Commun. Math. Phys. 187, 327 – 340 (1997)

Communications in
Mathematical Physics
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Existence of Solutions to Hyperbolic Conservation Laws with a Source

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Received: 19 July 1995 / Accepted: 3 December 1996

Abstract: Existence of solutions to three different systems of Eqs. (1.1), (1.2) and (1.3) coming from physically relevant models is shown, each needing a different proof which are given in Sects. 2, 3 and 4. The unifying theme is the presence of source terms and the general method of proof is vanishing viscosity together with compensated compactness. For system (1.2) entropy-entropy flux pairs of Lax type are constructed and estimates from singular perturbation theory of ODEs are used. For (1.1) and (1.3) weak entropy-entropy flux pairs are constructed following the compensated compactness framework set up by Diperna [4].

1. Introduction

In this paper we consider the Cauchy problem for an extended model of isothermal flow [15]:

$$\left.\begin{aligned} \rho_t + (\rho u)_x + h(x,t)\rho u &= 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + h(x,t)\rho u^2 &= 0 \end{aligned}\right\}, \tag{1.1}$$

where ρ is the density, u the velocity, $p = p(\rho)$ is the pressure. For the special case $h(x,t) = \dfrac{a'(x)}{a(x)}$ the function a (which depends on x only) represents the cross-sectional area of a variable duct.

We also consider the related system

$$\left.\begin{aligned} (a\rho)_t + (a\rho u)_x &= 0 \\ u_t + (u^2 + \int_0^\rho \frac{p'(s)}{s} ds)_x &= 0 \end{aligned}\right\}, \tag{1.2}$$

where $a = a(x)$. For $a(x) = 1$, system (1.2) was first derived by S. Earnshaw [5] for isentropic flow (see also [22], p. 168).

Systems (1.1) for $h(x,t) = \dfrac{a'(x)}{a(x)}$ and (1.2) are of interest because resonance occurs. This means there is a coincidence of wave speeds from different families of waves. To see this, we may augment either system by the equation

$$a_t = 0.$$

Thus one wave of the augmented system has zero wave speed $\lambda_0 = 0$. One of the other waves may have coinciding wave speed in the transsonic regime:

$$\lambda_k(\tilde{U}) = \lambda_0 \quad with \quad \nabla\lambda_k \cdot R_k \mid_{\tilde{U}} \neq 0.$$

(Here R_k is an eigenvector and $\tilde{U} = (\tilde{a}, \tilde{u}, \tilde{\rho}\tilde{u})$, thus we may have non-linear resonance.)

Finally, we consider an extended river flow equation [22]

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + a(x)\rho + c\rho u|u| &= 0 \end{aligned}\right\} . \tag{1.3}$$

System (1.3) also appears in the paper [12], where the function $a(x)$ corresponds physically to the slope of the topography and $c\rho|u|$ to a friction term.

T.P. Liu studied existence and qualitative behaviour of solutions for near constant data to resonant systems of this type by using Glimm's random choice method [15, 16]. In [7] Glimm, Marshall and Plohr solved (1.1) numerically by using the random choice method. In [6] Embid, Goodman and Majda considered steady state solutions to varying channel flow. Recently in [8], Isaacson and Temple solved the Riemann problem for a general inhomogeneous system of this type. Our interest in studying this resonant problem is motivated by their papers.

Remark 1. After announcing Theorem 1 and 2 at the Conference on Hyperbolic Problems in Stony Brook in 1994 (as set down in our preprint [10]), we learned that independently Chen and Glimm had arrived at a theorem related to our Theorem 2, namely existence of solutions to (1.1) with $h(x,t) = \dfrac{a'(x)}{a(x)}$ (see their preprint [2]). Their method of proof involves approximating the solution with a Godunov scheme which incorporates the steady state solutions. In our proof of Theorem 2 the viscosity method is used. Here the main difficulty was to find upper bound estimates $(\rho^\epsilon, u^\epsilon)$ and positive lower bound estimates $\rho^\epsilon \geq c(t,\epsilon) > 0$ for this viscous approximation.

Our Theorems 1 and 3 are not touched in [2].

