INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 3, Pages 372–381

CONTINUOUS SEDIMENTATION IN CLARIFIER-THICKENER UNITS: MODELING MACROSCOPIC CONSERVATION LAWS FROM MICROSCOPIC INTERACTING PARTICLES

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Abstract. We study a model of continuous sedimentation. Under idealizing assumptions, the settling of the solid particles under the influence of gravity can be described by the initial value problem for a one-dimensional scalar conservation law with a flux function that depends discontinuously on the spatial position.
There exists several entropy conditions related to the same conservation law in the literature giving rise to uniqueness. The same initial data may give rise to different entropy solutions, depending on the criteria one selects. This motivated us to derive the PDE together with an entropy condition as a hydrodynamic limit from a microscopic interacting particle system.We are inclined to prefer the entropy solution selected by this method. It turns out that this is an entropy condition suggested by Audusse and Pethame in a different context.

Key words. hyperbolic conservation laws, discontinuous flux functions, hydrodynamic limits, microscopic particle systems

1. Modeling a clarifyer-thickener model

Given a suspension of small solid particles dispersed in a fluid in a box. Under gravity these particles settle to the bottom of the box. The suspension shall be modeled by a mixture of two superimposed continuous media. Let v_s be the velocity of the solid phase, v_f the velocity of the fluid phase, and ρ the local solid concentration. The equations of continuity give:

$$u_t + (uv_s)_x = 0$$
$$u_t - ((1-u)v_f)_x = 0$$

Now we introduce a volume average velocity

$$q := uv_s + (1-u)v_f$$

This allows the above two conservation of mass equations to be combined to a single scalar equation of the type:

$$q_t + g(x,t,q)_x = 0 \qquad x \in R, t > 0$$

. Here the flux function in general depends in a discontinuous fashion on the space variable x. For more details see [3].

2. Scalar conservation laws with discontinuous flux

We are concerned with the following class of scalar conservation laws, describing flow through porous media

$$\text{(2.1)} \qquad \begin{cases} \partial_t \rho(t,x) + \partial_x F(x,\rho(t,x)) = 0 \\ \rho|_{t=0} = \rho_0(x) \end{cases}$$

Received by the editors January 13, 2013 and, in revised form, September 30, 2013.
2000 *Mathematics Subject Classification.* 35L60, 60K35, 82C22.

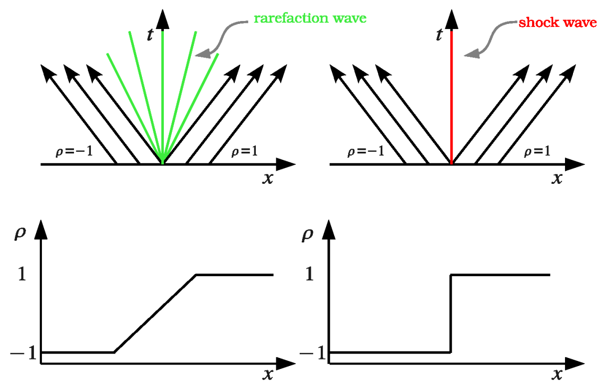

FIGURE 1

where $F(\cdot,\rho)$ is continuous except on a set of measure zero. This discontinuity may appear for example, if there is a sudden change of the porosity.

Recall that even if the space dependency of the flux is continuous, weak solutions of this partial differential equation may not be unique. The most known example of this is probably the inviscid Burger equation which reads as

$$\begin{cases} \partial_t \rho(t,x) + \frac{1}{2}\partial_x \rho^2(t,x) = 0 \\ \rho_0(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} \end{cases}$$

This equation may allow two solutions. The first solution produces a rarefaction at the origin, the second solution produces a shock at the origin:

Therefore the solutions have to be considered together with an admissibility criteria called entropy condition which picks a unique solution. The hope is, that the entropy condition naturally leads to the physical relevant one.

