A new grazing collision approximation to the Boltzmann collision operator for plasmas

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Quick Outline

1. Motivation – state of kinetic modeling of plasmas
2. Overview: spectral methods for the Boltzmann and Landau-Fokker-Planck operators
3. Derivation and analysis of the Landau approximation to Boltzmann via spectral methods

What is a kinetic model?

- Kinetic equations model the dynamics of *distributions* of particles

$$\frac{\partial f}{\partial t} + \mathbf{c} \cdot \nabla_{\mathbf{x}} f + \mathbf{a} \cdot \nabla_{\mathbf{c}} f = \frac{1}{K_n} \frac{\partial f}{\partial t}\bigg|_{\text{coll}}$$

$$f = f(\mathbf{x}, \mathbf{c}, t)$$

$$K_n = \frac{\lambda_{mfp}}{l_0} = \frac{t_{mft}}{t_0}$$

- This is typically referred to as 'non-equilibrium' modeling
  - Equilibrium = hydrodynamics

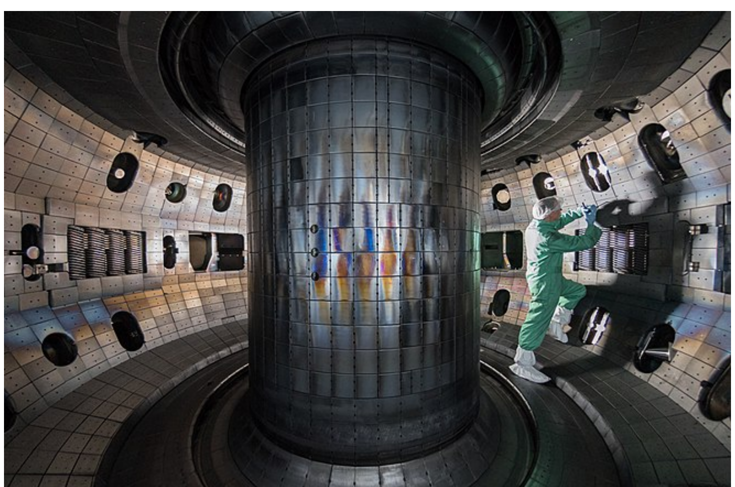
- Kinetic models are interesting in their own right, but are also useful for deriving quantities (transport coefficients) used in Navier-Stokes, etc.

Three different fields of kinetic modeling

- We all solve the same equations, but speak different languages…

Aerospace – Boltzmann equation, DSMC

Astrophysics: Linear Boltzmann, iterative linear algebra



Plasma physics: Vlasov, maybe Fokker-Planck, Particle in Cell

Kinetic modeling of plasmas is dominated by the Fokker-Planck (FP) operator

- Plasma physics tends to approach collisions from the collisionless 'side'
  - Focus is often more on e.g. E-field effects, waves, etc in dilute plasmas
- As you introduce collisions into your system they are relatively weak
- After a Taylor expansion, you obtain a nonlinear collision term*:

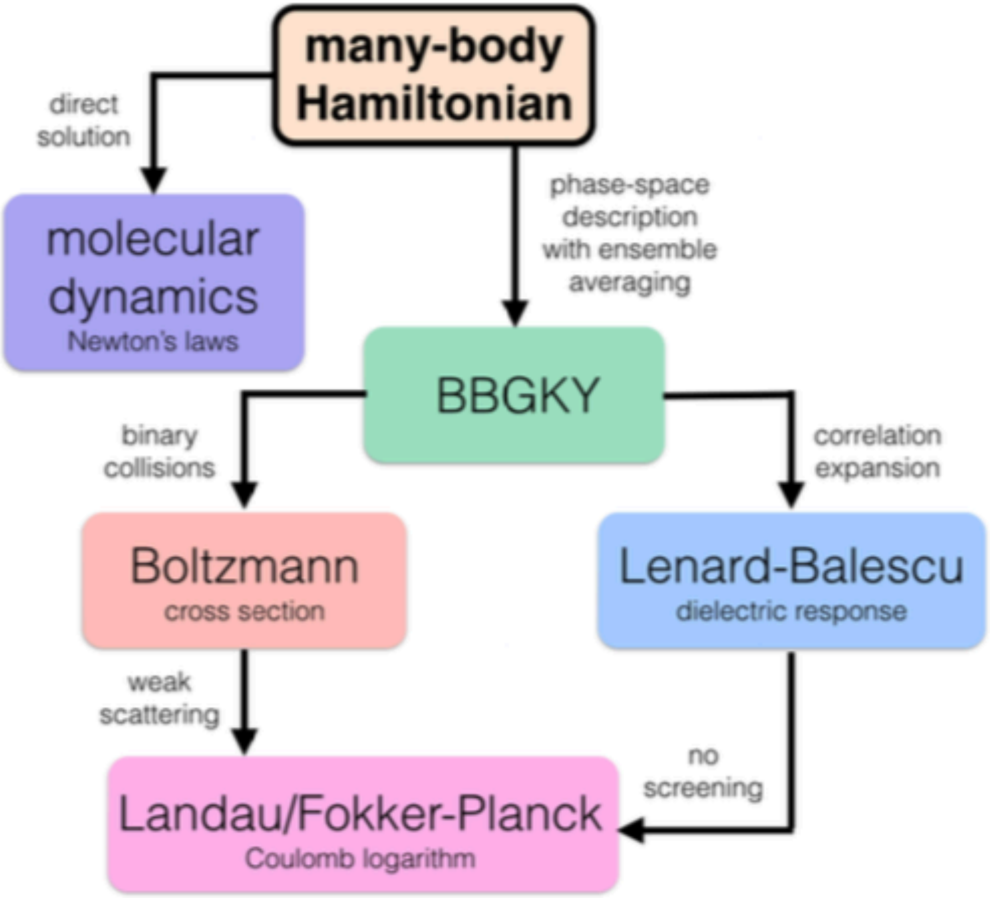
$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} = \gamma \nabla_{\mathbf{v}} \cdot \left[-\mathbf{F}[f] f + \mathsf{D}[f] \cdot \nabla_{\mathbf{v}} f\right]$$

