MATHEMATICS OF COMPUTATION
Volume 73, Number 247, Pages 1235–1259
S 0025-5718(03)01625-9
Article electronically published on December 22, 2003

A RELAXATION SCHEME FOR CONSERVATION LAWS WITH A DISCONTINUOUS COEFFICIENT

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ABSTRACT. We study a relaxation scheme of the Jin and Xin type for conservation laws with a flux function that depends discontinuously on the spatial location through a coefficient $k(x)$. If $k \in BV$, we show that the relaxation scheme produces a sequence of approximate solutions that converge to a weak solution. The Murat–Tartar compensated compactness method is used to establish convergence. We present numerical experiments with the relaxation scheme, and comparisons are made with a front tracking scheme based on an exact 2×2 Riemann solver.

1. INTRODUCTION

In this paper we want to construct a "simple" numerical scheme for conservation laws with a discontinuous coefficient $k(x)$ of bounded variation, i.e., for nonlinear PDEs of the form

$$u_t + f(k(x), u)_x = 0, \qquad (x,t) \in \mathbf{R} \times (0,T), \tag{1.1}$$

where $T > 0$ is a fixed time, $u(x,t)$ is the scalar unknown function that is sought, and the flux function $f(k,u)$ and the coefficient $k(x)$ are given functions. We are particularly interested in the *multiplicative case*

$$f(k,u) = kf(u), \qquad \text{for some function } f(u), \tag{1.2}$$

which occurs frequently in applications. Regarding the nonlinear function $f(u)$, we assume that there exist some finite constants $\underline{u}$, $\overline{u}$, $\underline{f}$, and $\overline{f}$ such that

$$f \in C^2\left[\underline{u}, \overline{u}\right] \text{ with } f\left(\underline{u}\right) = 0,\ f\left(\overline{u}\right) = 0;\ f \text{ genuinely nonlinear}, \tag{1.3}$$

but no convexity condition is assumed. As usual, "f genuinely nonlinear" means that there is no subinterval on which f is linear. Regarding the coefficient $k(x)$, we make the assumption that

$$\underline{k} \le k(x) \le \overline{k} \text{ on } \mathbf{R} \text{ for some constants } \underline{k},\ \overline{k};\ |k(x)| > 0 \text{ a.e. on } \mathbf{R};\ k \in BV(\mathbf{R}). \tag{1.4}$$

Hence the *convection part* of (1.1) depends explicitly on the spatial location through $k(x)$ and this dependency may be discontinuous.

Received by the editor June 5, 2002.
2000 *Mathematics Subject Classification.* Primary 35L65, 35L45, 65M06, 65M12.
Key words and phrases. Conservation law, discontinuous coefficient, relaxation scheme, convergence compensated compactness, numerical example.
This work has been supported by the BeMatA program of the Research Council of Norway.

Under investigation is the Cauchy problem for (1.1), and we specify an initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbf{R}, \tag{1.5}$$

where we assume that the initial function $u_0 \in L^\infty$ satisfies

(1.6) $\underline{u} \le u_0(x) \le \overline{u}$ on $\mathbf{R}$, where the constants $\underline{u}$ and $\overline{u}$ are defined in (1.3).

Nonlinear PDEs of the form (1.1) occur in several applications. We mention here briefly flow in porous media [15], sedimentation processes [5, 13, 14], and traffic flow on a highway [57, 17]. They also arise in radar shape-from-shading problems [44] and as building blocks in numerical methods for Hamilton–Jacobi equations [21] based on dimensional splitting. In view of their applications, there is great demand for accurate, efficient, and, at the same time, easy-to-implement numerical methods for conservation laws with discontinuous coefficients.

Independently of the smoothness of $k(x)$, solutions to (1.1) are in general not smooth and weak solutions must be sought. A *weak solution* is here defined as a function $u \in L^\infty$ which satisfies (1.1) in the sense of distributions, i.e., in $\mathcal{D}'$. When we speak here of a weak solution, we mean that the initial condition is included in the definition of a weak solution when the test function does not vanish at $t = 0$. If $k(x)$ is *smooth*, a weak solution u of (1.1) satisfies the *entropy condition* if for all convex C^2 functions $\eta : \mathbf{R} \to \mathbf{R}$,

$$\eta(u)_t + q(k(x),u)_x + k'(x)\big(\eta'(u) f_k(k(x),u) - q_k(k(x),u)\big) \le 0 \quad \text{in } \mathcal{D}', \tag{1.7}$$

where $q(k(x)u)$ is defined by

$$q_u(k(x),u) = \eta'(u) f_u(k(x),u).$$

We call (η, q) a convex C^2 entropy/entropy-flux pair for (1.1). Provided $f(k,u), k(x)$ are sufficiently smooth functions and $u_0 \in L^\infty$, Kružkov's theory [32] tells us that there exists a unique weak solution to the initial value problem (1.1)-(1.5) which satisfies the entropy condition (1.7).

In the case of a discontinuous coefficient $k(x)$, the notion of entropy solution as well as the accompanying existence and uniqueness theory breaks down. In this case, (1.1) has often been written as a 2×2 system of equations:

$$k_t = 0, \qquad u_t + f(k,u)_x = 0. \tag{1.8}$$

If $u \mapsto f_u(k,u)$ changes sign, then this system is nonstrictly hyperbolic, a situation described as resonance. A dramatic consequence of resonance is that no a priori bound on the spatial total variation of the conserved quantity u is available [50, 53].

Since there is generally no spatial BV (bounded variation) bound for the conserved variable u itself, the *singular mapping* approach has been used up to now as the analytical vehicle for proving convergence of numerical methods. Temple [50] was the first to use this approach when he established convergence of the Glimm scheme for a 2×2 resonant system of conservation laws modeling the displacement of oil in a reservoir by water and polymer, which is now known to be equivalent to a conservation law with a discontinuous coefficient (see, e.g., [30]). More recent convergence results for the 2×2 Glimm method can be found in Hong [18]. Convergence has been established for the 2×2 Godunov method by Lin, Temple, and Wang [36, 37], while the 2×2 front tracking method has been analyzed by Gimse and Risebro [15] and Klingenberg and Risebro [31, 30]. In Bürger et al. [6] (see also [5]), the 2×2 front tracking method is analyzed and applied to a model of continuous sedimentation in ideal clarifier-thickener units. This model consists of a particular conservation law with two discontinuous coefficients.

Regarding uniqueness of weak solutions to (1.1) when $k(x)$ is allowed to be discontinuous, this was first studied in [30], using a variant of Oleĭnik's technique in the case of a multiplicative k-dependence and a convex $f(u)$. Since the solution operator is L^1 contractive if $k(x)$ is sufficiently smooth, this contraction property holds also for solutions that are limits of solutions with smoothed coefficients, hence such limits are unique. This was shown by Klausen and Risebro in [28] for multiplicative/convex flux functions. More recently, L^1-contractivity was shown for piecewise smooth solutions in the case of convex flux functions by Towers [52], and in a more general case by Karlsen, Risebro, and Towers [24]. Finally, Seguin and Vovelle [45] proved uniqueness for L^∞ solutions for a special case of (1.1)-(1.2) with $k(\cdot)$ taking two values separated by a jump discontinuity. The authors of [52, 24, 45] use a Kružkov-type entropy condition.

The 2×2 Glimm, Godunov, and front tracking methods are very accurate since they rely on an exact 2×2 Riemann solver. However, the price to pay for using a 2×2 Riemann solver is that the numerical methods become complicated to implement. As simpler alternatives to these methods, Towers [52, 51] devised appropriate scalar versions of the Godunov and Engquist–Osher methods. He also established convergence of these methods by the singular mapping approach. The work of Towers was extended to strongly degenerate convection-diffusion equations in Karlsen, Risebro, and Towers [22]. For some other recent papers dealing with numerical methods for conservation laws with a discontinuous coefficient (but without rigorous analysis), see Bale, LeVeque, Mitran, and Rossmanith [3].

The purpose of the present paper is to continue the search for "simple" numerical methods for conservation laws with discontinuous coefficients. The starting point herein is to approximate (1.1) by a 2×2 semilinear hyperbolic system with a stiff relaxation term containing the discontinuous flux function $f(k(x), u)$:

$$u_t^\tau + v_x^\tau = 0, \qquad v_t^\tau + a^2 u_x^\tau = \frac{1}{\tau}\left(f(k(x), u^\tau) - v^\tau\right), \tag{1.9}$$

where $\tau > 0$ is the relaxation parameter and a satisfies the so-called subcharacteristic condition due to Whitham [57], Liu [39], and Chen, Levermore, and Liu [9] (see Section 2). Note that the variable v in (1.9) can be eliminated. The result is a conservation law (with a discontinuous coefficient) that has been regularized by a wave operator:

$$u_t^\tau + f(k(x), u^\tau)_x = -\tau\left(u_{tt}^\tau - a^2 u_{xx}^\tau\right).$$

Hence we expect (1.9) to be a first order approximation to (1.1) as $\tau \downarrow 0$. To build a numerical scheme, we now discretize (1.9) by an upwind scheme. The resulting scheme, which is called the relaxation scheme, has the advantage of not relying on a Riemann solver. This a consequence of the special semilinear structure of (1.9). In characteristic variables, (1.9) reduces to a diagonal system which is trivial to discretize with an upwind scheme without resorting to a Riemann solver. The stiff relaxation term is discretized implicitly.

In the k-independent case ($k \equiv 1$), our relaxation scheme reduces to the relaxation scheme first suggested by Jin and Xin [20]. Convergence results for various relaxation systems and relaxation schemes in the k-independent scalar conservation law case can be found in [9, 41, 1, 54, 25, 26, 27, 2, 42, 7, 33, 4, 38, 56, 47, 48].

These papers deal with convergence as well as convergence rates for relaxation approximations, most of them work within the L^1 framework of Kružkov [32] and rely on uniform BV estimates. An exception being the paper by Katsaounis and Makridakis [25], in which error estimates for finite volume relaxation schemes are derived with no uniform BV estimates available.

