Relaxation-projection schemes, the ultimate approximate Riemann solvers

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Abstract A relaxation approach pioneered by Frédérique Coquel leads to very stable approximate Riemann solvers with positivity preserving properties. We outline this approach and and give examples of its usefulness in practical applications.

1 Introduction

A classical method to numerically solve hyperbolic conservation laws is the finite volume method, which involves as its essential ingredient the numerical solution of a Riemann problem. Phil Roe noticed in 1981 [12] that an approximation of the Riemann solver suffices. This led to a quest for finding particularly useful approximate Riemann solvers.

Such an approximate Riemann solver was developed in Paris by Frédérique Coquel and co-workers around the turn of the century, see e.g. [4], [1]. It is inspired by an idea of Shi Jin and Zhou-ping Xin [10], where the solutions to a system of conservation laws are approximated by a straightforward relaxation system. The ensuing French idea was two-fold:

- find a particularly clever relaxation system that approximates a given system of conservation laws
- translate this into a numerical scheme by first solving the left hand side of the relaxation system (a linear transport and thus numerically easy to do) and then projecting the thus found solution to the equilibrium variables (again easy).

We shall show how this leads to approximate Riemann solvers with good properties, like stability and entropy consistency, which implies positivity of density and

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temperature for the shallow water- (see [15]), Euler- (see [5, 13, 14, 16]) and the ideal magnetohydrodynamics equations (see [2, 3, 17]).

2 Approximate Riemann solversg

In the classical Godunov numerical algorithm to solve one-dimensional systems of hyperbolic conservation laws

$$u_t + f(u)_x = 0 \quad ,$$

(see e.g. [11]), the exact solution to the Riemann problem was originally solved numerically by Godunov in the 1950s via an iterative method. Phil Roe noticed in 1981 [12] that for this algorithm to be successful it is enough to approximate the Riemann solution by an approximate Riemann solver consisting of jumps separating constant states. Roe's approach can be made quite accurate, but suffers from not necessarily satisfying a discrete entropy inequality. Thus soon after approximate Riemann solvers were introduced of so called HLL type [8] that were entropy stable but not quite accurate. A way forward was then shown by Coquel and co-workers (see e.g. [4]) from the late 1990s on, who developed approximate Riemann solvers that were inspired by a relaxation approach (see [10]).

To illustrate the underlying principle, consider a scalar conservation law $u_t + f(u)_x = 0$ which gets approximated à la Jin-Xin [10] by

$$u_t^\epsilon + v^\epsilon = 0 \tag{1}$$

$$v_t^\epsilon + a^2 u_x^\epsilon = \frac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon) \quad . \tag{2}$$

This gets solved numerically by a particular splitting approach. First solve the homogeneous, linearly degenerate equation

$$\begin{aligned} u_t^\epsilon + v^\epsilon &= 0 \\ v_t^\epsilon + a^2 u_x^\epsilon &= 0 \quad , \end{aligned}$$

and in the second so-called **projection** step, solve

$$\begin{aligned} u_t^\epsilon &= 0 \\ v_t^\epsilon &= \frac{1}{\epsilon}(f(u^\epsilon) - v^\epsilon) \quad , \end{aligned}$$

by setting ϵ to zero.

Notice that this gives rise to an approximate Riemann solver for $u_t + f(u)_x = 0$, which turns out to be equivalent to the Lax-Friedrichs solver. Also note that this splitting procedure (transport - projection) is nothing but an analytic tool to determine the speed of the jumps and the intermediate constant states of the approximate Reimann solver for the original equation $u_t + f(u)_x = 0$. In the numerical algorithm never is there an ϵ present.

In order to prove $lim_{\epsilon\to 0} u^\epsilon = u$ (where u^ϵ is the solution to (1), (2), and u is the solution to $u_t + f(u)_x = 0$) one needs to impose $-a < f'(u) < a$, the so called stability condition (see [10]). This condition ensures the entropy dissipative property of the associated approximate Riemann solver. Thus one can show that the relaxation-projection solver satisfies the discrete entropy inequality, which is at the core of their extreme usefulness also in more general cases.

Historically the first relaxation-projection solver of this type for the compressible Euler equations is what Coquel called the Siliciu solver. It is described in [1], section 2.4.4. Note that this way we obtain an approximate Riemann solver of HLLC type (three waves with constant states in between) that gives rise to a finite volume solver which satisfies the discrete entropy inequality. From this one deduces that this finite volume scheme gives positive density and internal energy. Thus one has found an approximate Riemann solver which can be made both quite accurate and stable.

As an aside it is interesting to note that in [6] it was found that even though the Siliciu solver from the previous paragraph is positivity preserving for density and internal energy, it will not maintain all invariant domains of the Euler equations. This means, given a set of initial data in phase space for the Euler equations, consider the solution set in phase space. Does the finite volume scheme based on the approximate Riemann solver obtained by the Siliciu relaxation always stay in this solution set? It is shown in [6] that even though this property is true for the set positive density and internal energy, in general this will not be the case.

3 Applications

We give a few examples of approximate Riemann solvers of relaxation-projection type, that have proven themselves quite useful in applications because of their accuracy and good stability property. Note that for each case there is no systematic procedure for finding the relaxation system analogous to (1), (2). This is a bit of an art, in particular proving the crucial stability estimates (the estimates corresponding to $-a < f'(u) < a$ in the previous section) may turn out to be a lengthy calculation. Nonetheless, as the next examples show, it is worth the effort because they lead to approximate Riemann solvers that are very useful in practice.

We begin with the Euler equations of compressible gas dynamics, and seek a relaxation-projection approximate Riemann solver that is able to work for all Mach numbers, in particular in the limit of the Mach number going to zero, where the incompressible Euler equations are reached. This was achieved in [14, 16] by splitting the pressure in to two parts and and then finding a relaxation system that relaxes these two different parts separately.

Next we consider the Euler equations with gravity and seek a finite volume solver that maintains hydrostatic equilibria, which are stationary solutions with the velocity set to zero. These are called well-balanced numerical methods. In [5, 13] the relaxation system ensures that the discrete form of the hydrostatic equilibrium is always maintained.

For the equations of ideal magnetohydrodynamics a relaxation-projection solver has been found in [2, 3, 17]. The relaxation system takes its inspiration from the Siliciu solver for the Euler equations. In particular it has been possible to extend the positivity preserving property to second order in multiple space dimensions on a cartesian mesh. The solonoidal property of the magnetic field is taken into account by writing the equations in symmetrizable form via a Powell term, and all of this is achieved while still maintaining the positivity preserving property.

4 Conclusion

The stable property of these solvers has been put to test in astrophysical codes. To name only one, the magnetohydrodynamic relaxation solver [2, 3, 17] was introduced into the FLASH code, giving rise to simulations (see e.g. [9]) where flows with Mach number 100 could be simulated in a stable way. That these solvers may be useful also in a discontinuous Galerkin context can bee seen e.g. in [7].

These relaxation-projection solvers deserve to be better known, which we hope to have shown in this overview.

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