- Arguments made based on the small angle scattering assumption give yield to the formulas for the parameters in this model
  - No additional kinetic models appear – they jump right to FP
- Once derived, it became the 'ground truth' for plasma physics
  - It works well in the regimes where it works well!

Landau (1937), Rosenbluth, McDonald & Judd (1957)

Many assumptions/approximations are made along the way – be careful!

- Once FP became 'ground truth', it became applied to many situations where it might not be valid
- How to check the assumptions?
- In particular, what even *is* the small angle scattering assumption? When does it break down?
- Open NRL plasma formulary, it says FP is 'fine for CL ~ 10-20'
  - This is not typical for many kinetic plasmas where FP is used



The Coulomb Logarithm is a catch-all factor used in the FP operator

- So what is this Coulomb Logarithm, and where does it appear?

$$\gamma \sim \log(\rho^{-1/2} T^{3/2})$$

  - (glossing over some dimensional constants here)
- This prefactor is large for **hot and/or dilute plasmas**
- There are some issues with this definition if you think for a minute
  - E.g. can go negative, for one

- See Gericke, Murillo, Schlanges (Phys Rev E, 2002) for more CL details than you can shake a stick at

- These fixups are mostly cosmetic however – if you ask the authors (I have) they will tell you that this is just a band-aid to try to extend a model in an area that you shouldn't extend it.

The Boltzmann operator can be more accurate, but much more difficult to work with

- Boltzmann collisions

$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} = \int_{\mathbb{R}^3}\int_{\mathcal{S}^2} |\mathbf{g}|\,\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega})(f(\mathbf{c}')f(\mathbf{c}'_*) - f(\mathbf{c})f(\mathbf{c}_*))d\boldsymbol{\omega}d\mathbf{c}_*$$

$$\mathbf{g} = \mathbf{c} - \mathbf{c}_*$$

$$\mathbf{c}' = \mathbf{c} + \frac{1}{2}(|\mathbf{g}|\boldsymbol{\omega} - \mathbf{g}) \qquad \mathbf{c}'_* = \mathbf{c}_* - \frac{1}{2}(|\mathbf{g}|\boldsymbol{\omega} - \mathbf{g})$$

- $\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega})$ is the *differential cross section*

- In plasmas, this has a messy angular dependence that can be tricky to deal with:

$$\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega}) \sim |\mathbf{g}|^{-4}\sin^{-4}(\theta/2)$$

How can we quantify when we can use FP, and when we need Boltzmann?

- We need an estimate on the error made when approximating Boltzmann by Fokker-Planck
- We also need numerical evidence to convince the FP crowd that what they are doing may be suspect

- **The answer to both**: spectral formulation of the kinetic equations

Overview: spectral methods for kinetic equations

Spectral analysis of Boltzmann leads to more 'efficient' numerics and a clearer Landau limit

- Spectral methods: Fourier transform turns collisions into a weighted convolution
  - Pareschi + Perthame (1996), Bobylev and Rjasanow (1998), Pareschi and Russo (2000)
  - Later work by Gamba and Tharkabhushanam (2008)
  - An equivalent form for FP also exists

- This provides a common framework that we will use to examine the limit, and give us a numerical method to boot.

- This is more amenable to numerical computation
  - Boltzmann evaluation requires $O(N^6)$ operations
  - FP and some* Boltzmann cases can be reduced to $O(N^3 \log N)$
  - Solutions converge spectrally quickly

The weak form is the key to spectral methods for collision operators

- Symmetries in Boltzmann lead to the generic form

$$\int \frac{\partial f}{\partial t}\bigg|_{\text{coll}} d\mathbf{c} := \int Q(f)\phi(\mathbf{c})d\mathbf{c} = \iiint f(\mathbf{c})f(\mathbf{c}_*)|\mathbf{g}|\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega})(\phi(\mathbf{c}') - \phi(\mathbf{c}))d\boldsymbol{\omega}d\mathbf{c}_*d\mathbf{c}$$

- The key step is to choose your test functions to be the Fourier modes

$$\phi = (2\pi)^{-3/2}e^{-i\mathbf{k}\cdot\mathbf{c}}$$

- Exponentials have nice properties, and we can rearrange things such that time-independent stuff can be precomputed:

$$\widehat{Q}(\mathbf{k}) = \int \mathcal{F}(f(\mathbf{c})f(\mathbf{c}-\mathbf{g}))(\mathbf{k}) \int |\mathbf{g}|\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega}) \left(e^{-i\frac{\mathbf{k}}{2}\cdot(|\mathbf{g}|\boldsymbol{\omega}-\mathbf{g})} - 1\right) d\boldsymbol{\omega}d\mathbf{g}$$

$$= \int \mathcal{F}(f(\mathbf{c})f(\mathbf{c}-\mathbf{g}))(\mathbf{k})G(\mathbf{g},\mathbf{k})d\mathbf{g}$$