Various results for hyperbolic systems of conservation laws can be found in [39, 10, 9, 55, 16, 35, 46, 34, 40, 29, 11], see also [58, 19, 12]. Since it is difficult to obtain uniform BV estimates for systems, most of the papers dealing with systems use the compensated compactness method to establish strong convergence of relaxation approximations. Among the papers cited we emphasize those analyzing numerical approximations; namely, the paper by Lattanzio and Serre [35], in which compensated compactness is used to prove convergence of the relaxation scheme for systems of conservation laws (their result does not cover our problem), and the paper by Gosse and Tzavaras [16], in which certain relaxation schemes for the equations of one-dimensional elastodynamics is analyzed using the L^p theory of compensated compactness. We refer to the lecture notes by Natalini [43] for an overview of the *relaxation approach* to hyperbolic problems.

As mentioned before, until now convergence of numerical methods for conservation laws with discontinuous coefficients has been established by the singular mapping approach. Herein we use instead the Murat–Tartar compensated compactness approach [49] to prove convergence of our relaxation approximations. A significant aspect of the compensated compactness method is that it applies to approximate solutions that do not yield entropy solutions (in the sense of Kružkov), which is the case here. As was pointed out in [23], the use of compensated compactness has the notable advantage of being easier to apply than the singular mapping approach when $u \mapsto f(k,u)$ is nonconvex and/or when $k(x)$ changes sign. The case where f is nonconvex has received less attention in the literature than the convex/concave case, which is probably due to additional analytical complexity with the singular mapping approach. An attractive feature of the compensated compactness approach employed herein is that no convexity condition is required for the flux $u \mapsto f(k,u)$. Also, sign changes of $k(x)$ are handled without any special considerations. Sign changes in $k(x)$ are usually ruled out with the singular mapping approach due to added analytical technicalities, see, e.g., [31, 30, 52, 51].

The remaining part of this paper is organized as follows. In Section 2, we present the relaxation scheme. A priori estimates can be found in Section 3, while our main convergence result is proved in Section 4. Finally, we present numerical experiments in Section 5.

2. The relaxation scheme

We are interested in constructing a numerical scheme for the initial value problem (1.1). Inspired by Jin and Xin [20], we consider the relaxation system

$$\begin{cases} u_t^\tau + v_x^\tau = 0, \\ v_t^\tau + a^2 u_x^\tau = \dfrac{1}{\tau}\left(f(k(x),u^\tau) - v^\tau\right), \end{cases} \tag{2.1}$$

where $\tau > 0$ is the relaxation parameter and a satisfies the subcharacteristic condition [57, 39, 9]

$$0 < \max_{k,u} |f_u(k,u)| < a. \tag{2.2}$$

The maximum is taken over the set $(k,u) \in \left[\underline{k}, \overline{k}\right] \times [\underline{u}, \overline{u}]$ (which is specified in Section 1).

To motivate (2.2), we suppose for the moment that u^τ, v^τ, $k(x)$ are smooth functions and make the usual ansatz

$$v^\tau = f\left(k(x), u^\tau\right) + \tau \tilde{v}^\tau + \mathcal{O}\left(\tau^2\right), \tag{2.3}$$

for some function $\tilde{v}^\tau$. This turns the second equation in (2.1) into

$$v_t^\tau + a^2 u_x^\tau = \tilde{v}^\tau + \mathcal{O}(\tau). \tag{2.4}$$

From the first equation in (2.1),

$$u_t^\tau = -f_u\left(k(x), u^\tau\right) u_x^\tau - f_k\left(k(x), u^\tau\right) k'(x) + \mathcal{O}(\tau).$$

From our ansatz (2.3), it then follows that

$$\begin{aligned} v_t^\tau &= f_u\left(k(x), u^\tau\right) u_t^\tau + \mathcal{O}(\tau) \\ &= -\left[f_u\left(k(x), u^\tau\right)\right]^2 u_x^\tau - f_u\left(k(x), u^\tau\right) f_k\left(k(x), u^\tau\right) k'(x) + \mathcal{O}(\tau). \end{aligned}$$

Plugging this into the second equation in (2.1), we find that

$$\tilde{v}^\tau = \left(a^2 - \left[f_u\left(k(x), u^\tau\right)\right]^2\right) u_x^\tau - f_u\left(k(x), u^\tau\right) f_k\left(k(x), u^\tau\right) k'(x) + \mathcal{O}(\tau),$$

and using this in the first equation in (2.1) the final result is, within an $\mathcal{O}\left(\tau^2\right)$ term,

$$\begin{aligned} &u_t^\tau + \Big(f\left(k(x), u^\tau\right) - \tau f_u\left(k(x), u^\tau\right) f_k\left(k(x), u^\tau\right) k'(x)\Big)_x \\ &\qquad = \tau \left(\left(a^2 - \left[f_u\left(k(x), u^\tau\right)\right]^2\right) u_x^\tau\right)_x, \end{aligned} \tag{2.5}$$

which is a first order correction to (1.1). To ensure that this equation is parabolic we need to assume that the subcharacteristic condition (2.2) holds. Observe that (2.5) contains an $\mathcal{O}(\tau)$ *diffusion correction* as well as an $\mathcal{O}(\tau)$ *convection correction.*

For (2.1), we specify the following initial data:

$$u^\tau(x,0) = u_0(x), \qquad v^\tau(x,0) = f(k(x), u_0(x)).$$

In characteristic variables

$$w = u + \frac{v}{a}, \quad z = u - \frac{v}{a} \qquad \Longleftrightarrow \qquad u = \frac{1}{2}(w+z), \quad v = \frac{a}{2}(w-z), \tag{2.6}$$

the system (2.1) simplifies to a diagonal system

$$\begin{cases} w_t^\tau + a w_x^\tau = \dfrac{1}{a\tau}\left(f(k(x), u^\tau) - v^\tau\right) \\ \qquad\qquad\quad = \dfrac{1}{a\tau}\left(f\left(k(x), \frac{1}{2}(w^\tau + z^\tau)\right) - \frac{a}{2}(w^\tau - z^\tau)\right), \\ z_t^\tau - a z_x^\tau = \dfrac{-1}{a\tau}\left(f(k(x), u^\tau) - v^\tau\right) \\ \qquad\qquad\quad = \dfrac{-1}{a\tau}\left(f\left(k(x), \frac{1}{2}(w^\tau + z^\tau)\right) - \frac{a}{2}(w^\tau - z^\tau)\right), \end{cases} \tag{2.7}$$

with

$$w^\tau(x,0) = u_0(x) + \frac{f(k(x), u_0(x))}{a}, \qquad z^\tau(x,0) = u_0(x) - \frac{f(k(x), u_0(x))}{a}.$$

Let $h > 0$ and $\Delta t > 0$ be the spatial and temporal discretization parameters, respectively. The spatial domain $\mathbf{R}$ is discretized into cells

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \qquad j \in \mathbf{Z},$$

where $x_k = kh$ for $k = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$. Similarly, the time interval $(0, T)$ is discretized via $t_n = n\Delta t$ for $n = 0, 1, \dots, N$, where the integer N is chosen such that $N\Delta t = T$, resulting in the time strips

$$(t_n, t_{n+1}), \qquad n = 0, 1, \dots, N-1.$$

We let $u_j^n \approx u^\tau(x_j, t_n)$, $v_j^n \approx v^\tau(x_j, t_n)$, $w_j^n \approx w^\tau(x_j, t_n)$, and $z_j^n \approx z^\tau(x_j, t_n)$, and consider a semi-implicit upwind scheme discretization of (2.7):

$$\begin{cases} \dfrac{1}{\Delta t}\left(w_j^{n+1} - w_j^n\right) + \dfrac{a}{h}\left(w_j^n - w_{j-1}^n\right) = \dfrac{1}{a\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right), \\ \dfrac{1}{\Delta t}\left(z_j^{n+1} - z_j^n\right) - \dfrac{a}{h}\left(z_{j+1}^n - z_j^n\right) = -\dfrac{1}{a\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right). \end{cases} \tag{2.8}$$

In the original variables, this finite difference scheme reads

$$\begin{cases} \dfrac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \dfrac{1}{2h}\left(v_{j+1}^n - v_{j-1}^n\right) - \dfrac{a}{2h}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) = 0, \\ \dfrac{1}{\Delta t}\left(v_j^{n+1} - v_j^n\right) + \dfrac{a^2}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) - \dfrac{a}{2h}\left(v_{j-1}^n - 2v_j^n + v_{j+1}^n\right) \\ \qquad = \dfrac{1}{\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right). \end{cases} \tag{2.9}$$

Note that although this is an implicit method, we do not have to solve a system of equations in order to update u^n and v^n, since we can use the first equation to update u^n, and then the source term is linear in v^{n+1}. We start the iterations (2.8) and (2.9) by defining

$$u_j^0 = \frac{1}{h}\int_{I_j} u_0(x)\,dx, \quad v_j^0 = \frac{1}{h}\int_{I_j} f(k(x), u_0(x))\,dx,$$
$$w_j^0 = u_j^0 + \frac{v_j^0}{a}, \quad z_j^0 = u_j^0 - \frac{v_j^0}{a}, \quad k_j = \frac{1}{h}\int_{I_j} k(x)\,dx.$$

For the difference scheme (2.9), we assume that the following CFL condition holds:

$$a\lambda \le 1, \qquad \lambda = \frac{\Delta t}{h}. \tag{2.10}$$

From now on, it is always understood that the space step h and the time step Δt are comparable, i.e., there are constants $c_1, c_2 > 0$ such that

$$c_1 \le \frac{\Delta t}{h} \le c_2.$$

3. A priori estimates

In this section we derive some a priori estimates (discrete L^∞ and L^2_{loc} estimates) to be used later for the convergence proof of the relaxation scheme. To derive these estimates, we do not actually need the assumption that $f(k, u)$ is of multiplicative form (1.2), and therefore we choose work with the more general form

$$f(k(x), u)$$

and not $k(x)f(u)$. Note that some of the formulas below simplify (slightly) if we replace the "general" $f(k(x),u)$ by $k(x)f(u)$. We replace the first part of (1.3) by the more general condition

$$f(k,\underline{u}) = \underline{f} \text{ and } f(k,\overline{u}) = \overline{f} \text{ for all } k \in \left[\underline{k},\overline{k}\,\right], \tag{3.1}$$

where $\underline{f}$ and $\overline{f}$ are constants (this assumption is used in Lemma 3.1 below), and

$$f(\cdot,u) \in C^1 \text{ on } \left[\underline{k},\overline{k}\,\right] \ \forall u, \qquad f(k,\cdot) \in C^1 \text{ on } [\underline{u},\overline{u}\,] \ \forall k.$$

We continue to assume that $k(x)$ and u_0 satisfy (1.4) and (1.5), respectively. Unless otherwise stated, we always assume that the subcharacteristic condition (2.2) and the CFL condition (2.10) are fulfilled.

