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Invariant region on a non-isentropic gas dynamics system

Yong Chen [a], Christian Klingenberg [b], Yun-guang Lu [a,*], Xianting Wang [c], Guoqiao You [d]

[a] *School of Mathematics, Hangzhou Normal University, China*
[b] *Department of Mathematics, Wuerzburg University, Germany*
[c] *Wuxi Institute of Technology, China*
[d] *School of Mathematics, Nanjing Audit University, China*

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ABSTRACT

In this paper, we found a special non-isentropic gas dynamics system, whose invariant region is the opposite of the corresponding isentropic case. This shows that the powerful invariant region theory introduced by Chueh, Conley and Smoller for general hyperbolic system of two conservation laws cannot be obviously applied to obtain the a priori L^∞ estimates for systems of more than two equations.

1. Introduction

For general hyperbolic system of conservation laws, a powerful method to obtain the a priori L^∞ estimates is the invariant region theory introduced by Chueh, Conley and Smoller [1] in 1977. However, this method is mainly valid for the following conservation systems of two equations

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \tag{1.1}$$

where u and v are in R (See [2] for more results about the invariant region theory on 2×2 systems of conservation laws). After that, many people tried to apply this technique to obtain the L^∞ estimates for systems of more than two equations, but did not obtain the obvious progress. An open question is whether the invariant region theory is still feasible for large systems, in which, the number of equations is more than two.

* Corresponding author.
E-mail address: ylu2005@ustc.edu.cn (Y.-g. Lu).

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In this paper, we consider the following full Euler system of gas dynamics, with a special equation of state: $P(\rho, s) = e^s e^{-\frac{1}{\rho}}$, ρ, P and s denoting the mass density, the pressure and the specific entropy, in which the temperature θ, the specific internal energy ε are given by $\theta = \varepsilon = P(\rho, s)$

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + e^s e^{-\frac{1}{\rho}})_x = 0 \\ (\frac{1}{2}\rho u^2 + \rho e^s e^{-\frac{1}{\rho}})_t + (u(\frac{1}{2}\rho u^2 + (\rho+1)e^s e^{-\frac{1}{\rho}}))_x = 0. \end{cases} \tag{1.2}$$

System (1.2) is interesting because it is the unique diagonalizable system we can find in the family of full non-isentropic gas dynamics systems [2,3]. The smooth solution for the Cauchy problem of system (1.2) with suitable smooth, monotonic initial data is studied by Zhu in [3] (See also [4]). More results about the diagonalizable hyperbolic systems can be found in [5].

In this paper, we study the invariant region of system (1.2).

The corresponding isentropic case ($s = 0$) of system (1.2) is as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{1.3}$$

where ρ is the density of gas, u the velocity and the pressure $P = P(\rho) = e^{-\frac{1}{\rho}}$.

Numerous papers deal with the analysis of weak solutions of the Cauchy problem (1.2). The first existence theorem for large initial data of locally finite total variation was proved in [6] for $\gamma = 1$ and in [7] for $\gamma \in (1, 1+\delta)$ in Lagrangian coordinates, where δ is small. The Glimm scheme [8] was used in these papers.

The ideas of compensated compactness developed in [9,10] were used in [11] to established a global existence theorem for the Cauchy problem (1.3) with large initial data for $\gamma = 1 + \frac{2}{N}$, where $N \geq 5$ odd, with the use of the viscosity method. The convergence of the Lax–Friedrichs scheme and the existence of a global solution in L^∞ for large initial data with adiabatic exponent $\gamma \in (1, \frac{5}{3}]$ were proved in [12,13]. In [14], the global existence of a weak solution was proved for $\gamma \geq 3$ with the use of the kinetic setting in combination with the compensated compactness method. The method in [14] was finally improved in [15] to fill the gap $\gamma \in (\frac{5}{3}, 3)$, and a new proof of the existence of a global solution for all $\gamma > 1$ was given there.

By simple calculations, two eigenvalues of system (1.3) are

$$\lambda_1 = \frac{m}{\rho} - \sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \sqrt{P'(\rho)}, \tag{1.4}$$

where $m = \rho u$ denotes the momentum, with corresponding two Riemann invariants

$$z(\rho, m) = \int_0^\rho \frac{\sqrt{P'(\tau)}}{\tau} d\tau - \frac{m}{\rho}, \quad z(\rho, m) = \int_0^\rho \frac{\sqrt{P'(\tau)}}{\tau} d\tau + \frac{m}{\rho}. \tag{1.5}$$

Consider the related parabolic system

$$\begin{cases} \rho_t + m_x = \varepsilon \rho_{xx} \\ m_t + (\frac{m^2}{\rho} + P(\rho))_x = \varepsilon m_{xx}. \end{cases} \tag{1.6}$$