The collision operators are convolutions in Fourier space

$$\widehat{Q}(\mathbf{k}) = \int \mathcal{F}(f(\mathbf{c})f(\mathbf{c}-\mathbf{g}))(\mathbf{k})G(\mathbf{g},\mathbf{k})d\mathbf{g}$$

- A few more identities gives

$$\widehat{Q}(\mathbf{k}) = \int \hat{f}(\mathbf{k}-\mathbf{m})\hat{f}(\mathbf{m})\widehat{G}(\mathbf{k},\mathbf{m})d\mathbf{m}$$

- This is easily amenable to numerical approximation and parallelization, though you do not get the usual $O(N^3 \log N)$ speedup for convolutions
  - Except in special cases

Convolution weight formulas for simple cross sections

- A few simple examples

- Hard spheres: $\sigma = C$

$$\widehat{G}(\mathbf{k},\mathbf{m}) = \frac{2C}{(2\pi)^{1/2}} \int_0^\infty r^3 \left(\text{sinc}\left(\frac{r|\mathbf{k}|}{2}\right) \text{sinc}(r|\mathbf{m} - \frac{1}{2}\mathbf{k}|) - \text{sinc}(r\mathbf{m}) \right) dr$$

- Maxwell molecules: $|\mathbf{g}|\sigma = C$

$$\widehat{G}(\mathbf{k},\mathbf{m}) = \frac{2C}{(2\pi)^{1/2}} \int_0^\infty r^2 \left(\text{sinc}\left(\frac{r|\mathbf{k}|}{2}\right) \text{sinc}(r|\mathbf{m} - \frac{1}{2}\mathbf{k}|) - \text{sinc}(r\mathbf{m}) \right) dr$$

- Isotropic cross sections -> adjust power on r in formula

Convolution weight formulas for less simple cross sections

- For angularly dependent cross sections, we have (Gamba, Haack 2016)

$$\widehat{G}(\mathbf{m},\mathbf{k}) = (2\pi)^{1/2}\int_0^\infty r^3 \int_0^\pi \int_0^\pi \sigma(r,\cos\theta)\sin\theta\sin\gamma J_0\left(r\left|\mathbf{m}-\frac{\mathbf{m}\cdot\mathbf{k}}{|\mathbf{k}|^2}\mathbf{k}\right|\sin\gamma\right)$$
$$\times\left[\cos\left(r\left(\mathbf{m}-\frac{\mathbf{k}}{2}(1-\cos\theta)\right)\cdot\frac{\mathbf{k}}{|\mathbf{k}|}\cos\gamma\right)J_0\left(\frac{1}{2}r|\mathbf{k}|\sin\gamma\sin\theta\right)\right.$$
$$\left.-\cos\left(r\mathbf{m}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}\cos\gamma\right)\right]d\theta d\gamma dr,$$

- This is more expensive to precompute and store, but you only have to do it once*

Numerical implementation has a few gotchas

- How big to take your velocity domain?
  - Distribution functions are infinite, computational grids are not
  - $\widehat{G}(\mathbf{m}, \mathbf{k})$ is a tempered distribution
  - Generally speaking, taking $L \sim 5\sqrt{k_b T/m} = 5|\mathbf{v}_{th}|$ is 'good enough' due to decay of distributions
  - Analysis of cutoff: Alonso & Gamba (2018)
  - Numerical analysis of cutoff effects: check accuracy of known moment calculations
  - This cutoff also appears in the weight calculation

- What integrators to use to precompute $\widehat{G}(\mathbf{m}, \mathbf{k})$?
  - I use prepackaged integrators from the GNU scientific library
  - Singularities at integral endpoints require additional treatment

- Weight storage requires memory allocation of $O(N^6)$

We can ensure that the conserved quantities stay conserved*

- One of the key constraints/symmetries of kinetic models is **conservation** – is this enforced here?

$$\int Q(f)d\mathbf{c} = \int \mathbf{c}Q(F)d\mathbf{c} = \int \mathbf{c}^2 Q(f)d\mathbf{c} = 0$$

- We can ensure that the **discrete** moments are conserved through a simple optimization problem: given the output $\tilde{Q} = (\tilde{Q}_1, \ldots, \tilde{Q}_M),\ M = N^3$

$$\text{Find } Q = (Q_1, \ldots, Q_M)^T \text{ that minimizes } \frac{1}{2}\|\tilde{Q} - Q\| \text{ such that } CQ = 0$$

$$C = \begin{pmatrix} w_j \\ c_j^x w_j \\ c_j^y w_j \\ c_j^z w_j \\ |\mathbf{c}_j|^2 w_j \end{pmatrix}$$

- Results in an extra projection step $Q = (I + C(CC^T)^{-1}C)\tilde{Q}$
- Inexpensive solve relative to the convolution

*Gamba + Tharkabhushanam 2009

Some algorithmic speedup tricks

- Because of the structure of $\widehat{G}(\mathbf{m},\mathbf{k})$, we are stuck with $O(N^6)$ complexity
  - At least it is embarrassingly parallel
- But if we tweak the weight computation, we can create a few exploitable convolutions*

$$\widehat{G}(\mathbf{k},\mathbf{m}) \approx \sum_{p=1}^{N_p} G_p^{(1)}(\mathbf{k}) G_p^{(2)}(\mathbf{m}) G_p^{(3)}(\mathbf{k}-\mathbf{m})$$