In this paper we denote by C a generic constant, not depending on the parameters τ, h. The actual value of C may change from one line to the next during a calculation.

Let the functions $h^{\pm} : \left[\underline{k},\overline{k}\,\right] \times [\underline{u},\overline{u}] \to \mathbf{R}$ be defined by

$$h^{\pm}(k,u) = u \pm \frac{f(k,u)}{a},$$

and define the constants $\underline{h}^{\pm}, \overline{h}^{\pm}$ by

$$\underline{h}^{\pm} = h^{\pm}\left(k,\underline{u}\right) = \underline{u} \pm \underline{f}/a, \qquad \overline{h}^{\pm} = h^{\pm}\left(k,\overline{u}\right) = \overline{u} \pm \overline{f}/a.$$

Next we introduce the sets

$$\mathcal{K}_w = \left[\underline{h}^{+},\overline{h}^{+}\right], \qquad \mathcal{K}_z = \left[\underline{h}^{-},\overline{h}^{-}\right], \qquad \mathcal{K}_{w,z} = \mathcal{K}_w \times \mathcal{K}_z,$$

as well as the sets

$$\mathcal{K}_u = \frac{1}{2}\left(\mathcal{K}_w + \mathcal{K}_z\right) = [\underline{u},\overline{u}], \qquad \mathcal{K}_v = \frac{a}{2}\left(\mathcal{K}_w - \mathcal{K}_z\right), \qquad \mathcal{K}_{u,v} = \mathcal{K}_u \times \mathcal{K}_v.$$

Let (u,v) and (w,z) be related as in (2.6). Then

$$(u,v) \in \mathcal{K}_{u,v} \quad\Longleftrightarrow\quad (w,z) \in \mathcal{K}_{w,z}.$$

We are now ready to state and prove the following invariant region result, which provides us with a uniform L^∞ estimate on the approximate solutions. The proof is similar to that in [35].

Lemma 3.1 (Discrete L^∞ estimate)**.** *Assume that* (2.10) *holds. Then*

$$\left(u_j^0, u_j^0\right) \in \mathcal{K}_{u,v} \quad \forall j \in \mathbf{Z} \quad\Longleftrightarrow\quad \left(w_j^0, z_j^0\right) \in \mathcal{K}_{w,z} \quad \forall j \in \mathbf{Z}$$

implies

$$\left(u_j^n, v_j^n\right) \in \mathcal{K}_{u,v} \quad \forall j \in \mathbf{Z} \quad\Longleftrightarrow\quad \left(w_j^n, z_j^n\right) \in \mathcal{K}_{w,z} \quad \forall j \in \mathbf{Z},$$

for $n = 1,2,\ldots,N$. In particular, there is a constant C, independent of τ and h, such that

$$\left|u_j^n\right| \le C, \qquad \forall (j,n) \in \mathbf{Z} \times \{0,1,\ldots,N\}. \tag{3.2}$$

Proof. We use induction on n, and assume that $(u_j^n, v_j^n) \in \mathcal{K}_{u,v}$ for some $n = 0, 1, \ldots, N-1$ (which is equivalent to $(w_j^n, z_j^n) \in \mathcal{K}_{w,z}$). Using the difference scheme (2.9), we find

$$\begin{aligned} u_j^{n+1} &= (1 - a\lambda)\, u_j^n + \frac{a\lambda}{2}\left(u_{j-1}^n + \frac{v_{j-1}^n}{a} + u_{j+1}^n - \frac{v_{j+1}^n}{a}\right) \\ &= \frac{1}{2}(1 - a\lambda)\left(w_j^n + z_j^n\right) + \frac{a\lambda}{2}\left(w_{j-1}^n + z_{j+1}^n\right). \end{aligned}$$

By the CFL condition (2.10), u_j^{n+1} is a convex combination of points in $\mathcal{K}_u$ (a convex set). Hence u_j^{n+1} is in $\mathcal{K}_u$. Using the functions $h^{\pm}$ we can also write

$$\begin{aligned} \frac{1}{a\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right) &= \frac{1}{\tau} h^+\left(k_j, u_j^{n+1}\right) - \frac{1}{\tau} w_j^{n+1}, \\ -\frac{1}{a\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right) &= \frac{1}{\tau} h^-\left(k_j, u_j^{n+1}\right) - \frac{1}{\tau} z_j^{n+1}, \end{aligned}$$

and thus the characteristic difference scheme (2.8) reads

$$\begin{aligned} w_j^{n+1} &= \left(1 + \frac{\Delta t}{\tau}\right)^{-1}\left((1 - a\lambda) w_j^n + a\lambda w_{j-1}^n + \frac{\Delta t}{\tau} h^+\left(k_j, u_j^{n+1}\right)\right), \\ z_j^{n+1} &= \left(1 + \frac{\Delta t}{\tau}\right)^{-1}\left((1 - a\lambda) z_j^n + a\lambda z_{j+1}^n + \frac{\Delta t}{\tau} h^-\left(k_j, u_j^{n+1}\right)\right). \end{aligned}$$

Since $u_j^{n+1} \in \mathcal{K}_u$, we then have that $h^+\left(k_j, u_j^{n+1}\right) \in \mathcal{K}_w$ and $h^-\left(k_j, u_j^{n+1}\right) \in \mathcal{K}_z$ by definition. By the CFL condition we then have that

$$w_j^{n+1} = \frac{a + \sigma b}{1 + \sigma},$$

where a and b are in $\mathcal{K}_w$ and $\sigma > 0$. Then $a \le w_j^{n+1} \le b$ and thus $w_j^{n+1} \in \mathcal{K}_w$. Similarly we find that $z_j^{n+1} \in K_z$. Consequently,

$$\left(w_j^{n+1}, z_j^{n+1}\right) \in \mathcal{K}_{w,z} \qquad \Longleftrightarrow \qquad \left(u_j^{n+1}, v_j^{n+1}\right) \in \mathcal{K}_{u,v}.$$

This concludes the proof of the lemma. $\square$

If $k(x)$ is *smooth*, a weak solution (u^τ, v^τ) of (2.1) is said to satisfy the entropy condition if

$$\begin{aligned} & E\left(k(x), u^\tau, v^\tau\right)_t + Q\left(k(x), u^\tau, v^\tau\right)_x - Q_k\left(k(x), u^\tau, v^\tau\right) k'(x) \\ & \qquad \le \frac{1}{\tau} E_v\left(k(x), u^\tau, v^\tau\right)\left(f\left(k(x), u^\tau, v^\tau\right) - v^\tau\right) \quad \text{in } \mathcal{D}', \end{aligned} \tag{3.3}$$

for all functions $E, Q : \left[\underline{k}, \overline{k}\right] \times \mathcal{K}_{u,v} \to \mathbf{R}$ satisfying the compatibility conditions

$$Q_u(k, u, v) = a^2 E_v(k, u, v), \qquad Q_v(k, u, v) = E_u(k, u, v). \tag{3.4}$$

We refer to (E, Q) as an entropy/entropy-flux pair for (2.1). In addition, we shall require that the entropy/entropy-flux pair (E, Q) reduces on the equilibrium curve $v = f(k, u)$ to a entropy/entropy-flux pair (η, q) for (1.1).

By the subcharacteristic condition (2.2) we have that $u \mapsto h^{\pm}(k, u)$ are nondecreasing functions on $\mathcal{K}_u$ for each $k \in \left[\underline{k}, \overline{k}\right]$. Hence, we can define their inverses $g^{\pm}(\cdot, \cdot)$ such that

$$g^{\pm}\left(k, h^{\pm}(k, u)\right) = u, \qquad \forall (k, u) \in \left[\underline{k}, \overline{k}\right] \times \mathcal{K}_u.$$

The next lemma tells us how to extend an arbitrary entropy/entropy-flux pair (η, q) for (1.1) to an entropy/entropy-flux pair for (2.1) by viewing (η, q) as an *equilibrium entropy/entropy-flux pair* for (2.1). The idea goes back to Chen, Levermore, and Liu [9, 10], see also [11, 41, 43, 46].