If the flux function $F(\cdot,\rho)$ is a smooth function in space, the equation is well understood. In this case Kruzkov in [10] proposed an entropy inequality for which uniqueness and existence have been shown. The entropy inequality, in the weak sense, reads as

$$\partial_t |\rho - c| + \partial_x \{\text{sign}(\rho - c)(F(x,\rho) - F(x,c))\} + \text{sign}(\rho - c)\partial_x F(x,c) \leq 0,$$

for each constant c. Obviously, if there are discontinuities in the space dependency of the flux function F, the third term of the last expression does not make sense and hence the Kruzkov inequality does not hold anymore. For this case several admissibility criteria have been proposed in the literature, for example see [6, 9, 7], for which uniqueness and existence have been shown within the various classes. In [1] the authors proved uniqueness of a Kruzkov-type entropy inequality, but not existence. Unfortunately these entropy conditions are in general not equivalent, that means, it may happen that one admissibility criteria selects a unique entropy solution, which is different from the unique entropy solution selected by an other admissibility criteria.

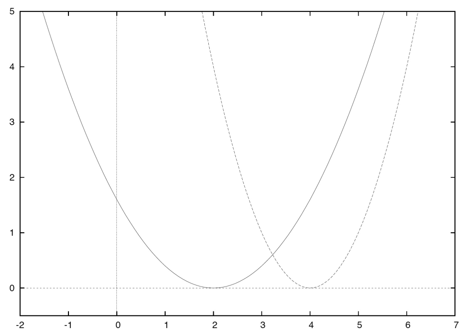

FIGURE 2

For example, let us consider the following initial value problem with a Heavyside type flux:

$$(2.2)\qquad \begin{cases} \partial_t u + \partial_x \left[H(x)\frac{u^2}{2} + (1-H(x))\frac{(u-1)^2}{2}\right] = 0 \\ u_0(x) = \frac{1}{2} \end{cases}$$

where $H(x)$ denotes the Heavyside function. A weak solution of this Riemann problem is the function given by

$$(2.3)\qquad u(t,x) = \begin{cases} \frac{1}{2} & x \le -\frac{t}{2} \\ 1+\frac{x}{t} & -\frac{t}{2} < x \le -u_i t \\ 1-u_i & -u_i t < x \le 0 \\ u_i & 0 < x \le u_i t \\ \frac{x}{t} & u_i t < x \le \frac{t}{2} \\ \frac{1}{2} & \frac{t}{2} < x \end{cases}$$

for any $u_i \in [0, \frac{1}{2}]$. For this particular crossing convex flux function, the interface condition in [7] selects the solution with $u_i = \frac{1}{2}$, hence the constant solution

$$u(t,x) = \frac{1}{2}$$

as the unique entropy solution. On the other hand, the Kruzkov-type entropy condition in [1] selects the solution with $u_i = 0$, that means

$$u(t,x) = \frac{x}{t} + 1 - H(x)$$

as unique entropy solution. The question arises which entropy condition is the one choosing the solution which is of physical interest?

3. The work plan

The idea is now to look at these partial differential equation models as the hydrodynamic limit of microscopic interacting particle systems. This is in order to find out which macroscopic entropy is selected by this limit. This is an ongoing project. Thus in order to show a proof of concept, we have begun with a specific class of scalar discontinuous flux conservation laws, for which we were able to prove this limit. This class does not include the example of the previous section, and again the example from the previous section is simpler than the clarifier thickener model in the section before that. In what follows, we are able to show the hydrodynamic limit only for a microscopic zero range process. We hope to be able to eventually extend such limits to more general situations. We find that the entropy found from our simple process is an entropy that does not rely on traces. We expect this to lead the way in the future.

4. Notion of measure-valued entropy solutions

The entropy condition of [1] is interesting for two reasons: the first reason is, that the condition effects the solution globally, that means it does not impose admissibility criteria at the shocks. The second reason is, that there are microscopic interacting particle systems called Zero Range Processes (**ZRP**) which satisfy such an inequality similar to the entropy inequality proposed in [1] at microscopic level. The particle densities of such ZRP, in the hydrodynamic limit satisfy a special case of the conservation law (2.1)

To apply the uniqueness theorem in [1], we have to do some restrictions on the flux function:

- (H1) $F(x, \rho)$ is continuous at all points of $(\mathbb{R}\backslash\mathcal{N}) \times \mathbb{R}$ with $\mathcal{N}$ a closed set of measure zero;
- (H2) $\exists$ continuous functions f, g such that, for any $x \in \mathbb{R}$ and large ρ, $f(\rho) \leq |F(x,\rho)| \leq g(\rho)$ with $f(\rho) \geq 0$ and $f(\pm\infty) = \infty$;
- (H3) There exists a function $\rho_m(x)$from $\mathbb{R}$ to $\mathbb{R}$ and a constant M_0 such that, for $x \in \mathbb{R}\backslash\mathcal{N}$, $F(x, \rho)$ is a locally Lipschitz, one to one function from $(-\infty, \rho_m]$ and $[\rho_m, \infty)$ to $[M_0, \infty)$ (or $(-\infty, M_0]$) with $F(x, \rho_m(x)) = M_0$ and with common Lipschitz constant L_I for all $x \in \mathbb{R}\backslash\mathcal{N}$ and all $\rho \in I$ that is any bounded interval in $\mathbb{R}$;

or

- (H3') For $x \in \mathbb{R}\backslash\mathcal{N}$, $F(x, \cdot)$ is a locally Lipschitz, one to one function from $\mathbb{R}$ to $\mathbb{R}$ with common Lipschitz constant L_I for all $x \in \mathbb{R}\backslash\mathcal{N}$ and all $\rho \in I$ that is any bounded interval in $\mathbb{R}$.

One example of the flux functions satisfying (H1)–(H2) and (H3) or (H3') is

$$F(x, \rho) = \lambda(x)h(\rho), \tag{4.1}$$

where $\lambda(x)$ is continuous in $x \in \mathbb{R}$ with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for some constants λ_1 and λ_2, except on a closed set $\mathcal{N}$ of measure zero, and $h(\rho)$ is locally Lipschitz and is either monotone or convex (or concave) with $h(\rho_m) = 0$ for some ρ_m in which case $M_0 = 0$.

It is easy to check that, if the flux function $F(x, \rho)$ satisfies (H1)–(H3), then, for any constant $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$), there are two steady-state solutions m^+_α from $\mathbb{R}$ to $[\rho_m(x), \infty)$ and m^-_α from $\mathbb{R}$ to $(-\infty, \rho_m(x)]$ of (2.1) such that

$$F(x, m^\pm_\alpha(x)) = \alpha \qquad \text{for a.e. } x \in \mathbb{R}. \tag{4.2}$$

In the case (H1)–(H2) and (H3'), $m^+_\alpha(x) = m^-_\alpha(x)$ which is even simpler.

The notion of entropy solutions in L^∞ introduced in Audusse-Perthame [1] and Baiti-Jenssen [2] can be further formulated into the following.

Definition 4.1 (Notion of entropy solutions in L^∞)**.** *We say that an L^∞ function $\rho : \mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is an entropy solution of* (2.1) *provided that, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions*

$m^{\pm}_{\alpha}(x)$ *of* (2.1),

$$\int \Big(|\rho(t,x) - m^{\pm}_{\alpha}(x)|\, \partial_t J + sign(\rho(t,x) - m^{\pm}_{\alpha}(x))\big(F(x,\rho(t,x)) - \alpha\big)\partial_x J \Big)\, dtdx$$

$$+ \int |\rho_0(x) - m^{\pm}_{\alpha}(x)| J(0,x)\, dx \geq 0 \tag{4.3}$$

for any test function $J : \mathbb{R}^2_+ \to \mathbb{R}_+$.