- This now costs $O(N_p N^3 \log(N))$
- To get this split: discretize the weight function integrals on quadrature points

$$N_p = NM, \quad M << N^2$$

- Not going to work well with singularities

N	direct spectral	fast spectral $M = 14$
8	6.91e-04	7.33e-04
16	7.83e-05	7.63e-05
32	3.90e-08	3.90e-08
64	—	3.81e-08

N	direct spectral	fast spectral $M = 14$
8	0.09s	0.14s
16	6.31s	0.26s
32	542.34s	1.78s
64	—	33.15s

*Gamba, Haack, Hauck, Hu (2017)

These same ideas also apply to Fokker-Planck, resulting in a convolution with different weights

- The FP operator has an equivalent integral form, called the Landau operator

$$Q_L = \log \Lambda \nabla_{\mathbf{c}} \cdot \int |\mathbf{g}|^{-1} \left(I - \frac{1}{|g|^2} \mathbf{g} \otimes \mathbf{g} \right) (f(\mathbf{c}_*) \nabla_{\mathbf{c}} f(\mathbf{c}) - f(\mathbf{c})(\nabla_{\mathbf{c}} f)(\mathbf{c}_*)) d\mathbf{c}_*$$

- Run it through the same 'spectral car wash' and obtain

$$\widehat{Q}_L = \int \widehat{G}_L(\mathbf{k}, \mathbf{m}) \hat{f}(\mathbf{k} - \mathbf{m}) \hat{f}(\mathbf{m}) d\mathbf{m}$$

$$\widehat{G}_L(\mathbf{k}, \mathbf{m}) = \log \Lambda \int |\mathbf{g}|^{-3} (4i(\mathbf{g} \cdot \mathbf{k}) - |\mathbf{g}|^2 |\mathbf{k}^{\perp}|^2|) e^{-i\mathbf{g} \cdot m} d\mathbf{g}$$

$$\mathbf{k}^{\perp} = \mathbf{k} - \frac{\mathbf{k} \cdot \mathbf{g}}{|\mathbf{g}|^2} \mathbf{g}$$

- This convolution **can** be computed in $O(N^3 \log N)$ operations*

- We have a similar form for both operators – how do we connect them?

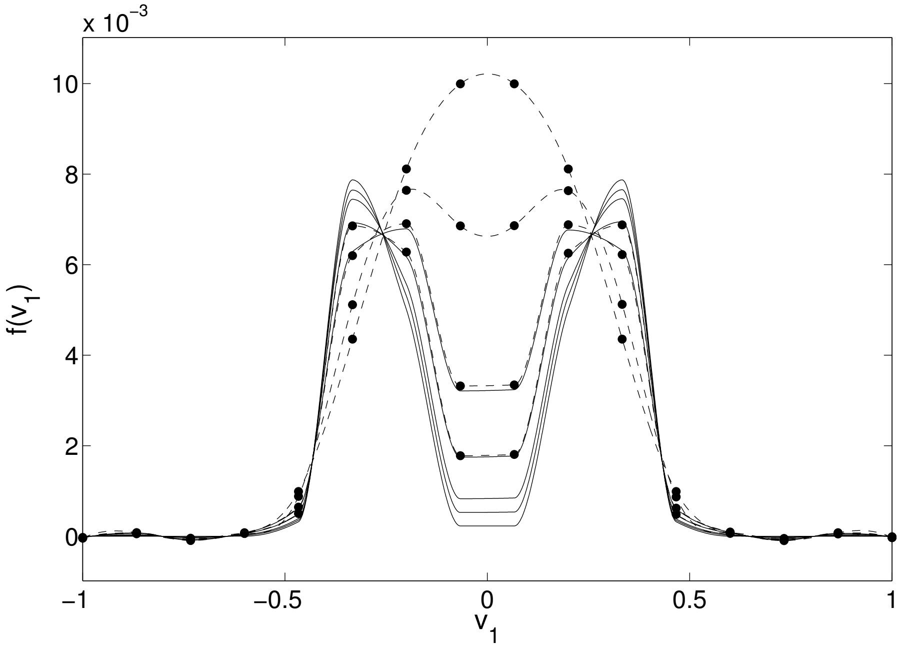
*Pareschi, Russo, Toscani (2000), Gamba & Zhang (2018)

Previous results: Haack + Gamba

- The difference between B and FP has been 'known' by some in the plasma community, but not quantified mostly because Boltzmann was too expensive to compute
- Using spectral methods, the first direct comparison was demonstrated*