Theorem 1 (Existence of Solutions to (1.2)). *Let*
1.$a(x) \geq c > 0$ *are bounded and nondecreasing (or nonincreasing) on* **R**, *where* c *is a constant.*
2. $w_0(x) \leq M$, $z_0(x) \geq 0$ *(or* $w_0(x) \leq 0$, $z_0(x) \geq -M$*), where*

$$w = u + \int_0^\rho \frac{\sqrt{p'(s)}}{s} ds \qquad z = u - \int_0^\rho \frac{\sqrt{p'(s)}}{s} ds \tag{1.4}$$

are two Riemann invariants of the systems (1.1), (1.2), (1.3). We use $w_0(x) = w|_{t=0}$, $z_0(x) = z|_{t=0}$.

3. $g(\rho) := \frac{\sqrt{p'(\rho)}}{\rho} \in C^2(0,\infty)$, $g \geq d > 0$, $g' \geq 0$ *for* $\rho \geq 0$, *where* d *is a positive constant.*

Then the Cauchy problem (1.2) with bounded initial data $(\rho, u)|_{t=0} = (\rho_0(x), u_0(x))$, $(\rho_0(x) \geq 0)$ *has a global weak solution in the sense of distributions.*

Remark 2. For the special choice of $p(\rho) = \int^{\rho} s^2(\rho+\delta)^{\gamma-3}ds$ $(\gamma > 3, \delta > 0)$ (see [17]), assumption 3 in Theorem 1 is satisfied.

Theorem 2 (Existence of Solutions to (1.1)). *Let*
1. h(x,t) be continuous, bounded in $\mathbf{R}\times\mathbf{R}^+$, $h(x,t) \geq 0$ *(or* $h(x,t) \leq 0$*) and* $|h(x,t)| \leq \alpha(1+t)^{-1}$, *where* α *is a positive constant depending on the bound of the initial data.*
2.$w_0(x) \leq M$, $z_0(x) \geq 0$ *(or* $w_0(x) \leq 0$, $z_0(x) \geq -M$*);* $\rho_0(x) \geq 0$.
3.$p(\rho) = \rho^{\gamma}$, $1 < \gamma < \frac{5}{3}$.
Then the Cauchy problem (1.1) with bounded initial data $(\rho, \rho u)|_{t=0} = \big(\rho_0(x), \rho_0(x) u_0(x)\big)$ *(where* $\rho_0(x) \geq 0$*) has a global weak solution.*

Remark 3. : Assumption 2 in Theorem 2 without brackets refers to supersonic flow, with brackets to subsonic flow.

Theorem 3 (Existence of Solutions to (1.3)). *Let*
1.$|a(x)| \leq M$, $a'(x) \geq 0$ *, and let c be a nonnegative constant.*
2.$w_0(x) \leq M$, $z_0(x) \geq -M$, $\rho_0(x) \geq 0$.
3.$p(\rho) = \dfrac{\rho^{\gamma}}{\gamma}$, $1 < \gamma \leq \frac{5}{3}$.
Then the Cauchy problem (1.3) with bounded initial data $(\rho, \rho u)|_{t=0} = \big(\rho_0(x),\ \rho_0(x) u_0(x)\big)$ *(where* $\rho_0(x) \geq 0$*) has a global weak solution.*

Remark 4. All the existence Theorems 1, 2, 3 above include the vacuum case $\rho = 0$.

2. Proof of Theorem 1

We rewrite system (1.2) in the form:

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= -\frac{a'(x)}{a(x)}\rho u \\ u_t + \left(\frac{u^2}{2} + \int_0^{\rho} s g^2(s) ds\right)_x &= 0 \end{aligned}\right\} \tag{2.1}$$

with g given in assumption 3 of Theorem 1. First smoothing $a(x)$ by a mollifier, we get $a^{\delta}(x) \in C^{\infty}(R)$, $c \leq a^{\delta}(x) \leq M$, $0 \leq \delta(a^{\delta}(x))' \leq M$ (or $-M \leq \delta(a^{\delta}(x))' \leq 0$) by assumption 1 in Theorem 1, and $a^{\delta}(x) \to a(x)$, a.e. in any compact set of R as $\delta \to 0$. Adding viscosity terms to the right-hand side of the system (2.1) yields the following parabolic system:

$$\left.\begin{aligned} {\rho^{\epsilon}}_t + (\rho^{\epsilon} u^{\epsilon})_x &= -\frac{(a^{\delta}(x))'}{a^{\delta}(x)}\rho^{\epsilon} u^{\epsilon} + \epsilon {\rho^{\epsilon}}_{xx} \\ {u^{\epsilon}}_t + (\tfrac{(u^{\epsilon})^2}{2} + f(\rho^{\epsilon}))_x &= \epsilon {u^{\epsilon}}_{xx} \end{aligned}\right\} \tag{2.2}$$

with the initial data

$$(\rho^\epsilon, u^\epsilon)_{t=0} = (\rho_0(x), u_0(x)), \tag{2.3}$$

where $f(\rho) = \int_0^\rho sg^2(s)ds$.

Lemma 1. *If the conditions in Theorem 1 are satisfied and the solutions $(\rho^\epsilon, u^\epsilon)$ of the Cauchy problem (2.2) , (2.3) exist in $R \times [0, T]$, then $(\rho^\epsilon, u^\epsilon)$ satisfy the following estimates:*

$$w(\rho^\epsilon, u^\epsilon) \leq M \quad , \quad z(\rho^\epsilon, u^\epsilon) \geq 0 \quad , \quad \rho^\epsilon \geq 0 \tag{2.4}$$

or

$$w(\rho^\epsilon, u^\epsilon) \leq 0 \quad , \quad z(\rho^\epsilon, u^\epsilon) \geq -M \quad , \quad \rho^\epsilon \geq 0, \tag{2.5}$$

where w and z are the Riemann invariants.

Proof. If $(a^\delta)'(x) \geq 0$, we have from (2.2),

$$\begin{aligned}
w_t + \left(u^\epsilon + \sqrt{p'(\rho^\varepsilon)}\right) w_x &= \varepsilon w_{xx} - \varepsilon g'(\rho^\epsilon)(\rho^\epsilon)_x{}^2 - \frac{(a^\delta)'}{a^\delta} u^\epsilon \sqrt{p'(\rho^\varepsilon)} \\
&\leq \epsilon w_{xx} - \frac{(a^\delta)'}{a^\delta} \sqrt{p'(\rho^\varepsilon)} z - \frac{(a^\delta)'}{a^\delta} \sqrt{p'(\rho^\varepsilon)} \int_0^{\rho^\epsilon} g(s)ds, \\
z_t + \left(u^\epsilon - \sqrt{p'(\rho^\varepsilon)}\right) z_x &= \epsilon z_{xx} + \epsilon g'(\rho^\epsilon)(\rho^\epsilon)_x{}^2 + \frac{(a^\delta)'}{a^\delta} u^\epsilon \sqrt{p'(\rho^\varepsilon)} \\
&\geq \epsilon z_{xx} + \frac{(a^\delta)'}{a^\delta} \sqrt{p'(\rho^\varepsilon)} z + \frac{(a^\delta)'}{a^\delta} \sqrt{p'(\rho^\varepsilon)} \int_0^{\rho^\epsilon} g(s)ds \\
&\geq \epsilon z_{xx} + \frac{(a^\delta)'}{a^\delta} \sqrt{p'(\rho^\varepsilon)} z.
\end{aligned} \tag{2.6}$$

So the second equation in (2.6) and the maximum principle [18] give the estimate $z \geq 0$ first and then we have from the first equation in (2.6) that

$$w_t + \left(u^\epsilon + \sqrt{p'(\rho^\varepsilon)}\right) w_x \quad \leq \quad \epsilon w_{xx}, \tag{2.7}$$

so applying the maximum principle again to (2.7), we obtain $w \leq M$. $\rho^\epsilon \geq 0$ is trivial by the first equation in (2.2). □

Similarily we can get the estimates (2.5) if $(a^\delta)'(x) \leq 0$, thus ending the proof of Lemma 1.