Following the notion of entropy solutions, we introduce the corresponding notion of measure-valued entropy solutions. We denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

Definition 4.2 (Notion of measure-valued entropy solutions)**.** *We say that a measurable map*

$$\pi : \mathbb{R}^2_+ \to \mathcal{P}(\mathbb{R})$$

is a measure-valued entropy solution of (2.1) *provided that* $\langle \pi_{0,x}; k \rangle = \rho_0(x)$ *for a.e.* $x \in \mathbb{R}$ *and, for each* $\alpha \in [M_0, \infty)$ *(or* $\alpha \in (-\infty, M_0]$*) and the corresponding two steady-state solutions* $m^{\pm}_{\alpha}(x)$,

$$\int \big(\langle \pi_{t,x}; |k - m^{\pm}_{\alpha}(x)| \rangle\, \partial_t J + \langle \pi_{t,x}; sign(k - m^{\pm}_{\alpha}(x))\, (F(x,k) - \alpha) \rangle\, \partial_x J \big)\, dxdt$$

$$+ \int |\rho_0(x) - m^{\pm}_{\alpha}(x)|\, J(0,x)\, dx \geq 0 \tag{4.4}$$

for any test function $J : \mathbb{R}^2_+ \to \mathbb{R}_+$.

If a measure-valued entropy solution $\pi_{t,x}(k)$ is a Dirac mass with the associated profile $\rho(t,x)$, i.e. $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$, then $\rho(t,x)$ is an entropy solution of (2.1), which is unique as shown in [1]. A detailed proof of this reduction theorem can be found in [4].

5. The hydrodynamic limit from a microscopic interacting particle system to a scalar conservation law with discontinuous flux

We consider a one dimensional discrete lattice with N sites and of macroscopic length one. On each site there may sit a finite number of particles. According to a Markovian law one particle can jump from site u to site v with a jump rate

$$\lambda\left(\frac{u}{N}\right) g(\eta(u))$$

depending on the macroscopic position $\frac{u}{N}$ and on the number of particles at the jumping off site $\eta(u)$. Thus particles only interact with particles sitting at the same site. This is where the name ZRP comes from. The function g is a monotone function, and the porosity here is expressed through the discontinuous function λ, continuous at all points of $\mathbb{R} \neq \mathcal{N}$ where $\mathcal{N}$ is a closed set of measure zero. Site v is chosen by the transition probability $p(v-u)$ which is asymmetric in our case. More detailed properties of the zero range process will be described in what follows.
We now want to send N to infinity and consequently the distance between particles tends to zero. Notice that if N goes to infinity, a particle initially at the origin, at time t only had time to move a distance of order $\frac{1}{N}$ and hence macroscopically did not have time to evolve. Therefore we need to rescale time by the Euler scaling factor N in order to have a macroscopic evolution in macroscopic time. With this scaling, we call the limit as $N \to \infty$ *the hydrodynamic limit* (**HDL**). It turns out

that the microscopic particle density of a ZRP, in the HDL, satisfies the initial value problem

$$\tag{5.1} \begin{cases} \partial_t \rho(t,x) + \partial_x \{\lambda(x) h(\rho)\} = 0 \\ \\ \rho|_{t=0} = \rho_0(x) \end{cases}$$

where $h(\rho)$ is a monotone function of ρ. More precisely, we can construct from the microscopic interacting particle system a microscopic inequality which as $N \to \infty$ naturally leads us to a macroscopic measure-valued entropy inequality defined in 4.2 and thereby proves the existence of measure valued entropy solutions. Then we can use the the results from [4] and the uniqueness result from [1] to prove convergence of the particle densities to the unique weak entropy solution given in Definition 4.1. In other words, we prove convergence to a deterministic function ρ satisfying (2.1) together with entropy inequality of Definition 4.1 and thereby implies uniqueness of the limit points.
Rezakhanlou in [11] first established the hydrodynamic limit of the processus des misanthropes (PdM) with constant speed-parameter. Covert-Rezakhanlou [5] provided a proof of the hydrodynamic limit of a PdM with non-constant but continuous speed-parameter λ.

We first analyze some properties of the ZRP. Obviously we have two space scales: The discrete lattice $\mathbb{Z}$ as embedded in $\mathbb{R}$ with vertices $\frac{u}{N}$ and $u \in \mathbb{Z}$. In this way, the distances between particles tend to zero if N increases to infinity. Sites of the microscopic scale $\mathbb{Z}$ are denoted by the letters u, v and correspond to the points $\frac{u}{N}$, $\frac{v}{N}$ in the macroscopic scale $\mathbb{R}$. Points of the macroscopic space scale $\mathbb{R}$ are denoted by the letters x, y and correspond to the sites $[xN]$, $[yN]$ in the microscopic space scale, where [z] is the integer part of z. We denote by $\eta_t(u)$ the number of particles at time $t > 0$ at site u. Then the vector $\eta_t = (\eta_t(u) : u \in \mathbb{Z})$ is called a configuration at time t with configuration space $E := \mathbb{N}^{\mathbb{Z}}$.