- 0D relaxation to equilibrium
  - Dashed line: Boltzmann
  - Solid Line: Landau
  - Corresponds to CL of ~4
    - **BAD REGIME?**

$$Q_B - Q_L \approx \frac{1}{\log \varepsilon} + \ldots\ldots$$

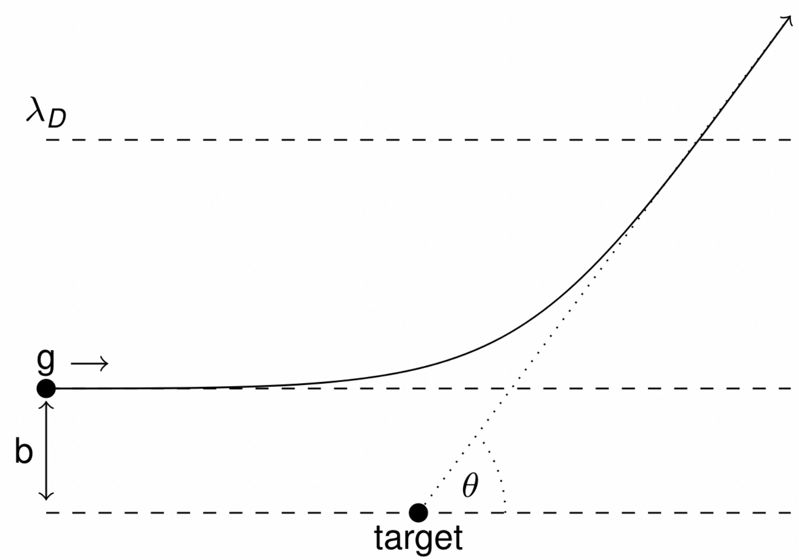


*Gamba & Haack (2016)

The Landau approximation to Boltzmann, via spectral methods

Boltzmann cross sections arise from two-body interactions

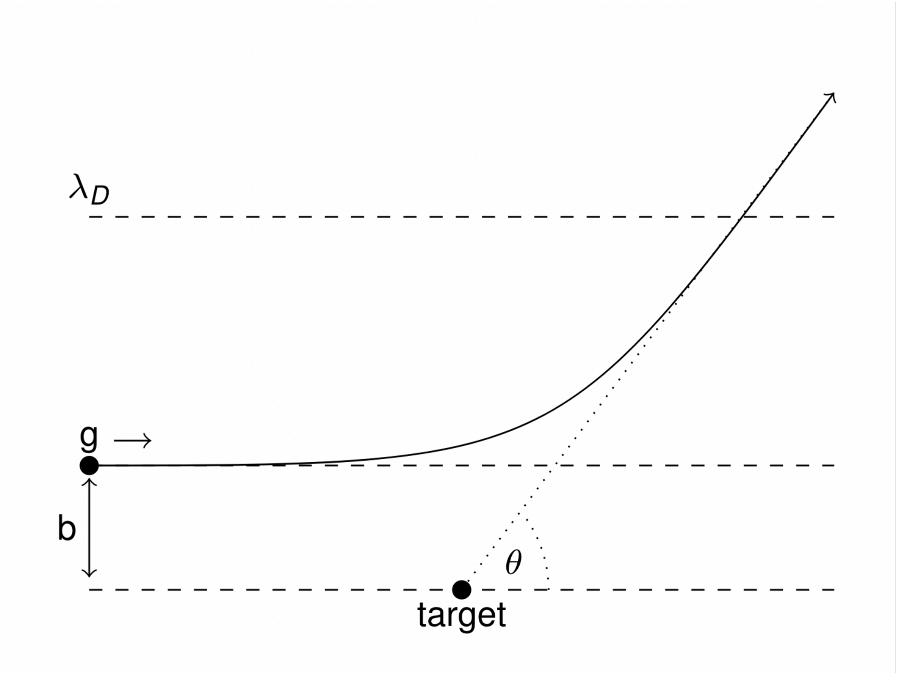
- The key term in Boltzmann is the cross section $\sigma(|\mathbf{g}|, \mathbf{g} \cdot \boldsymbol{\omega})$
- Cross sections follow from two-body interactions, which have three key parameters, $(|\mathbf{g}|, b, \theta)$, plus a spherical interaction potential $\phi(r)$
- Generally speaking, the solution to this interaction does not have a closed form solution



Boltzmann cross sections arise from two-body interactions

- For Coulomb interactions, $\phi(r) = \frac{C_0}{r}$, we *can* obtain a formula for the angular deflection:

$$\theta = 2\arctan\left(\frac{C_0}{|\mathbf{g}|^2 b}\right)$$



This result gives the classic Rutherford cross section, originally discovered by experiment

- The cross section formula is

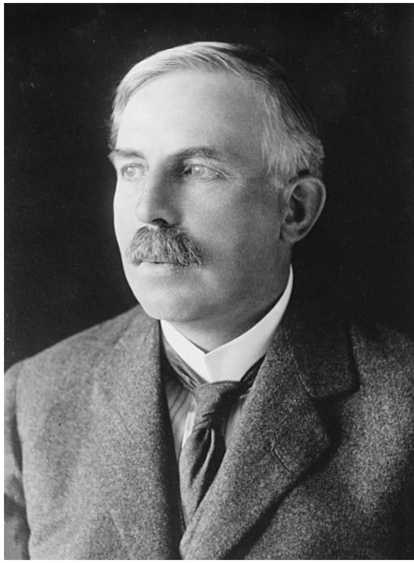
$$\sigma = \frac{b}{\sin(\theta)} \left| \frac{db}{d\theta} \right|$$

- Since we can easily solve for the impact parameter, we obtain

$$\sigma(\mathbf{g}, \theta) = \left(\frac{C_0}{2|\mathbf{g}|^2 \sin^2(\theta/2)} \right)^2$$

- This is exactly the form derived by Rutherford in 1911 in experiments to nail down the size/existence of the nucleus

- But there is a problem if you try to use it in Boltzmann
  - The angular piece of the collision operator is **not integrable**
  - How do we correct this?