Lemma 3.2 (Entropy/entropy-flux pair for relaxation system)**.** *Let (η, q) be a C^2 entropy/entropy-flux pair for (1.1). Then there exists a C^2 entropy/entropy-flux pair (E, Q) for (2.1), and the functions $E, Q : \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_{u,v} \to \mathbf{R}$ are given explicitly as*

$$(3.5)\qquad \begin{cases} E(k,u,v) = e^{+}\left(k, u + \dfrac{v}{a}\right) + e^{-}\left(k, u - \dfrac{v}{a}\right) = e^{+}(k,w) + e^{-}(k,z), \\ Q(k,u,v) = ae^{+}\left(k, u + \dfrac{v}{a}\right) - ae^{-}\left(k, u - \dfrac{v}{a}\right) = ae^{+}(k,w) - ae^{-}(k,z), \end{cases}$$

where $e^{+} : \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_w \to \mathbf{R}$ and $e^{-} : \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_z \to \mathbf{R}$ take the form

$$(3.6)\qquad \begin{cases} e^{+}(k,w) = \dfrac{1}{2}\left(\eta\left(g^{+}(k,w)\right) + \dfrac{1}{a} q\left(k, g^{+}(k,w)\right)\right), \\ e^{-}(k,z) = \dfrac{1}{2}\left(\eta\left(g^{-}(k,z)\right) - \dfrac{1}{a} q\left(k, g^{-}(k,z)\right)\right). \end{cases}$$

Moreover, the following properties hold:

(a) *$E(k,u,f(u)) = \eta(u)$ and $Q(k,u,f(u)) = q(k,u)$ $\forall u \in \mathcal{K}_u$.*

(b) *$\eta''(u) \geq (>) 0$ $\forall u \in \mathcal{K}_u$ implies*

$$E_{vv}(k,u,v) \geq (>) 0, \qquad \forall (k,u,v) \in \left[\underline{k}, \overline{k}\,\right] \times \in \mathcal{K}_{u,v}.$$

(c) *For all $(k,u,v) \in \left[\underline{k}, \overline{k}\right] \times \mathcal{K}_{u,v}$ and with η convex,*

$$(3.7)\qquad E_v(k,u,v)(f(k,u) - v) \leq 0.$$

(d) *For all $(k,u,v) \in \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_{u,v}$ and with η strictly convex,*

$$(3.8)\qquad E_v(k,u,v)(f(k,u) - v) \leq -\frac{\alpha^2}{2}\left(f(k,u) - v\right)^2,$$

where α only depends on η and f.

(e) *For all $(k,u,v) \in \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_{u,v}$,*

$$(3.9)\qquad |E_v(k,u,v)| \leq C|f(k,u) - v|,$$

for some constant C depending on η and f.

Proof. The proof is more or less the same as in the k-independent case. First, from (3.4) and D'Alemberts formula, we notice that E and Q are solutions of the wave equations

$$E_{uu} - a^2 E_{vv} = 0, \qquad E|_{v=f(k,u)} = \eta(u), \quad E_v|_{v=f(k,u)} = 0,$$

and

$$Q_{uu} - a^2 Q_{vv} = 0, \qquad Q|_{v=f(k,u)} = q(k,u), \quad Q_v|_{v=f(k,u)} = 0.$$

One can easily verify that the functions E and Q as specified in the lemma solve these two Cauchy problems. This explains how the pair (E, Q) is constructed. It is also clear that (a) holds.

Next, we have that

$$
\begin{aligned}
e_w^+(k,w) &= \frac{1}{2}\left(\eta'\left(g^+(k,w)\right) + \frac{1}{a}q_u\left(k, g^+(k,w)\right)\right) g_w^+(k,w) \\
&= \frac{1}{2}\eta'\left(g^+(k,w)\right)\left(1 + \frac{1}{a}f_u\left(k, g^+(k,w)\right)\right)\frac{1}{h_u^+\left(k, g^+(k,w)\right)} \\
&= \frac{1}{2}\eta'\left(g^+(k,w)\right).
\end{aligned}
\tag{3.10}
$$

Similarly, $e_z^-(k,z) = \frac{1}{2}\eta'\left(g^-(k,z)\right)$. Therefore,

$$
\begin{aligned}
E_{vv}(k,u,v) &= \frac{1}{a^2}\left(e_{ww}^+(k,w) + e_{zz}^-(k,z)\right) \\
&\qquad = \frac{1}{a^2}\left(\eta''\left(g^+(k,w)\right)g_w^+(k,w) + \eta''\left(g^-(k,z)\right)g_z^-(k,z)\right),
\end{aligned}
\tag{3.11}
$$

and thus

$$
E_{vv}(k,u,v)\begin{cases} \geq 0, & \eta''(\cdot) \geq 0, \\ > 0, & \eta''(\cdot) > 0. \end{cases}
$$

This proves (b). Since

$$
E_v(k,u,f(k,u)) = 0, \tag{3.12}
$$

we see that

$$
\operatorname{sign}\left(E_v(k,u,v)\right) = -\operatorname{sign}\left(f(k,u) - v\right),
$$

and thus (c) holds. Since (3.12) holds and E is convex in v, $v \mapsto E(k,u,v)$ has a unique minimum for $v = f(k,u)$. Furthermore, whenever $\eta''(\cdot)$ is strictly positive, we can find a constant $\alpha > 0$ such that

$$
\frac{\alpha}{2} \leq \min\left\{E_{vv}(k,u,v) : (k,u,v) \in \left[\underline{k}, \overline{k}\,\right] \times \mathcal{K}_{u,v}\right\}.
$$

Therefore, by (3.12) and the mean value theorem,

$$
\begin{aligned}
E_v(k,u,v)(f(k,u) - v) &= \left(E_v(k,u,v) - E_v(k,u,f(u))\right)\left(f(k,u) - v\right) \\
&\leq -\frac{\alpha^2}{2}(f(k,u) - v)^2.
\end{aligned}
$$

This is the proof of (d). Finally, (e) follows from (3.12). □

We now derive a discrete L^2_{loc} estimate of the difference between v_j^{n+1} and the equilibrium value $f\left(k_j, u_j^{n+1}\right)$. We also derive weak dissipation estimates for w_j^n, z_j^n. To derive the dissipation estimates we need a stronger CFL condition; see (3.18) and (3.23) below.

Lemma 3.3 (Discrete L^2_{loc} estimates)**.** *Assume that the CFL condition* (3.18) *(found in the proof below) holds. Then, for any positive integer J,*

$$
\frac{h\,\Delta t}{\tau}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right)^2 \leq C, \tag{3.13}
$$

$$
h\,\Delta t\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left\{\left(w_j^{n+1} - w_j^n\right)^2 + \left(z_j^{n+1} - z_j^n\right)^2\right\} \leq C\Delta t, \tag{3.14}
$$

$$h\,\Delta t \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left\{ \left(w_j^n - w_{j-1}^n\right)^2 + \left(z_{j+1}^n - z_j^n\right)^2 \right\} \leq Ch, \tag{3.15}$$

for some constant C that is independent of τ, h.

Proof. Let (η, q) be an entropy/entropy-flux pair for (1.1) with η being strictly convex and C^2, and let (E, Q) be the corresponding entropy/entropy-flux pair for (2.1) (see Lemma 3.2).

We shall repeatedly make use of the following simple identity. Let h be a C^2 function on $\mathbf{R}$. Then, for any $b_1, b_2 \in \mathbf{R}$,

$$(b_2 - b_1)h'(b_2) = h(b_2) - h(b_1) + \frac{1}{2}h''(\xi)(b_2 - b_1)^2,$$

for some number ξ between b_1 and b_2.

We start by multiplying the first equation in (2.8) by $e_w^+(k_j, w_j^{n+1})$, obtaining

$$\begin{aligned}
&e^+\left(k_j, w_j^{n+1}\right) - e^+\left(k_j, w_j^n\right) + \frac{1}{2}e_{ww}^+\left(k_j, \bar{w}_j^{n+1/2}\right)\left(w_j^{n+1} - w_j^n\right)^2 \\
&\quad + a\lambda\left(e^+\left(k_j, w_j^n\right) - e^+\left(k_{j-1}, w_{j-1}^n\right)\right) \\
&\quad + \underbrace{a\lambda\left(e^+\left(k_{j-1}, w_{j-1}^n\right) - e^+\left(k_j, w_{j-1}^n\right)\right)}_{A_j^{+,n}} \\
&\quad + \frac{a\lambda}{2}e_{ww}^+\left(k_j, \bar{w}_{j-1/2}^n\right)\left(w_j^n - w_{j-1}^n\right)^2 \\
&\quad + \underbrace{a\lambda\left(e_w^+\left(k_j, w_j^{n+1}\right) - e_w^+\left(k_j, w_j^n\right)\right)\left(w_j^n - w_{j-1}^n\right)}_{B_j^{n+1/2}} \\
&\quad = \frac{\Delta t}{a\tau}e_w^+\left(k_j, w_j^{n+1}\right)\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right),
\end{aligned} \tag{3.16}$$

for some intermediate values $\bar{w}_j^{n+1/2}$ and $\bar{w}_{j-1/2}^n$. For later use we note that

$$\sum_{j \in \mathbf{Z}} \left|A_j^{+,n}\right| \leq C\lambda \left|k\right|_{BV},$$

for some constant C independent of Δt and n. Using the mean value theorem and Cauchy's inequality with ϵ, we have that

$$\begin{aligned}
B_j^{n+1/2} &= a\lambda e_{ww}^+\left(k_j, \tilde{w}_j^{n+1/2}\right)\left(w_j^{n+1} - w_j^n\right)\left(w_j^n - w_{j-1}^n\right) \\
&\geq -a\lambda e_{ww}^+\left(k_j, \tilde{w}_j^{n+1/2}\right)\left[\epsilon\left(w_j^{n+1} - w_j^n\right)^2 + \frac{1}{4\epsilon}\left(w_j^n - w_{j-1}^n\right)^2\right].
\end{aligned}$$

With this and (3.16) we find that

$$\begin{aligned} &e_j^{+,n+1} - e_j^{+,n} + a\lambda\left(e_j^{+,n} - e_{j-1}^{+,n}\right) + A_j^{+,n} \\ &\quad + \left\{\frac{1}{2}e_{ww}^{+}\left(k_j, \bar{w}_j^{n+1/2}\right) - a\lambda\epsilon e_{ww}^{+}\left(k_j, \tilde{w}_j^{n+1/2}\right)\right\}\left(w_j^{n+1} - w_j^n\right)^2 \\ &\quad + a\lambda\left\{\frac{1}{2}e_{ww}^{+}\left(k_j, \bar{w}_j^{n+1/2}\right) - \frac{1}{4\epsilon}e_{ww}^{+}\left(k_j, \tilde{w}_j^{n+1/2}\right)\right\}\left(w_j^{n} - w_{j-1}^n\right)^2 \\ &\qquad \le \frac{\Delta t}{a\tau}e_w^{+}\left(k_j, w_j^{n+1}\right)\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right), \end{aligned}$$

where we have adopted the notation

$$e_j^{+,n} = e^{+}\left(k_j, w_j^n\right), \qquad e_j^{-,n} = e^{-}\left(k_j, z_j^n\right). \tag{3.17}$$