First let us define the standard mollification in space of $\lambda(x, \rho)$:

$$\tag{5.2} \lambda_\varepsilon(x, \rho) := (\lambda(x, \rho) * \theta^\varepsilon)(x) \to \lambda(x, \rho) \quad a.e.$$

with $\theta^\varepsilon(x) := \theta(\frac{x}{\varepsilon})$, $\theta(x) \geq 0$, $\operatorname{supp} \theta(x) \subset [-1, 1]$, $\int_{-1}^{1} \theta(x) dx = 1$.

In general, the ZRP can be described as follows: For each $\varepsilon > 0$ fixed, infinitely many indistinguishable particles are distributed on a 1-dimensional lattice. Any site of the lattice may be occupied by a finite number of particles. Associated to a given site u there is an exponential clock with rate $\lambda_\varepsilon(\frac{u}{N}) g(\eta(u))$ depending on the macroscopic spatial coordinates. Each time the clock rings on the site u, one of the particles jumps to the site v chosen with probability $p(u, v)$. The elementary transition probabilities p: $\mathbb{Z} \to [0, 1]$ are supposed to be

(i) translation invariant: $p(x, y) = p(0, y - x) =: p(y - x)$;
(ii) normalized: $\sum_y p(x, y) = 1$, $p(x, x) = 0$;
(iii) assumed to be of finite range: $p(x, y) = 0$ for $|y - x|$ sufficiently large;
(iv) irreducible: $p(0, 1) > 0$.

Without loss of generality, we assume that $\sum_z p(z) z = \gamma = 1$; otherwise, for $\gamma \neq 1$, we replace the function $h(\rho)$ by $h(\rho)/\gamma$ in the following argument. The rate $g : \mathbb{N} \to \mathbb{R}_+$ is a positive, nondecreasing function with $g(0) = 0$, $g(\infty) = \infty$, and

$$\tag{5.3} \frac{g(k)}{k^2} \to 0 \qquad \text{when } k \to \infty.$$

With this description, the Markov process η_t is generated by

$$NL_\varepsilon^N f(\eta) = N \sum_{u,v} \lambda_\varepsilon(\frac{u}{N}) g(\eta(u)) p(v-u)(f(\eta^{u,v}) - f(\eta)). \tag{5.4}$$

Here N comes from the Euler scaling factor speeding the generator, thus η_t denotes a configuration on which the speeded generator NL_ε^N has acted for time t, and $\eta^{u,v}$ represents the configuration η where one particle jumped from u to v:

$$\eta^{u,v}(w) = \begin{cases} \eta(w) & \text{if } w \neq u, v, \\ \eta(u) - 1 & \text{if } w = u, \\ \eta(v) + 1 & \text{if } w = v. \end{cases}$$

For any $\varepsilon = \varepsilon(N) > 0$ and for any constant $\alpha \geq 0$, we define a product measure given by

$$\mu_{m_\alpha}^N(\eta) = \prod_u \frac{1}{Z\left(h\left(m_\alpha(\frac{u}{N})\right)\right)} \frac{\left(h\left(m_\alpha(\frac{u}{N})\right)\right)^{\eta(u)}}{g(\eta(u))!}, \tag{5.5}$$

where Z is a partition function equal to

$$Z(h) = \sum_{n=0}^{\infty} \frac{(h)^n}{g(n)!}. \tag{5.6}$$

With these settings, it turns out, that the expected value of the occupation variable $\eta(u)$ is equal to

$$E_{\mu_{m_\alpha}^N(\eta)}[\eta(u)] = m_\alpha(\frac{u}{N}),$$

where m_α is a steady-state solution to

$$\partial_t \rho + \partial_x (\lambda_\varepsilon(x) h(\rho)) = 0. \tag{5.7}$$

and

$$E_{\mu_{m_\alpha}^N(\eta)}[g(\eta(u))] = h\left(m_\alpha(\frac{u}{N})\right).$$