We multiply (1.6) by (w_ρ, w_m) and (z_ρ, z_m), respectively, to obtain

$$w_t + \lambda_2 w_x = \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho} \rho_x w_x - \tfrac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2 \tag{1.7}$$

and

$$z_t + \lambda_1 z_x = \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho} \rho_x z_x - \tfrac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2. \tag{1.8}$$

Then the assumption $P(\rho) = e^{-\frac{1}{\rho}}$ yields

$$w_t + \lambda_2 w_x \le \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x \tag{1.9}$$

and

$$z_t + \lambda_1 z_x \le \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x. \tag{1.10}$$

If we consider (1.9) and (1.10) as inequalities about the variables w and z, then we can get the estimates $w(\rho^\varepsilon, m^\varepsilon) \le M, z(\rho^\varepsilon, m^\varepsilon) \le M$ by applying the maximum principle to (1.9) and (1.10). This shows the following conclusion. The region

$$\Sigma = \{(\rho^\varepsilon, m^\varepsilon) : w(\rho^\varepsilon, m^\varepsilon) \le M,\ z(\rho^\varepsilon, m^\varepsilon) \le M\} \tag{1.11}$$

is an invariant region [16].

However, we shall verify that the corresponding Riemann invariants for the non-isentropic gas dynamics system (1.2) have the opposite estimates.

Theorem 1. *The viscosity approximate solutions $(\rho^\varepsilon, m^\varepsilon, s^\varepsilon)$, of system (1.2), given by the parabolic system (2.5), satisfy the following estimates*

$$\Sigma = \{(\rho^\varepsilon, m^\varepsilon, s^\varepsilon) : w_1(\rho^\varepsilon, m^\varepsilon, s^\varepsilon) \ge M_1,\ w_2(\rho^\varepsilon, m^\varepsilon, s^\varepsilon) \ge M_2,\ s^\varepsilon \ge M_3, \} \tag{1.12}$$

where the functions w_1, w_2 and s are the Riemann invariants of system (1.2):

$$w_1 = 2e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} - \frac{m}{\rho}, \quad w_2 = 2e^{\frac{s}{2}} e^{-\frac{1}{2\rho}} + \frac{m}{\rho}, \tag{1.13}$$

and $M_i, i = 1, 2, 3$ are constants.

This shows that the powerful invariant region theory introduced by Chueh, Conley and Smoller for general hyperbolic system of two conservation laws cannot be obviously applied to obtain the a priori L^∞ estimates for systems of more than two equations.

Remark 1. A positive invariant region $s \ge M_3$ for general full Euler system of gas dynamics is established in Theorem 8.2.2 of [2].

Remark 2. In the unpublished paper [17], the authors are able to carry out Compensated Compactness for a 2×2 system without assuming L^∞ bounds by making rather restrictive assumptions on the nonlinearity of the momentum equations. This contrasts with our result pertaining to systems of equations with more than two equations.

We shall prove Theorem 1 in the next section.

2. Proof of Theorem 1

Substituting the first equation in (1.2) into the second, and substituting the first, the second equations in (1.2) into the third, respectively, we obtain, for the smooth solution, the following equivalent system about the variables (ρ, u, s),

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + \frac{1}{\rho^3} e^{s - \frac{1}{\rho}} \rho_x + u u_x + \frac{1}{\rho} e^{s - \frac{1}{\rho}} s_x = 0 \\ s_t + u s_x = 0. \end{cases} \tag{2.1}$$

Let the matrix $A(U)$ be

$$A(U)=\begin{pmatrix} u & \rho & 0 \\ \frac{1}{\rho^3}e^{s-\frac{1}{\rho}} & u & \frac{1}{\rho}e^{s-\frac{1}{\rho}} \\ 0 & 0 & u \end{pmatrix}. \tag{2.2}$$

Then three eigenvalues of (2.1) are

$$\lambda_1=u-\frac{1}{\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}},\quad \lambda_2=u+\frac{1}{\rho}e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}\quad \lambda_3=u \tag{2.3}$$

with corresponding three Riemann invariants

$$z=w_1=2e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}-u,\quad w=w_2=2e^{\frac{s}{2}}e^{-\frac{1}{2\rho}}+u,\quad w_3=s. \tag{2.4}$$

We consider the following parabolic system of (1.2)

$$\begin{cases} \rho_t+(\rho u)_x=\varepsilon\rho_{xx} \\ (\rho u)_t+(\rho u^2+e^se^{-\frac{1}{\rho}})_x=\varepsilon(\rho u)_{xx} \\ (\frac{1}{2}\rho u^2+\rho e^se^{-\frac{1}{\rho}})_t+(u(\frac{1}{2}\rho u^2+(\rho+1)e^se^{-\frac{1}{\rho}}))_x=\varepsilon(\frac{1}{2}\rho u^2+\rho e^se^{-\frac{1}{\rho}})_{xx}. \end{cases} \tag{2.5}$$