Now by (3.10),

$$e_{ww}^{+}(k,w) = \frac{\eta''(g^{+}(k,w))}{1 + f_u(k, g^{+}(k,w))/a},$$

and thus there exists constants m, M such that

$$0 < m \le e_{ww}^{+}(k,w) \le M, \qquad \forall (k,w) \in \left[\underline{k}, \overline{k}\right] \times \mathcal{K}_w.$$

We can also choose m and M such that

$$0 < m \le e_{zz}^{-}(k,z) \le M, \qquad \forall (k,z) \in \left[\underline{k}, \overline{k}\right] \times \mathcal{K}_z.$$

Next we choose $\epsilon = M/m$, and then

$$\frac{1}{2}e_{ww}^{+}\left(k_j, \bar{w}_j^{n+1/2}\right) - \frac{1}{4\epsilon}e_{ww}^{+}\left(k_j, \tilde{w}_j^{n+1/2}\right) \ge \frac{m}{4}.$$

In order to bound the other quadratic term, we demand that the strengthened CFL condition

$$a\lambda \le \frac{m^2}{4M^2} \tag{3.18}$$

holds; see also (3.23) below. Then

$$\frac{1}{2}e_{ww}^{+}\left(k_j, \bar{w}_j^{n+1/2}\right) - a\lambda\epsilon e_{ww}^{+}\left(k_j, \tilde{w}_j^{n+1/2}\right) \ge \frac{m}{4}.$$

Thus,

$$\begin{aligned} &e_j^{+,n+1} - e_j^{+,n} + a\lambda\left(e_j^{+,n} - e_{j-1}^{+,n}\right) + A_j^{+,n} + \frac{m}{4}\left(w_j^{n+1} - w_j^n\right)^2 \\ &\quad + \frac{ma\lambda}{4}\left(w_j^n - w_{j-1}^n\right)^2 \le \frac{\Delta t}{a\tau}e_w^{+}\left(k_j, w_j^{n+1}\right)\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right). \end{aligned} \tag{3.19}$$

Similarly, multiplying the second equation in (2.8) with $e_z^{-}\left(k_j, z_j^{n+1}\right)$, we show that

$$\begin{aligned} &e_j^{-,n+1} - e_j^{-,n} - a\lambda\left(e_{j+1}^{-,n} - e_j^{-,n}\right) + A_j^{-,n} + \frac{m}{4}\left(z_j^{n+1} - z_j^n\right)^2 + \frac{ma\lambda}{4}\left(z_{j+1}^n - z_j^n\right)^2 \\ &\qquad \le -\frac{\Delta t}{a\tau}e_z^{-}\left(k_j, z_j^{n+1}\right)\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right), \end{aligned} \tag{3.20}$$

where $e_j^{-,n}$ is defined in (3.17) and

$$A_j^{-,n} = a\lambda\left(e^{-}\left(k_{j+1}, z_{j+1}^n\right) - e^{-}\left(k_j, z_{j+1}^n\right)\right).$$

Next, adding (3.19) and (3.20), and rearranging, we find that

(3.21)
$$
\begin{aligned}
&E_j^{n+1} - E_j^n + \lambda\left(Q_{j+1/2}^n - Q_{j-1/2}^n\right) \\
&\quad + \frac{m}{2}\left\{\left(w_j^{n+1} - w_j^n\right)^2 + \left(z_j^{n+1} - z_j^n\right)^2 + a\lambda\left(w_j^n - w_{j-1}^n\right)^2 + a\lambda\left(z_{j+1}^n - z_j^n\right)^2\right\} \\
&\qquad + \frac{\Delta t\alpha^2}{2a\tau}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right)^2 \le \left|A_j^{+,n}\right| + \left|A_j^{-,n}\right|,
\end{aligned}
$$

where we have used the simplifying notation

$$
E_j^n = e_j^{+,n} + e_j^{-,n}, \qquad Q_{j+1/2}^n = ae_j^{+,n} - ae_{j+1}^{-,n}. \tag{3.22}
$$

Since the terms in (3.21) telescope, we can multiply by h and sum over j, n to obtain

$$
\begin{aligned}
&h\sum_{j=-J}^{J} E_j^N + h\frac{m}{2}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left\{\left(w_j^{n+1} - w_j^n\right)^2 + \left(z_j^{n+1} - z_j^n\right)^2\right. \\
&\qquad\qquad\qquad\qquad\qquad\left. + a\lambda\left(w_j^n - w_{j-1}^n\right)^2 + a\lambda\left(z_{j+1}^n - z_j^n\right)^2\right\} \\
&\quad + \frac{\alpha^2}{2a}\frac{h\,\Delta t}{\tau}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(f\left(k_j, u_j^{n+1}\right) - v_j^{n+1}\right)^2 \\
&\qquad\qquad \le CNh\lambda\,|k|_{BV} + h\sum_{j=-J}^{J} E_j^0 + a\Delta t\sum_{n=0}^{N-1}\left(Q_{-J-1/2}^n - Q_{J+1/2}^n\right).
\end{aligned}
$$

Since w_j^n and z_j^n are uniformly bounded, $Q_{j+1/2}^n$ are also uniformly bounded, and since $Nh\lambda = T$ and Lemma 3.1 holds, we obtain (3.13), (3.14), and (3.15). $\square$

Remark 3.1. By choosing $\eta(u) = \frac{u^2}{2}$, we find that

$$
\frac{a}{a + \max|f_u|} \le m \le M \le \frac{a}{a - \max|f_u|},
$$

and thus the CFL condition (3.18) holds if (consult (2.2))

$$
a\lambda \le \frac{1}{4}\left(\frac{a - \max|f_u|}{a + \max|f_u|}\right)^2. \tag{3.23}
$$

Lemma 3.3 with $\eta(u) = \frac{u^2}{2}$ is all that we need for the convergence proof in Section 4. Although the strengthened CFL condition (3.23) is needed for the convergence theory, the usual CFL condition (2.10), which implies stability, is sufficient for the numerical implementation.

4. Convergence

For the convergence analysis, we need to extend the finite difference solutions $\left(w_j^n\right)$ and $\left(z_j^n\right)$ to functions $w^{\tau,h}$ and $z^{\tau,h}$ defined a.e. on $\mathbf{R} \times (0, T)$. Let

$$
w^{\tau,h}(x,t) = w_j^n, \qquad \text{for } (x,t) \in [x_{j-1/2}, x_{j+1/2}) \times [t_n, t_{n+1}) =: \chi_j^n,
$$

and similarly for $z^{\tau,h}(x,t), k^h(x)$. We shall refer to the functions $w^{\tau,h}$ and $z^{\tau,h}$, or, equivalently,

$$
u^{\tau,h} = \frac{1}{2}\left(w^{\tau,h} + z^{\tau,h}\right), \qquad v^{\tau,h} = \frac{a}{2}\left(w^{\tau,h} - z^{\tau,h}\right)
$$

as our approximate solutions.

Let us briefly describe our strategy for proving convergence of the approximate solution $u^{\tau,h}$. Although the entropy condition (1.7) (as well as Kružkov's theory [32]) breaks down when $k(x)$ is discontinuous, the observation in [23] is that one can still establish strong compactness of approximate solutions via the compensated compactness method due to Murat and Tartar [49], at least when $k \in BV$ and $f(k,u)$ has multiplicative form (1.2).

First of all, it turns that the a priori estimates derived in Section 3 imply for any (not necessarily convex) C^2 entropy/entropy-flux pair (η, q) that

$$\eta\left(u^{\tau,h}\right)_t + q\left(k(x), u^{\tau,h}\right)_x = m^{\eta}_{\tau,h} + E^{\eta}_{\tau,h},$$

where $m^{\eta}_{\tau,h} \in \mathcal{M}_{\mathrm{loc}}$ (with no control of the sign of this measure when η is convex) and $E^{\eta}_{\tau,h} \to 0$ in H^{-1}_{loc} as $\tau, h \downarrow 0$. This yields strong H^{-1}_{loc} compactness of the sequence of distributions

$$\left\{\eta\left(u^{\tau,h}\right)_t + q\left(k(x), u^{\tau,h}\right)_x\right\}_{\tau,h>0},$$

for any entropy/entropy-flux pair (η, q). We refer to Lemma 4.2 below for details. Denote by $\nu_{x,t}$ the Young measure corresponding to the sequence of approximate solutions

$$\left\{u^{\tau,h}\right\}_{\tau,h>0} \subset L^{\infty}.$$

When $f(k,u)$ has multiplicative form $kf(u)$ and $k(x)$ is bounded,

$$k(x)f(u) \overset{\star}{\rightharpoonup} k(x)\bar{f} \quad \text{in } L^{\infty}, \qquad \bar{f} = \bar{f}(x,t) = \langle \nu_{x,t}, f\rangle\,,$$

and an application of the div-curl lemma [49] then yields a k-dependent version of the Murat–Tartar commutation relation. Therefore when the multiplicative form (1.2) holds, the commutation relation can be used (as in the k-independent case) to reduce the Young measure $\nu_{x,t}$ to a Dirac measure, which implies the desired strong compactness of $\left\{u^{\tau,h}\right\}_{\tau,h>0}$.

The chain of arguments just sketched leads to a general compensated compactness theorem for conservation laws with a multiplicative discontinuous coefficient.

