{TV}\left(u^\varepsilon\right)$. Similarly for the other term

$$\begin{aligned}\left|\int_{Q_T}\mathcal{Q}_{2,2}\right|dX &= \varepsilon\int_{Q_T}\operatorname{sign}(u^\varepsilon-u)(u^\varepsilon)_yK_\beta(x)\,\Psi(y,t)\,\omega'_r(y-z)\,\omega_{r_0}(t-s)\,dX \\ &\le \frac{K\varepsilon}{r}\int_{\Pi_T}(u^\varepsilon)_y(x,y,t)\chi^{\alpha_0}_{(\nu,\tau)}(t)\,\chi^{\alpha}_{(L_l,L_r)}(y,t)\,dxdydt \\ &\le CK\frac{\varepsilon}{r},\end{aligned}$$

where again $K = \|K_\beta\|_\infty$ and $C = \mathrm{TV}(u^\varepsilon)$.

Therefore

$$\left| \int_{\Pi_T^2} \mathcal{Q}_1 + \mathcal{Q}_2 \, dX \right| \leq C\varepsilon + \frac{C}{\beta\alpha} + \frac{C\varepsilon}{r}. \tag{17}$$

where C depends on (among other things) L and T, but not on the parameters α_0, α, r_0, r, β or ε.

Now we have proved the follwoing Lemma:

Lemma 3.1. *Assume that u and u^ε take values in the interval $[-K, K]$ for some positive K. Let $M > \max_{v\in[-K,K]} |f'(v)|$. Then if $T \geq \tau > \nu \geq 0$ and $L - M\tau > 0$, we have*

$$\begin{aligned} \int_{\mathbb{R}} \int_{-L+M\tau}^{L-M\tau} & |u^\varepsilon(x, y, \tau) - u(x, y, \tau)| \, dxdy \\ & \leq \int_{\mathbb{R}^2} |u^\varepsilon(x, y, \nu) - u(x, y, \nu)| \, dxdy \\ & \quad + C\left[r_0 + r + \alpha + \varepsilon + \frac{1}{\alpha\beta} + \frac{\varepsilon}{r} \right]. \end{aligned} \tag{18}$$

This follows from (16) and (17), observing that we can send α_0 to zero. Now we let $u(x, y, t)$ be the unique entropy solution of (1). Also note that since we let β tends to ∞ so $\gamma = \frac{1}{\beta}$ is small. We set $\alpha = r = r_0$ and $\varepsilon = \gamma$, and assume that α is sufficiently small, then

$$\int_{\mathbb{R}} \int_{-L+Mt}^{L-Mt} |u_{\Delta x}(x, t) - v(x, t)| \, dx \, dy \leq C \left(\alpha + \frac{\varepsilon}{\alpha} \right), \tag{19}$$

for some constant C which is independent of the small parameters α, ε. This follows from (18). Then setting $\varepsilon = \alpha^2$ proves the main theorem.

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