The important attribute of the ZRP with non-constant speed-parameter is that the *product* measure $\mu_{m_\alpha}^N(\eta)$ is invariant under the generator NL_ε^N, i.e.,

$$\int L_\varepsilon^N(f(\eta)) d\mu_{m_\alpha}^N(\eta) = 0. \tag{5.8}$$

As initial distribution of our system , we choose the local equilibrium product measure $\mu_0^N(\eta)$ associated to a bounded density profile $\rho_0 \geq 0$ as follows:

$$\mu_0^N(\eta) := \prod_u \frac{1}{Z(h(\rho_{u,N}))} \frac{(h(\rho_{u,N}))^{\eta(u)}}{g(\eta(u))!}, \tag{5.9}$$

where $\rho_{u,N} \geq 0$ is a sequence satisfying $\lim_{N\to\infty} \int |\rho_{[Nx],N} - \rho_0(x)| dx = 0$ for $[Nx]$ as the integer part of Nx and

$$\lim_{N\to\infty} \langle \mu_0^N(\eta)\,;\, \Big|\frac{1}{N}\sum_u J(\frac{u}{N})\eta(u) - \int J(x)\rho_0(x)dx\Big|\rangle = 0 \quad \text{for any test function } J.$$

Furthermore, let μ_t^N denote the distribution of a configuration at time t initially distributed by μ_0^N:

$$\mu_t^N = S_t^N * \mu_0^N, \tag{5.10}$$

where $S_t^N = e^{tNL_\varepsilon^N}$ is the semigroup corresponding to the generator NL_ε^N. Since we consider a nondecreasing function g, we obtain that for two initial measures

$\mu_{\rho_0}^N$ and $\mu_{\omega_0}^N$ on E with profiles ρ_t and ω_t, respectively, the following monotonicity holds:

$$\mu_{\rho_0}^N \leq \mu_{\omega_0}^N \;\;\Rightarrow\;\; S_t^N \mu_0^N \leq S_t^N \mu_{m_\alpha}^N = \mu_{m_\alpha}^N. \tag{5.11}$$

Since our initial distribution has a bounded density profile, then the density profile remains bounded at later time t.

The goal in proving the hydrodynamic limit of a ZRP is that, if we start from a configuration η_0 distributed with an initial measure μ_0^N associated to the bounded density profile ρ_0, then the distribution μ_t^N of the configuration η_t at later time t is associated to the density profile $\rho(t,\cdot)$, where ρ is the solution of the Cauchy problem (5.1) in the sense of Definition 4.1. In other words, our main theorem is the following:

Theorem 5.1 (Hydrodynamic limit of an attractive ZRP with discontinuous speed–parameter)**.** *Let η_t be an attractive ZRP with rate $\lambda_\varepsilon(\frac{u}{N})g(\eta(u))$ and with* (5.3)*, initially distributed by the measure μ_0^N associated to a bounded density profile $\rho_0 : \mathbb{R}_+^2 \to \mathbb{R}_+$ as defined in* (5.9)*. Let $\varepsilon = \varepsilon(N) = N^{-\sigma}, \sigma \in (0,1)$. Then, at later time t,*

$$\lim_{N\to\infty} \langle \mu_t^N(\eta);\, \Big| \frac{1}{N} \sum_u J(\frac{u}{N})\eta_t(u) - \int J(x)\rho(t,x)dx \Big| \rangle = 0 \tag{5.12}$$

for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}$, where $\rho(t,x)$ is the unique solution of the Cauchy problem (5.1) *in the sense of Definition 4.1.*

To prove the entropy inequality at microscopic level, which is the most important step towards the proof of Theorem 5.1, we use the scaling relation $\varepsilon = \varepsilon(N) = N^{-\sigma}, \sigma \in (0,1)$. This is the statement of the proposition. The proof relies on coupling arguments and here the assumption of attractiveness, that means, that the function g is nondecreasing of is crucial.