Interactions are screened at the Debye length, an effective three-body effect

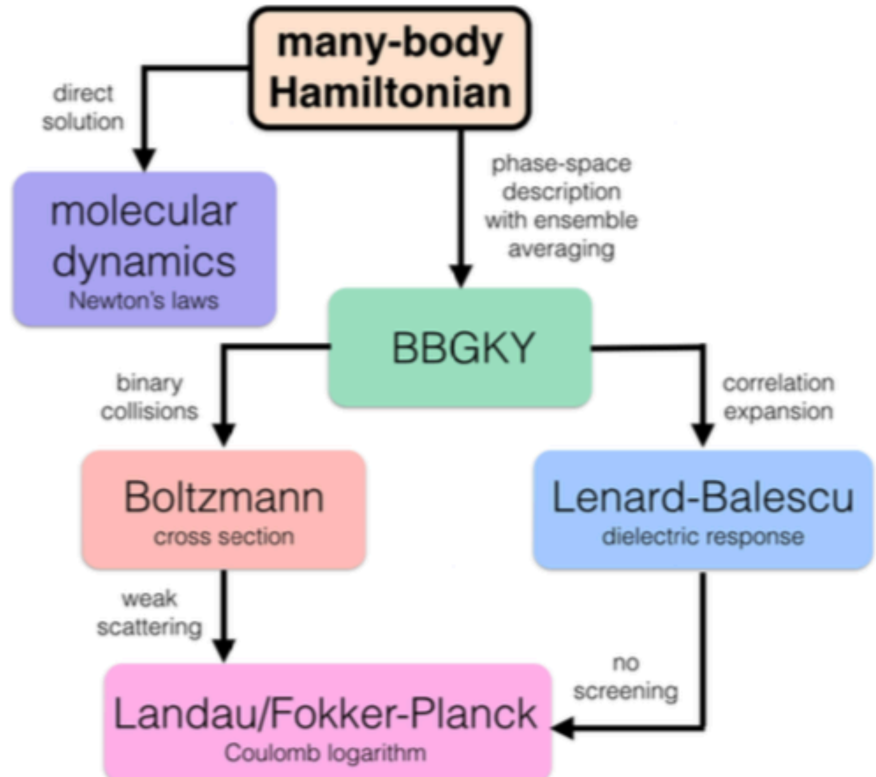
- Coulomb interactions are an odd duck
  - They decay quite slowly
  - But are fairly weak
- We are saved by another term that appears in plasmas: screening effects

- This comes from a 3-body term in the BBGKY hierarchy, folded into the Boltzmann equation via the cross section.
  - 'Effective Boltzmann equation'

- We cut off at

$$b = \lambda_D = \left(\frac{4\pi C_0 n}{T} \right)^{-1/2}$$

  - This corresponds to a minimum angle θ_m



The Debye length cutoff resolves the singularity in the Boltzmann cross section

- With the cutoff, we now have

$$\sigma(|\mathbf{g}|, \mathbf{g}\cdot\boldsymbol{\omega}) = \left(\frac{C_1}{|\mathbf{g}|^2 \sin^2(\theta/2)}\right)^2 \mathbb{1}_{\theta>\theta_m}$$

$$\theta_m = 2\arctan\left(\frac{C_1}{|\mathbf{g}|^2 \lambda_D}\right)$$

- Note here that we have a **velocity dependence** in θ_m
  - In the past this was typically ignored to make life a bit easier
  - The usual approximation is to take $|\mathbf{g}|^2 = |\mathbf{c}_{th}|^2 = T$ to make this a constant, as it makes the usual approximation procedure tractable
  - However, this will not be an issue in the spectral approach, so we will leave it in place

The spectral approach will shows that Landau is an approximation of the Boltzmann convolution weights

- Armed with this cross section, we can now write (easier in this form)

$$\widehat{Q}_B = \int \mathcal{F}\{f(\mathbf{c})f(\mathbf{c}-\mathbf{g})\}(\mathbf{k})G_B(\mathbf{g},\mathbf{k})d\mathbf{g}$$

$$G_B(\mathbf{g},\mathbf{k}) = \int_{\mathcal{S}^2} \frac{C_1^2}{|\mathbf{g}|^3 \sin^4(\theta/2)} \mathbb{1}_{\theta>\theta_m(|\mathbf{g}|)} \left(e^{-i\frac{\mathbf{k}}{2}\cdot(|\mathbf{g}|\boldsymbol{\omega}-\mathbf{g})} - 1\right) d\boldsymbol{\omega}$$

- For mathematical convenience, we rescale the cross section

$$\sigma_m(|\mathbf{g}|,\theta) := \frac{\sigma(|\mathbf{g}|,\theta)}{2\pi\log(\sin(\theta_m/2))}$$

- This ensures we have a mathematical limit
  - Physically, you can think of this limit as the two operators converging to each other, as they both diverge logarithmically

Theorem – rate of convergence

- **Theorem**

  Assume that f satisfies

$$|\mathcal{F}\{f(\mathbf{c})f(\mathbf{c}-\mathbf{g})\}(\mathbf{k})| \leq \frac{A(\mathbf{k},t)}{1+(\Gamma^{3/4}|\mathbf{g}|)^{3+\alpha}}$$

  with $A(\mathbf{k},t)$ uniformly bounded by $k_0(1+|\mathbf{k}|)^{-3}$, and $a > 0$