Theorem 4.1 (Compensated compactness theorem)**.** *Let $\gamma(x)$ be a function in BV (and hence also in L^{∞}). Let $\{u^{\varepsilon}\}_{\varepsilon>0}$ be sequence of functions that belongs to L^{∞} uniformly in ε. Assume that for any C^2 function*

$$\eta : \mathbf{R} \to \mathbf{R},$$

the sequence of distributions

$$\left\{\eta\left(u^{\varepsilon}\right)_t + \left(\gamma(x) q\left(u^{\varepsilon}\right)\right)_x\right\}_{\varepsilon>0} \quad \textit{lies in a compact subset of } H^{-1}_{\mathrm{loc}}, \tag{4.1}$$

where $q : \mathbf{R} \to \mathbf{R}$ is defined by $q'(u) = \eta'(u) f'(u)$. Then along a subsequence

$$\begin{cases} u^{\varepsilon} \overset{\star}{\rightharpoonup} u \quad \textit{in } L^{\infty} \textit{ as } \varepsilon \downarrow 0, \\ \gamma(x) f(u^{\varepsilon}) \overset{\star}{\rightharpoonup} \gamma(x) f(u) \quad \textit{in } L^{\infty} \textit{ as } \varepsilon \downarrow 0. \end{cases} \tag{4.2}$$

Furthermore, if $\gamma(x) \neq 0$ for a.e. x and there is no interval on which $f(u)$ is linear, then a subsequence of $\{u^{\varepsilon}\}_{\varepsilon>0}$ converges a.e. to u.

In the k-independent case, this theorem is due to Tartar [49]. Essentially the same proof carries over to the k-dependent case; see [23] for details. In the course of applying Theorem 4.1 to our approximate solution $u^{\tau,h}$, the following functional analysis lemma (known as Murat's lemma) is useful.

Lemma 4.1 (Murat)**.** *Suppose that* $\left\{\mathcal{L}^{\tau,h}\right\}_{\tau,h>0}$ *is bounded in* $W^{-1,\infty}_{\mathrm{loc}}$*. Suppose also that*

$$\mathcal{L}^{\tau,h} = \mathcal{L}_1^{\tau,h} + \mathcal{L}_2^{\tau,h},$$

where $\left\{\mathcal{L}_1^{\tau,h}\right\}_{\tau,h>0}$ *lies in a compact subset of* H^{-1}_{loc} *and* $\left\{\mathcal{L}_2^{\tau,h}\right\}_{\tau,h>0}$ *lies in a bounded subset of* $\mathcal{M}_{\mathrm{loc}}$*. Then* $\left\{\mathcal{L}^{\tau,h}\right\}_{\tau,h>0}$ *lies in a compact subset of* H^{-1}_{loc}*.*

For a nice overview of the compensated compactness method and its applications to scalar and systems of conservation laws, we refer to the lecture notes by Chen [8].

Before proceeding with the convergence analysis, we note that it follows immediately from (3.2) and (3.13) that the following a priori estimates hold:

$$\left\|u^{\tau,h}\right\|_{L^\infty}, \left\|v^{\tau,h}\right\|_{L^\infty}, \left\|w^{\tau,h}\right\|_{L^\infty}, \left\|z^{\tau,h}\right\|_{L^\infty} \le C_1, \tag{4.3}$$

$$\int_0^T \int_{-L}^{L} \left(f\left(k^h(x), u^{\tau,h}\right) - v^{\tau,h}\right)^2 \, dt\, dx \le C_2 \tau, \qquad L > 0, \tag{4.4}$$

$$\int_0^T \int_{-L}^{L} \left(f\left(k(x), u^{\tau,h}\right) - v^{\tau,h}\right)^2 \, dt\, dx \to 0, \qquad \text{as } \tau, h \downarrow 0, \tag{4.5}$$

for some constants C_1, C_2 that are independent of τ, h but C_2 depends on L. The bounds in (4.3) are immediate. The bound (4.4) holds by Lemma 3.3. To show (4.5) we observe that

$$\begin{aligned}&\left\|f\left(k, u^{\tau,h}\right) - v^{\tau,h}\right\|_{L^2(-L,L)} \\ &\qquad \le \left\|f\left(k^h, u^{\tau,h}\right) - v^{\tau,h}\right\|_{L^2(-L,L)} + C\left\|k - k^h\right\|_{L^2(-L,L)}.\end{aligned}$$

Since k^h is bounded and $k^h \to k$ a.e., Lebesgue's dominated convergence theorem implies that the last term on the right above vanishes when $h \downarrow 0$. The first term is $\mathcal{O}\left(\sqrt{\tau}\right)$ by (4.4).

We now prove the H^{-1}_{loc} compactness for the approximate solution $u^{\tau,h}$. Note that for this lemma to hold, it is not necessary to assume that $f(k,u)$ has multiplicative form (1.2). In fact, this lemma holds under the assumptions used in Section 3.

Lemma 4.2 (H^{-1}_{loc} compactness)**.** *Assume that the strengthened CFL condition* (3.23) *holds. Then the sequence of distributions*

$$\left\{\eta\left(u^{\tau,h}\right)_t + q\left(k(x), u^{\tau,h}\right)_x\right\}_{\tau,h>0} \textit{ lies in a compact subset of } H^{-1}_{\mathrm{loc}},$$

for any C^2 function $\eta : \mathbf{R} \to \mathbf{R}$ and corresponding q defined by

$$q_u(k,u) = \eta'(u) f_u(k,u).$$

Proof. Let us define the distribution $\mathcal{L}^{\tau,h}$ by

$$\left\langle \mathcal{L}^{\tau,h}, \varphi\right\rangle = \int_0^T \int_{\mathbf{R}} \Big(\eta\left(u^{\tau,h}\right)\varphi_t + q\left(k(x), u^{\tau,h}\right)\varphi_x\Big)\, dt\, dx, \qquad \varphi \in \mathcal{D}.$$

The goal is prove that $\left\{\mathcal{L}^{\tau,h}\right\}_{\tau,h>0} \subset \mathcal{D}'$ belongs to a compact subset of H^{-1}_{loc}. To this end, we write

$$\eta\left(u^{\tau,h}\right)_t + q\left(k(x), u^{\tau,h}\right)_x = \mathcal{L}_1^{\tau,h} + \mathcal{L}_2^{\tau,h},$$

where

$$\begin{aligned}
\mathcal{L}_1^{\tau,h} &= \Big(E\left(k(x), u^{\tau,h}, f\left(u^{\tau,h}\right)\right) - E\left(k(x), u^{\tau,h}, v^{\tau,h}\right)\Big)_t \\
&\qquad + \Big(Q\left(k(x), u^{\tau,h}, f\left(u^{\tau,h}\right)\right) - Q\left(k(x), u^{\tau,h}, v^{\tau,h}\right)\Big)_x, \\
\mathcal{L}_2^{\tau,h} &= E\left(k(x), u^{\tau,h}, v^{\tau,h}\right)_t + Q\left(k(x), u^{\tau,h}, v^{\tau,h}\right)_x.
\end{aligned}$$

Clearly, (4.5) implies

$$\left|\left\langle \mathcal{L}_1^{\tau,h}, \varphi\right\rangle\right| \le \mathrm{Const}\left\{\iint_{\mathrm{supp}(\varphi)} \left(f\left(k(x), u^{\tau,h}\right) - v^{\tau,h}\right)^2 \, dx\, dt\right\}^{\frac{1}{2}} \|\varphi\|_{H^1} \to 0$$

as $\tau, h \downarrow 0$, so that $\left\{\mathcal{L}_1^{\tau,h}\right\}_{\tau,h>0}$ is compact in H^{-1}_{loc}. Now we analyze $\mathcal{L}_2^{\tau,h}$. To this end, we first write

$$\begin{aligned}
\mathcal{L}_2^{\tau,h} &= E\left(k^h, u^{\tau,h}, v^{\tau,h}\right)_t + Q\left(k^h, u^{\tau,h}, v^{\tau,h}\right)_x \\
&\qquad + \left(E\left(k, u^{\tau,h}, v^{\tau,h}\right) - E\left(k^h, u^{\tau,h}, v^{\tau,h}\right)\right)_t \\
&\qquad\qquad + \left(Q\left(k, u^{\tau,h}, v^{\tau,h}\right) - Q\left(k^h, u^{\tau,h}, v^{\tau,h}\right)\right)_x \\
&=: \mathcal{L}_{2,1}^{\tau,h} + \mathcal{L}_{2,2}^{\tau,h}.
\end{aligned}$$

As observed above, $k^h \to k$ in L^2. Thus

$$\left|\left\langle \mathcal{L}_{2,2}^{\tau,h}, \varphi\right\rangle\right| \le C \|\varphi\|_{H^1} \left\|k^h - k\right\|_{L^2} \to 0 \quad \text{as } \tau, h \downarrow 0. \tag{4.6}$$

Thus also $\mathcal{L}_{2,2}^{\tau,h}$ belongs to a compact subset of H^{-1}_{loc}.

Recalling that $E = e^+ + e^-$ and $Q = ae^+ - ae^-$,

$$
\begin{aligned}
\left\langle \mathcal{L}^{\tau,h}_{2,1}, \varphi \right\rangle &= -\sum_{n,j} \iint\limits_{\chi^n_j} \Big(E\left(k_j, u^n_j, v^n_j\right) \varphi_t + Q\left(k_j, u^n_j, v^n_j\right) \varphi_x \Big)\, dt\, dx \\
&= -\sum_{n,j} e^{+,n}_j h \int_{t_n}^{t_{n+1}} \varphi_t(x_{j-1/2}, t)\, dt + a e^{+,n}_j \Delta t \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi_x(x, t_{n+1})\, dx \\
&\quad - \sum_{n,j} \Bigg(e^{+,n}_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x} \varphi_{tz}(z,t)\, dz\, dt\, dx \\
&\qquad\qquad + a e^{+,n}_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \int_{t_{n+1}}^{t} \varphi_{xs}(x,s)\, ds\, dt\, dx \Bigg) \\
&\quad - \sum_{n,j} e^{-,n}_j h \int_{t_n}^{t_{n+1}} \varphi_t(x_{j-1/2}, t)\, dt - a e^{-,n}_j \Delta t \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi_x(x, t_{n+1})\, dx \\
&\quad - \sum_{n,j} \Bigg(e^{-,n}_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x} \varphi_{tz}(z,t)\, dz\, dt\, dx \\
&\qquad\qquad + a e^{-,n}_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t_n}^{t_{n+1}} \int_{t_{n+1}}^{t} \varphi_{xs}(x,s)\, ds\, dt\, dx \Bigg),
\end{aligned}
$$

where $e^{-,n}_j$, $e^{+,n}_j$ are defined in (3.17). The four sums after the second equality are denoted by E_1, E_2, E_3, E_4, respectively.