Proposition 5.2 (Entropy inequality at microscopic level for $\varepsilon = N^{-\sigma}$ with $\sigma \in (0,1)$ when $N \to \infty$)**.** *Let m_α^ε be the steady-state solutions of* (5.7) *as defined in* (4.2)*. Let η_t be the ZRP generated by NL_ε^N defined by* (5.4) *and initially distributed by the measure μ_0^N defined by* (5.9)*. Let $\eta^l(u)$ be the average density of particles in large microscopic boxes of size $2l+1$ and centered at u:*

$$\eta^l(u) := \frac{1}{2l+1} \sum_{|u-v|\leq l} \eta(v).$$

Then, for every test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$\lim_{l\to\infty} \lim_{N\to\infty} \mu_t^N \Big\{ \int_0^t \frac{1}{N} \sum_u \Big(\partial_s J(s,\frac{u}{N}) \big|\eta_s^l(u) - m_\alpha^\varepsilon(\frac{u}{N})\big| + \partial_x J(s,\frac{u}{N}) \big|\lambda_\varepsilon(\frac{u}{N}) h(\eta_s^l(u)) - \alpha\big| \Big) ds + \frac{1}{N} \sum_u J(0,\frac{u}{N}) \big|\eta_0^l(u) - m_\alpha^\varepsilon(\frac{u}{N})\big| \geq -\delta \Big\} = 1. \tag{5.13}$$

Inequality (5.13) is the entropy inequality (4.3) with ρ replaced by the average density of particles in the microscopic boxes of length $2l+1$. To prove the microscopic entropy inequality, we consider the coupled zero range process (η_t, ξ_t) with initial distribution of (η_t, ξ_t) by $\bar{\mu}_0^N = \mu_0^N \times \mu_{m_\alpha^\varepsilon}^N$, where μ_0^N is the initial measure with density profile ρ_0 defined by (5.9) and $\mu_{m_\alpha^\varepsilon}^N$ denotes the invariant measure as defined in (5.5).

Since the ξ-marginal of $\bar{\mu}_0^N$ is the invariant measure $\mu_{m_\alpha^\varepsilon}^N$, at any time t the marginal remains the same. Thus the measure $\mu_{m_\alpha^\varepsilon}^N$ is always stochastically bounded since m_α^ε is bounded. Therefore by the law of large numbers for any limit point μ_m of $\mu_{m_\alpha^\varepsilon}^N$ and for each u fixed we can define μ_m a.s the density

$$\lim_{l \to \infty} \xi^l(u) := m(x).$$

Then (5.13) follows by the equivalence of ensembles.

A detailed proof of Proposition 5.13 can be found in [4]

Recall that a microscopic entropy inequality leading to the Kruzkov entropy inequality has been proved in [5] for the process of PdM with nonconstant but continuous speed-parameter λ_ε. Since there does not exist an invariant product measure for a PdM in general such that $E_{\mu_{m_\alpha^\varepsilon}^N}[\xi(u)] = m_\alpha^\varepsilon(\frac{u}{N})$, one has to apply the relative entropy method of Yau [12].

In our case of a space-dependent ZRP, the invariant product measure is available so that we can approximate steady-state solutions m_α^ε by a process ξ distributed by the invariant measure $\mu_{m_\alpha^\varepsilon}^N$ for any $\alpha \in (0, \infty)$ and hence the relative entropy method is not necessary in our case.

Following standard arguments from [11, 5, 8] one can now prove convergence to measure valued entropy solutions defined in 4.2, and thereby the existence of such measure valued entropy solutions. Notice, that the proof of existence of measure valued entropy solutions naturally follows from the ZRP we constructed. Then we can use the results from [4] to complete the proof, that means to reduce the measure valued entropy solutions to the unique entropy solution of (5.1) defined in 4.1.

6. Summary

After modeling a clarifier-thickener unit by a scalar conservation law where the flux function depends in a discontinuous fashion on the space variable, and observing that these equations allow for many possible entropy conditions, we proceed to derive a particlular PDE model of this class from a microscopic interacting particle model. We have shown that the hydrodynamic limit selects one admissibility condition for our conservation law and argue that the thus selected is the physically correct one.

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