Substituting the first equation in (2.5) into the second, we may obtain the following equation about the variable u

$$u_t+uu_x+\tfrac{1}{\rho^3}e^{s-\frac{1}{\rho}}\rho_x+\tfrac{1}{\rho}e^{s-\frac{1}{\rho}}s_x=\varepsilon u_{xx}+2\varepsilon\tfrac{\rho_x}{\rho}u_x. \tag{2.6}$$

We may rewrite the third equation in (2.5) as follows

$$\begin{aligned} &\rho e^{s-\frac{1}{\rho}}(s_t+us_x)+(\tfrac{1}{2}u^2+(1+\tfrac{1}{\rho})e^{s-\frac{1}{\rho}})(\rho_t+\rho u_x+u\rho_x) \\ &+\rho u(u_t+uu_x+\tfrac{1}{\rho^3}e^{s-\frac{1}{\rho}}\rho_x+\tfrac{1}{\rho}e^{s-\frac{1}{\rho}}s_x)=\varepsilon(\tfrac{1}{2}\rho u^2+\rho e^se^{-\frac{1}{\rho}})_{xx} \\ &=\varepsilon\rho uu_{xx}+2u\rho_xu_x+\rho u_x^2+\tfrac{1}{2}u^2\rho_{xx}+(1+\tfrac{1}{\rho})e^{s-\frac{1}{\rho}}\rho_{xx} \\ &+\tfrac{1}{\rho^3}e^{s-\frac{1}{\rho}}\rho_x^2+2(1+\tfrac{1}{\rho})e^{s-\frac{1}{\rho}}\rho_xs_x+\tfrac{1}{\rho}e^{s-\frac{1}{\rho}}s_x^2+\rho e^{s-\frac{1}{\rho}}s_{xx}. \end{aligned} \tag{2.7}$$

Thus we have from the first equation in (2.5), (2.6) and (2.7) that

$$\begin{aligned} s_t+us_x&=\varepsilon s_{xx}+\varepsilon(e^{\frac{1}{\rho}-s}u_x^2+\tfrac{1}{\rho^4}\rho_x^2+2\tfrac{1+\rho}{\rho^2}\rho_xs_x+s_x^2) \\ &\geq \varepsilon s_{xx}+2\varepsilon\tfrac{\rho_x}{\rho}s_x. \end{aligned} \tag{2.8}$$

By simple calculations,

$$\begin{cases} w_\rho=\frac{1}{\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}},\quad w_u=1,\quad w_s=e^{\frac{s}{2}-\frac{1}{2\rho}}, \\ w_{\rho\rho}=-\frac{2}{\rho^3}e^{\frac{s}{2}-\frac{1}{2\rho}}+\frac{1}{2\rho^4}e^{\frac{s}{2}-\frac{1}{2\rho}},\quad w_{\rho u}=0,\quad w_{\rho s}=\frac{1}{2\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}}, \\ w_{uu}=0,\quad w_{su}=0,\quad w_{ss}=\frac{1}{2}e^{\frac{s}{2}-\frac{1}{2\rho}}, \\ z_\rho=\frac{1}{\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}},\quad z_u=-1,\quad z_s=e^{\frac{s}{2}-\frac{1}{2\rho}}, \\ z_{\rho\rho}=-\frac{2}{\rho^3}e^{\frac{s}{2}-\frac{1}{2\rho}}+\frac{1}{2\rho^4}e^{\frac{s}{2}-\frac{1}{2\rho}},\quad z_{\rho u}=0,\quad z_{\rho s}=\frac{1}{2\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}}, \\ z_{uu}=0,\quad z_{su}=0,\quad z_{ss}=\frac{1}{2}e^{\frac{s}{2}-\frac{1}{2\rho}}. \end{cases} \tag{2.9}$$

Now, we multiply the first equation in (2.5) by w_ρ, (2.6) by w_u, (2.8) by w_s respectively, and add the results to obtain an equality, whose left-hand side L and the right R are respectively