After summation by parts (with $\varphi|_{t=0,T} = 0$),

$$
\begin{aligned}
E_1 + E_3 &= h \sum_{n,j} \varphi^{n+1}_{j-1/2} \left\{ \left(e^{+,n+1}_j - e^{+,n}_j \right) + a\lambda \left(e^{+,n}_j - e^{+,n}_{j-1} \right) \right\} \\
&\qquad + h \sum_{n,j} \varphi^{n+1}_{j-1/2} \left\{ \left(e^{-,n+1}_j - e^{-,n}_j \right) - a\lambda \left(e^{-,n}_{j+1} - e^{-,n}_j \right) \right\}, \\
&= h \sum_{n,j} \varphi^{n+1}_{j-1/2} \left\{ \left(E^{n+1}_j - E^n_j \right) + \lambda \left(Q^n_{j+1/2} - Q^n_{j-1/2} \right) \right\},
\end{aligned}
\tag{4.7}
$$

where $\varphi^n_j = \varphi(x_j, t_n)$ and E^n_j, $Q^n_{j+1/2}$ are defined in (3.22).

If we carefully go through the arguments leading to (3.21), it is not hard to derive the estimate

$$
\begin{aligned}
&\left| \left(E^{n+1}_j - E^n_j \right) + \lambda \left(Q^n_{j+1/2} - Q^n_{j-1/2} \right) \right| \\
&\quad \le \tilde{C} \Big\{ (1 + a\lambda) \left(w^{n+1}_j - w^n_j \right)^2 + (1 + a\lambda) \left(z^{n+1}_j - z^n_j \right)^2 \\
&\qquad\qquad + a\lambda \left(w^n_j - w^n_{j-1} \right)^2 + a\lambda \left(z^n_{j+1} - z^n_j \right)^2 \\
&\qquad\qquad\quad + \left| A^{+,m}_j \right| + \left| A^{-,n}_j \right| + \frac{\Delta t}{a\tau} \left(f\left(k_j, u^{n+1}_j\right) - v^{n+1}_j \right)^2 \Big\},
\end{aligned}
$$

for some constant $\tilde{C}$ independent of τ, h. In what follows, "$\sum$ without n, j" means summation over all n, j such that $(x_j, t_n) \in \operatorname{supp}(\varphi)$. Using Lemma 3.3, we thus get

$$
|E_1 + E_3| \le \|\varphi\|_{L^\infty} h \sum \left| \left(E^{n+1}_j - E^n_j \right) + \lambda \left(Q^n_{j+1/2} - Q^n_{j-1/2} \right) \right| \le C \|\varphi\|_{L^\infty},
$$

for some constant that is independent of τ, h but dependent on $\mathrm{supp}(\varphi)$.

To estimate E_2, we first perform summation by parts:

$$E_2 = \underbrace{\sum \left(e_j^{+,n+1} - e_j^{+,n}\right) \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\varphi(x, t_{n+1}) - \varphi(x_{j-1/2}, t_{n+1})\right) \, dx}_{E_{2,1}}$$

$$+ \underbrace{\sum a \left(e_j^{+,n} - e_{j-1}^{+,n}\right) \int_{t_n}^{t_{n+1}} \left(\varphi(x_{j-1/2}, t) - \varphi(x_{j-1/2}, t_{n+1})\right) \, dt}_{E_{2,2}}.$$

To estimate $E_{2,1}$ and $E_{2,2}$, we shall need the following (easy to derive) inequality. Let h be a H^1 function on $(b_1, b_2) \subset \mathbf{R}$. Then

$$\left| \int_{b_1}^{b_2} (h(z) - h(b)) \, dz \right| \leq \frac{1}{2}(b_2 - b_1)^2 \left\| h' \right\|_{L^2(b_1, b_2)}, \qquad \forall b \in [b_1, b_2].$$

Applying this inequality, we get

$$|E_{2,1}| \leq \frac{h^2}{2} \sum_n \left\| \varphi_x(\cdot, t_{n+1}) \right\|_{L^2} \left| e_j^{+,n+1} - e_j^{+,n} \right|.$$

Next, by a Taylor expansion,

$$e_j^{+,n+1} = e_j^{+,n} + e_w^+ \left(k_j, \bar{w}_j^{n+1/2}\right) \left(w_j^{n+1} - w_j^n\right),$$

where the quantity with a $\bar{\cdot}$ indicates an intermediate value, we find (using also $h = \mathcal{O}(\Delta t)$)

$$|E_{2,1}| \leq \tilde{C} \left\{ h \, \Delta t \sum \left\| \varphi_x(\cdot, t_{n+1}) \right\|_{L^2}^2 \right\}^{\frac{1}{2}} \left\{ h \, \Delta t \sum \left(w_j^{n+1} - w_j^n\right)^2 \right\}^{\frac{1}{2}}.$$

Appealing to (3.14), the final result is

$$|E_{2,1}| \leq C \left\{ \Delta t \sum_n \left\| \varphi_x(\cdot, t_{n+1}) \right\|_{L^2}^2 \right\}^{\frac{1}{2}} \sqrt{h} \to 0 \quad \text{as } \tau, h \downarrow 0,$$

for some constant C that is independent of τ, h but dependent on $\mathrm{supp}(\varphi)$. The term

$$\Delta t \sum_n \left\| \varphi_x(\cdot, t_{n+1}) \right\|_{L^2}^2$$

converges to $\|\varphi_x\|_{L^2(\mathbf{R} \times [0,T])}$ and is bounded uniformly in Δt and τ, since $\varphi \in \mathcal{D}$. Similarly,

$$|E_{2,2}| \leq \frac{a \Delta t^2}{2} \sum \left\| \varphi_t(x_{j-1/2}, \cdot) \right\|_{L^2} \left| e_j^{+,n} - e_{j-1}^{+,n} \right|.$$

After a Taylor expansion,

$$e_{j-1}^{+,n} = e_j^{+,n} - e_w^+ \left(k_j, \bar{w}_{j-1/2}^n\right) \left(w_j^n - w_{j-1}^n\right) + e_k^+ \left(\bar{k}_{j-1/2}, w_{j-1}^n\right) \left(k_j - k_{j-1}\right),$$

where again quantities with a $\bar{\cdot}$ indicate intermediate values, we get (using also $\Delta t = \mathcal{O}(h)$),

$$
\begin{aligned}
|E_{2,2}| &\leq \tilde{C}\left\{h\,\Delta t\sum\left\|\varphi_t(x_{j-1/2},\cdot)\right\|_{L^2}^2\right\}^{\frac{1}{2}} \\
&\qquad\times\left(\left\{h\,\Delta t\sum\left(w_j^n - w_{j-1}^n\right)^2\right\}^{\frac{1}{2}} + h\,\Delta t\sum|k_j - k_{j-1}|\right) \\
&\leq C\left\{h\sum_j\left\|\varphi_t(x_{j-1/2},\cdot)\right\|_{L^2}^2\right\}^{\frac{1}{2}}\left(\sqrt{h}+h\right) \to 0 \quad \text{as } \tau, h \downarrow 0,
\end{aligned}
$$

for some constant C that is independent of τ, h but dependent on $\operatorname{supp}(\varphi)$, $|k|_{BV}$, and T. To get the last estimate we used (3.15).

Summing up, we have now shown that the sequence of distributions $\left\{\mathcal{L}^{\tau,h}\right\}_{\tau,h>0}$ is the sum of many terms, either of which is compact in H^{-1}_{loc} or else bounded in $\mathcal{M}_{\text{loc}}$. In addition, the L^∞ estimate in (4.4) implies that $\left\{\mathcal{L}^{\tau,h}\right\}_{\tau,h>0}$ belongs to a bounded subset of $W^{-1,\infty}_{\text{loc}}$. Hence, the proof of the lemma is now finished by appealing to Murat's lemma (Lemma 4.1). $\square$

We are now ready to state and prove our main convergence theorem for the approximate solution $u^{\tau,h}$, which applies to the *multiplicative case* (1.2) since we need to use the compensated compactness theorem (Theorem 4.1).

Theorem 4.2 (Convergence)**.** *Assume that* (1.6), (1.2), (1.3), *and* (1.4) *hold. As soon as the subcharacteristic condition* (2.2) *and the strengthened CFL condition* (3.23) *are fulfilled, passing if necessary to a subsequence, we have*

$$u^{\tau,h} \to u \quad \text{in } L^p_{\text{loc}} \text{ as } \tau, h \downarrow 0, \quad \text{for any } p < \infty,$$

and u is a weak solution u of the Cauchy problem (1.1)-(1.5).