  Then the approximation error made by approximating Boltzmann by Landau is given by

$$\left|\widehat{Q}_B - \widehat{Q}_L\right| \leq C \left| \int_{\mathbb{R}^3} \mathcal{F}\{f_{\theta_m}(\mathbf{c}) f_{\theta_m}(\mathbf{c}-\mathbf{g})\} \left(\frac{\left|1-\sin^2(\theta_m/2)\right|}{\log(\sin(\theta_m/2))} \left(\frac{|\mathbf{k}|^2}{|\mathbf{g}|} + |\mathbf{k}|^3 \right) \right) d\mathbf{g} \right|$$

  Furthermore, we have that

$$\left|\widehat{Q}_B - \widehat{Q}_L\right| \leq C_2 \Gamma^{3/2}$$

  Giving the rate of convergence of the approximation

Proof: expand and bound error terms

- Expand the exponential in the weight function G

$$
\begin{aligned}
G_B(\mathbf{g},\mathbf{k}) &= |\mathbf{g}| \int_{S^2} \sigma_m \left(e^{-i\frac{\mathbf{k}}{2}\cdot(|\mathbf{g}|\boldsymbol{\omega}-\mathbf{g})} - 1 \right) d\boldsymbol{\omega} \\
&= |\mathbf{g}| \int_{S^2} \sigma_m \left[i \left(\frac{\mathbf{g}\cdot\mathbf{k}}{2} - |\mathbf{g}| \frac{\mathbf{g}\cdot\boldsymbol{\omega}}{2} \right) \right. \\
&\left. - \frac{1}{2} \left(\frac{\mathbf{g}\cdot\mathbf{k}}{2} - |\mathbf{g}| \frac{\mathbf{g}\cdot\boldsymbol{\omega}}{2} \right)^2 - \frac{ie^{ic}}{6} \left(\frac{\mathbf{g}\cdot\mathbf{k}}{2} - |\mathbf{g}| \frac{\mathbf{k}\cdot\boldsymbol{\omega}}{2} \right)^3 \right] d\boldsymbol{\omega} \\
&:= G_{B_1} + G_{B_2} + G_{B_3}
\end{aligned}
$$

- We will see the Landau weights ‘pop out’ of the first two terms, the rest can be bounded

Some Lemmas

- **Barely a Lemma 1**
  - Define the Landau weights

$$G_{L*}(\mathbf{g},\mathbf{k}) = |\mathbf{g}|^{-3}(4i(\mathbf{g}\cdot\mathbf{k}) - |\mathbf{g}|^2|\mathbf{k}^{\perp}|^2)$$

  - Then we have that

$$|G_{B_1} + G_{B_2} - G_{L*}| \leq C_4 \left(\frac{|\mathbf{g}|^{-1}|\mathbf{k}||1-\sin^2(\theta_m/2)|}{|\log(\sin(\theta_m/2))|} \right)$$

Sketch of Proof

- Define ω in terms of a pole at g, write in spherical coordinates and integrate in azimuthal direction.

$$G_{B_1} + G_{B_2} = \frac{|\mathbf{g}|}{-2C_1^2\log(\sin(\theta_m/2))}\int_{\theta_m}^{\pi} \sigma_m \sin\theta \Bigg(2i\sin^2(\theta/2)(\mathbf{g}\cdot\mathbf{k})$$

$$- (\mathbf{g}\cdot\mathbf{k})^2\sin^4(\theta/2) - \frac{1}{2}|\mathbf{g}|^2|\mathbf{k}^{\perp}|^2\cos^2(\theta/2)\sin^2(\theta/2)\Bigg) d\theta$$

$$= |\mathbf{g}|^{-3}\left[\boxed{4i(\mathbf{g}\cdot\mathbf{k})} + \frac{(\mathbf{g}\cdot\mathbf{k})^2(1-\sin^2(\theta_m/2))}{\log(\sin(\theta_m/2))} - \boxed{|\mathbf{g}|^2|\mathbf{k}^{\perp}|^2} - \frac{|\mathbf{g}|^2|\mathbf{k}^{\perp}|^2(1-\sin^2(\theta_m/2))}{\log(\sin(\theta_m/2))} \right]$$

Some Lemmas

- **Barely a Lemma 2**
  - Under the assumptions of the Theorem, the bound for G_{B_3} is given by

$$G_{B_3} \leq C_3 \frac{|k|^3 |1 - \sin^2(\theta_m/2)|}{|\log(\sin(\theta_m/2))|}$$

- **Sketch of proof:**

$$\begin{aligned}|G_{B_3}| &\leq \frac{5|\mathbf{k}|^3|\mathbf{g}|^4}{12C_1^2|\log(\sin(\theta_m/2))|}\int_{\theta_m}^{\pi} \sigma_m \sin\theta \sin^4(\theta/2) d\theta \\ &= \frac{5|\mathbf{k}|^3|1 - \sin^2(\theta_m/2)|}{12|\log(\sin(\theta_m/2))|}\end{aligned}$$

  - Combining these Lemmas gives the first estimate of the Theorem

Approximation of Boltzmann – what is the small parameter?