$$\begin{aligned} L&=w_t+\tfrac{1}{\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}}(\rho u_x+u\rho_x)+(uu_x+\tfrac{1}{\rho^3}e^{s-\frac{1}{\rho}}\rho_x+\tfrac{1}{\rho}e^{s-\frac{1}{\rho}}s_x)+e^{\frac{s}{2}-\frac{1}{2\rho}}us_x \\ &=w_t+(u+\tfrac{1}{\rho}e^{\frac{s}{2}-\frac{1}{2\rho}})(\tfrac{1}{\rho^2}e^{\frac{s}{2}-\frac{1}{2\rho}}\rho_x+u_x+e^{\frac{s}{2}-\frac{1}{2\rho}}s_x)=w_t+\lambda_2w_x \end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
R &= \varepsilon w_\rho \rho_{xx} + w_u(\varepsilon u_{xx} + 2\varepsilon \tfrac{\rho_x}{\rho} u_x) + w_s(\varepsilon s_{xx} + \varepsilon(e^{\frac{1}{\rho}-s} u_x^2 + \tfrac{1}{\rho^4}\rho_x^2 + 2\tfrac{1+\rho}{\rho^2}\rho_x s_x + s_x^2)) \\
&= \varepsilon w_{xx} - \varepsilon w_{\rho x}\rho_x - \varepsilon w_{ux} u_x - \varepsilon w_{sx} s_x + 2\varepsilon \tfrac{\rho_x}{\rho} u_x + \varepsilon w_s (e^{\frac{1}{\rho}-s} u_x^2 + \tfrac{1}{\rho^4}\rho_x^2 + 2\tfrac{1+\rho}{\rho^2}\rho_x s_x + s_x^2) \\
&= \varepsilon w_{xx} - \varepsilon[(\tfrac{1}{2\rho^4} - \tfrac{2}{\rho^3}) e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x^2 + \tfrac{1}{\rho^2} e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x s_x + \tfrac{1}{2} e^{\frac{s}{2}-\frac{1}{2\rho}} s_x^2 - 2\tfrac{\rho_x}{\rho} u_x \\
&\quad - e^{\frac{s}{2}-\frac{1}{2\rho}} (e^{\frac{1}{\rho}-s} u_x^2 + \tfrac{1}{\rho^4}\rho_x^2 + 2\tfrac{1+\rho}{\rho^2}\rho_x s_x + s_x^2)] \\
&= \varepsilon w_{xx} + 2\varepsilon \tfrac{\rho_x}{\rho} w_x + \varepsilon(\tfrac{1}{2\rho^4} e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x^2 + \tfrac{1}{\rho^2} e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x s_x + \tfrac{1}{2} e^{\frac{s}{2}-\frac{1}{2\rho}} s_x^2 + e^{\frac{1}{2\rho}-\frac{s}{2}} u_x^2) \\
&\geq \varepsilon w_{xx} + 2\varepsilon \tfrac{\rho_x}{\rho} w_x.
\end{aligned}
\tag{2.11}
$$

From (2.10) and (2.11), we have

$$
w_t + \lambda_2 w_x \geq \varepsilon w_{xx} + 2\varepsilon \frac{\rho_x}{\rho} w_x. \tag{2.12}
$$

Similarly, we multiply the first equation in (2.5) by z_ρ, (2.6) by z_u, (2.8) by z_s respectively, and add the results to obtain the following equality

$$
\begin{aligned}
& z_t + \lambda_1 z_x = \varepsilon z_{xx} + 2\varepsilon \tfrac{\rho_x}{\rho} z_x \\
& + \varepsilon(\tfrac{1}{2\rho^4} e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x^2 + \tfrac{1}{\rho^2} e^{\frac{s}{2}-\frac{1}{2\rho}} \rho_x s_x + \tfrac{1}{2} e^{\frac{s}{2}-\frac{1}{2\rho}} s_x^2 + e^{\frac{1}{2\rho}-\frac{s}{2}} u_x^2) \geq \varepsilon z_{xx} + 2\varepsilon \tfrac{\rho_x}{\rho} z_x.
\end{aligned}
\tag{2.13}
$$

Therefore we obtain the lower bounds of $w(\rho^\varepsilon, u^\varepsilon, s^\varepsilon) \geq M_2, z(\rho^\varepsilon, u^\varepsilon, s^\varepsilon) \geq M_1$ and $s(\rho^\varepsilon, u^\varepsilon, s^\varepsilon) \geq M_3$ by using the maximum principle to (2.12), (2.8) and (2.13) if we assume that the initial data have the same bounds, which are the opposite of the isentropic case given in (1.11). So, we obtain the proof of Theorem 1.

Remark 3. It seems that we can see more clearly the genuine reason why the domains $w, z \geq M$ are positively invariant as suggested by the anonymous reviewer. We denote Q for either w or z, and λ the corresponding velocity. The use of the artificial viscosity in (2.5) yields the transport-diffusion equation

$$
Q_t + \lambda Q_x = \varepsilon dQ \cdot U_{xx} = \varepsilon Q_{xx} - \varepsilon D^2 Q(U_x, U_x), \tag{2.14}
$$

where $U = (\rho, \rho u, \frac{1}{2}\rho u^2 + \rho e^s e^{-\frac{1}{\rho}})^T$. Therefore (2.11) or (2.13) amounts to proving that

$$
r D^2 Q(\xi, \xi) + 2dr \cdot \xi dQ \cdot \xi \leq 0, \ \forall \ \xi \in R^3, \tag{2.15}
$$

where r is a coordinate of U. The details refer to [1,2].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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