Proof. The strong L^p_{loc}-convergence of $\left\{u^{\tau,h}\right\}$ to a function $u \in L^\infty$ follows immediately from Lemma 4.2 and Theorem 4.1. It remains to prove that u is a weak solution, i.e., that the limit function u satisfies

$$\int_0^T\int_{\mathbf{R}}\left(u\varphi_t + k(x)f(u)\varphi_x\right)dx\,dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\,dx = 0,$$

for all $\varphi \in \mathcal{D}$ with $\varphi|_{t=T} \equiv 0$. Fix $X > 0$ such that φ vanishes for $|x| \geq X = Jh$. Multiplying the first difference equation in (2.9) by $\Delta t h\,\varphi_j^n = \Delta t h\,\varphi(x_j, t^n)$ and then summing by parts, we get

$$
\begin{aligned}
&h\,\Delta t\sum_{n=1}^N\sum_{j=-J}^J u_j^n\frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t} + h\sum_{j=-J}^J u_j^0\varphi_j^0 + h\,\Delta t\sum_{n,j} v_j^n\frac{\varphi_{j+1}^n - \varphi_{j-1}^n}{2h} \\
&\quad= \frac{a\Delta t}{2}\sum_{n=1}^N\sum_{j=-J}^J\left(u_{j+1}^n - u_j^n\right)\left(\varphi_{j+1}^n - \varphi_j^n\right) \\
&\quad= \underbrace{\frac{a\Delta t}{4}\sum_{n,j}\left\{\left(w_{j+1}^n - w_j^n\right) + \left(z_{j+1}^n - z_j^n\right)\right\}\left(\varphi_{j+1}^n - \varphi_j^n\right)}_{E(h)},
\end{aligned}
$$

where $Jh = X$ and $N\Delta t = T$. Therefore in the standard way we have that

$$\int_0^T \int_{\mathbf{R}} \left(u^{\tau,h}\varphi_t + v^{\tau,h}\varphi_x \right) dx\, dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\, dx = \mathcal{O}\left(\Delta t + h\right) + E(h).$$

Using that $u^{\tau,h} \to u$ and $v^{\tau,h} \to k(x)f(u)$ strongly as $\tau, h \downarrow 0$ (see (4.5)), we get

$$\int_0^T \int_{\mathbf{R}} \left(u\varphi_t + k(x)f(u)\varphi_x \right) dx\, dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\, dx = \lim_{h\downarrow 0} E(h).$$

By Cauchy's inequality

$$\begin{aligned}\left(E(h)\right)^2 &\leq \frac{a}{2}\Delta t^2 \sum \left\{ \left(w_{j+1}^n - w_j^n\right)^2 + \left(z_{j+1}^n - z_j^n\right)^2 \right\} h^2 \sum \left(\frac{\varphi_{j+1}^n - \varphi_j^n}{h} \right)^2 \\ &\leq Ch,\end{aligned}$$

where we have used (3.15) to get the final estimate, so that C is a constant that depends on X but not τ, h. Hence $\lim_{h\downarrow 0} E(h) = 0$, which concludes the proof of the theorem. □

Remark 4.1. One is interested in the behavior of the relaxation scheme (2.9) as the relaxation parameter τ tends to 0. Jin and Xin [20] refer to the limit scheme obtained in this fashion as the relaxed scheme (what we refer to as a "relaxation scheme" they refer to as a "relaxing scheme"). In view of the required quality of the numerical scheme in the under-resolved regime $\tau \ll \Delta t$, a reasonable criteria is that the relaxed scheme should be a stable and consistent discretization of the limit conservation law with a discontinuous coefficient (1.1). Thanks to (3.13), we see that the relaxed scheme takes the form

$$\left\{ \begin{aligned} & v_j^n = k_j f\left(u_j^n\right), \\ & \frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \frac{1}{2h}\left(v_{j+1}^n - v_{j-1}^n\right) - \frac{a}{2h}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = 0. \end{aligned} \right. \tag{4.8}$$

For example, if we specify $a\frac{\Delta t}{h} \equiv 1$ in Lemma 3.3, then (4.8) takes the form

$$\frac{1}{\Delta t}\left(u_j^{n+1} - \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) \right) + \frac{1}{2h}\left(k_{j+1} f\left(u_{j+1}^n\right) - k_{j-1} f\left(u_{j-1}^n\right) \right) = 0, \tag{4.9}$$

which is the scalar Lax–Lax-Friedrichs scheme for (1.1)-(1.2). If we do not specify $a\frac{\Delta t}{h} \equiv 1$, then the relaxed scheme coincides with the scalar generalized Lax–Friedrichs scheme with numerical viscosity $Q = 1$ replaced by $Q = a\frac{\Delta t}{h}$. Let us note that convergence of the Lax–Friedrichs scheme (4.9) is open and does not follow from our results since we need the strengthened CFL condition (3.23) to hold. The Lax–Friedrichs scheme (4.9) will be analyzed elsewhere.

5. Numerical results

In this section we present two numerical examples calculated by the relaxation scheme presented here, as well as a second order MUSCL modification [20]. In both examples we use the flux function

$$f(k,u) = ku(1-u),$$

and $\tau = 10^{-11}$. The first example is a Riemann problem with initial data given by

$$u(x,0) = 0.15, \qquad k(x) = \begin{cases} 3 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{5.1}$$

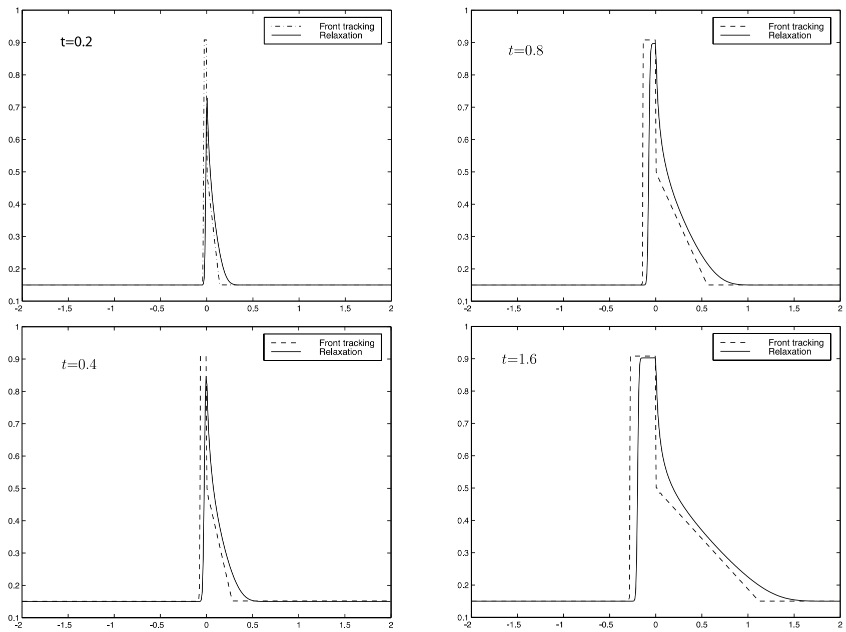


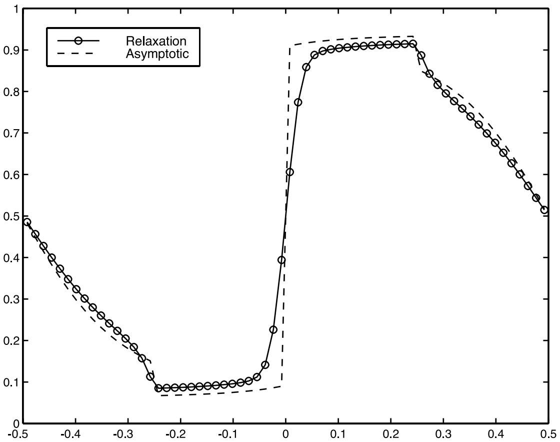
FIGURE 1. The relaxation solution at various times for the initial data (5.1).

The correct solution of this Riemann problem consists of a shock moving to the left separating u values $(0.15, 0.933)$, then a stationary discontinuity at $x = 0$ separating u values $(0.933, 0.5)$, and finally a rarefaction wave with positive speed from $u = 0.5$ to $u = 0.15$. To compare the approximate solution, we use the front tracking method described in [31]. This method is "almost exact" on Riemann problems. In Figure 1 we show the relaxation approximation and the front tracking approximation using 400 grid cells in the interval $[-2, 2]$, i.e., $h = 1/100$. We used $\Delta t = 0.95\|k\|_\infty h$. We see from the simulations that the relaxation method produces a shock with too little speed. Nevertheless, further experiments reveal that the method converges nicely to the correct solution.

The second example has periodic initial data given in the interval $[-1/2, 1/2]$, with

$$u(x,0) = 0.5, \qquad k(x) = \cos^2(\pi x) + 1 + 8\,|x|\,\chi_{[-1/4,1/4]}(x). \tag{5.2}$$

In this case one can calculate the asymptotic solution as $t \to \infty$ analytically [31], and we have calculated the approximate solution at $t = 5.0$. At this time the approximate solution is almost stationary, and therefore we have used the asymptotic solution as a reference solution to compute L^1 errors. In Figure 2 we show the relaxation approximation using $h = 1/64$ and $\Delta t = 0.95\,\|k\|_{L^\infty}\,h$, at $t = 5.0$, as well as the asymptotic solution to the conservation law. We have also implemented a second order MUSCL method (as reported in [20]) and calculated L^1 relative errors for two versions of this method; second order in space and first order in time, and second order in both space and time. These are given in the right part of Figure 2. In this table the first column gives the number of grid cells, and the remaining columns give the relative errors produced by the first order method, the second



N	1	3/2	2
16	20.55	8.95	6.51
32	11.71	5.50	5.04
64	6.83	1.55	2.13
128	3.86	0.71	0.76
256	2.09	0.54	0.46
512	1.09	0.29	0.16
1024	0.56	0.16	0.08

FIGURE 2. The relaxation approximation and the asymptotic solution (left), and the relative L^1 errors for (5.2).

order in space and first order in time method, and the full second order method, respectively. The relative error is defined by

$$e = \frac{\sum_j \left|u_\infty(x_j) - u_j^M\right|}{\sum_j \left|u_\infty(x_j)\right|},$$

and the second, third, and fourth columns show 100 times this number. We see that the numerical convergence rates of all these methods are the same, although the higher order methods produce consistently smaller errors.

Summing up, the relaxation schemes seem to give acceptable results for conservation laws with a discontinuous coefficient. Their main advantage compared with the front tracking method (or any other numerical method based on a 2×2 Riemann solver) is that the (first and high order) relaxation schemes are very easy to implement.

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