- The second estimate in the Theorem involves the nondimensional parameter Γ, the plasma coupling parameter

$$\Gamma = \frac{C_0 (4/3\pi n)^{1/3}}{T}$$

- Weak coupling, where $\Gamma << 1$, is the regime where we expect Landau to approximate Boltzmann well.
- Combining with other parameters running around this talk, we can also write

$$\Gamma = \left(\frac{4C_1^2}{3\lambda_D^2 |\mathbf{c}_{th}|^4} \right)^{1/3}$$

and

$$\theta_m = 2\arctan\left(\frac{\sqrt{3}\Gamma^{3/2}|\mathbf{c}_{th}|^2}{|\mathbf{g}|^2} \right)$$

$$\sin(\theta_m/2) = \left(1 + \frac{|\mathbf{g}|^4}{3\Gamma^3 |\mathbf{c}_{th}|^2} \right)^{-1/2}$$

Completing the convergence rate estimate

- Using this rewritten form of the equation and the assumption in the theorem, the estimate becomes

$$\left|\widehat{Q_B}[f_{\theta_m}] - \widehat{Q_{L*}}[f_{\theta_m}]\right| \leq \frac{\sqrt{6}|\mathbf{c}_{th}|^2\Gamma^{3/2}}{(1+|\mathbf{k}|)^3} \int_0^\infty \frac{|\mathbf{k}|^2 z + |\mathbf{k}|^3 3^{1/4}|\mathbf{c}_{th}|\Gamma^{3/4} z^2}{1+(3^{1/4}|\mathbf{c}_{th}|z)^{3+a}} dz$$

- This integral is bounded for all $\mathbf{k}$, which gives the result

The new Landau operator

- Ok, so what did we actually derive?

- Working backwards from $G_{L*}(\mathbf{g}, \mathbf{k})$, and un-scaling the cross section, we have the collision operator

$$Q_{L*}(\mathbf{c}) = 2\pi C_1^2 \nabla_{\mathbf{c}} \cdot \int |\mathbf{g}|^{-1} \left(I - \frac{\mathbf{g} \otimes \mathbf{g}}{|\mathbf{g}|^2} \right) (\nabla_{\mathbf{c}} - \nabla_{\mathbf{c}_*})(C_L(|\mathbf{g}|) f(\mathbf{c}) f(\mathbf{c}_*)) d\mathbf{c}_*$$

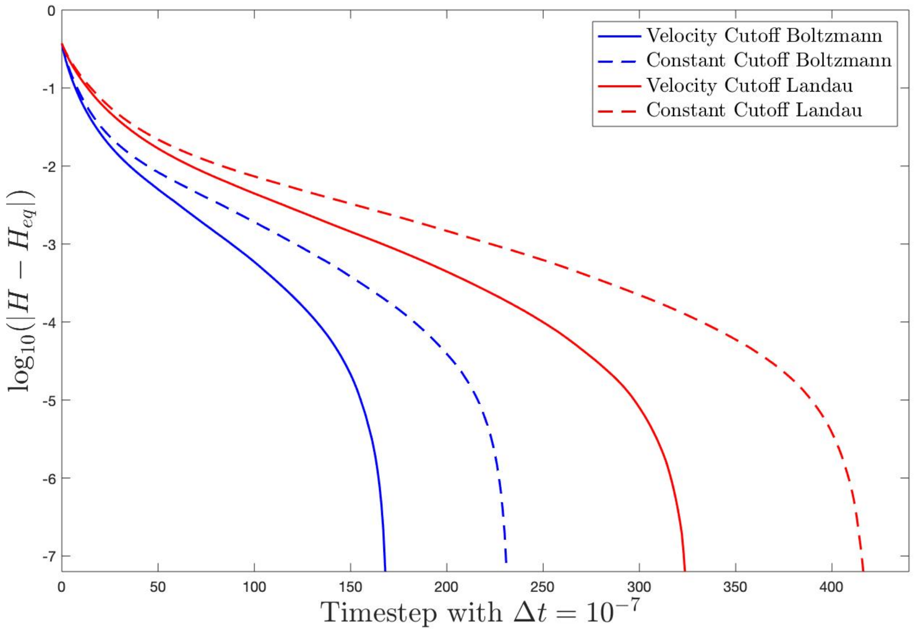
$$C_L = \frac{1}{2} \log \left(1 + \frac{\lambda_D^2 |\mathbf{g}|^4}{4C_1^2} \right) = \frac{1}{2} \log \left(1 + \frac{1}{3\Gamma^3} |\mathbf{g}/\mathbf{c}_{th}|^4 \right)$$

- The previously used operator with constant CL falls out immediately.
- This weights high speed collisions more heavily, and de-emphasizes low speed collisions

Some numerical results

- How does this impact convergence to equilibrium?
- Initial condition – ‘volcano’ inspired by legendre polynomial expansion

$$f(\mathbf{v},0) = 0.01 n_r \left(\frac{m}{k_b T_r}\right)^{3/2} \exp\left(-\frac{10m\left(|\mathbf{c}| - 0.3 v_r\right)^2}{0.09 k_b T_r}\right)$$



Summary and expected impact

- We derived a new version of the Landau/Fokker-Planck collision operator, now with a velocity dependent CL
  - This CL emphasizes collisions with high relative velocity and de-emphasizes the lower energy collisions in the FP framework
  - This provides a better approximation to Boltzmann, but it is still an approximation so YMMV

- Future work
  - Can we derive new plasma transport coefficients off of this model to improve NS modeling?
  - We tossed out some higher order terms, but could they lead to further improvements in the FP operator, which has nice numerical properties?
  - Investigate of the FP speedup algorithms apply to this new operator.
  - Analytical work predicts a convergence to Maxwellian of 2/3, does this model pick that up?

Thanks!!