Using the general contraction mapping principle and the estimate (2.4) or (2.5) we have

Lemma 2. *If the conditions in Theorem 1 are satisfied, then for any fixed $\epsilon > 0$, the Cauchy problem (2.2), (2.3) has a unique global solution $(\rho^\epsilon, u^\epsilon)$, satisfying*

$$0 \leq \int_0^{\rho^\epsilon} \sqrt{p'(s)} s ds \leq u^\epsilon \leq M$$

or

$$-M \leq u^\epsilon \leq -\int_0^{\rho^\epsilon} \sqrt{p'(s)} s ds \leq 0.$$

Remark 5. Dropping the zero$^{\text{th}}$ order term from the right hand side of (2.1) yields the system of conservation laws

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ u_t + \left(u^2 + \int_0^\rho s g^2(s) ds\right)_x &= 0 \end{aligned}\right\} . \tag{2.8}$$

Noticing the system (2.8) has a strictly convex entropy

$$\eta = \frac{1}{2}u^2 + \int_0^\rho \int_0^y g^2(s) ds dy,$$

we deduce that
$\sqrt{\epsilon}\partial_x \rho^\epsilon$, $\sqrt{\epsilon}\partial_x u^\epsilon$ are uniformly bounded in $L^2_{loc}(R \times R^+)$.

From Remark 5 and the boundedness of $(\rho^\epsilon, u^\epsilon)$, we have

Lemma 3. *For any C^2 entropy-entropy flux pair $(\eta(\rho, u), q(\rho, u))$ of the system (2.8)*
$\eta(\rho^\epsilon, u^\epsilon)_t + q(\rho^\epsilon, u^\epsilon)_x$ *is compact in* $H^{-1}_{loc}(R \times R^+)$
with respect to the vicosity solutions $(\rho^\epsilon, u^\epsilon)$ of the Cauchy problem (2.2), (2.3).

Lemma 3 guarantees the commutativity relation for the representing measure to be true. We are going to construct the entropy-entropy flux pairs of Lax type and give their desired estimates by using the theory of singular perturbations of ordinary differential Eqs. [19].

We recall that a pair (η, q) of real-valued maps is an entropy-entropy flux pair of (2.8) if all smooth solutions of (2.8) satisfy

$$\big(u\eta_\rho + f'(\rho)\eta_u, \rho\eta_\rho + u\eta_u\big) = (q_\rho, q_u) \quad . \tag{2.9}$$

Eliminating the q from (2.9), we have

$$\eta_{\rho\rho} = \frac{f'(\rho)}{\rho}\eta_{uu} \quad . \tag{2.10}$$

We choose special Lax entropies

$$\eta_k^1 = e^{kw(u,\rho)}\big(\alpha(\rho) + \frac{\beta(\rho, k)}{k}\big),$$

(here $k \in N$ and w is defined in (1.4)) and α and β are to be determined below. Now substitute η_k^1 into (2.10),

$$k[2g(\rho)\alpha' + g'(\rho)\alpha] + \alpha'' + sg(\rho)\beta' + g'(\rho)\beta + \frac{\beta''}{k} = 0.$$

Since this should hold for $k \in N$, we have

$$2g(\rho)\alpha' + g'(\rho)\alpha = 0, \tag{2.11}$$

$$\alpha'' + 2g(\rho)\beta' + g'(\rho)\beta + \frac{\beta''}{k} = 0, \tag{2.12}$$

then

$$\alpha = g^{-\frac{1}{2}}.$$

The existence of b and the uniformly bounded estimates of β and β' w.r.t. k can be obtained by (2.11), (2.12) and the following Lemma [21], also referred to in [9], p. 114.

Lemma 4. *Let $Y(x) \in C^2[0,h]$ be solutions of the equation*

$$F(x, Y, Y') = 0,$$

and let functions $f(x,y,\zeta,\lambda)$, $F(x,Y,Y')$ be continuous on the regions $0 \le x \le h$, $|y - Y(x)| \le l(x)$, $|\zeta - Y'(x)| \le m(x)$ for some positive functions $l(x)$ and $m(x)$ and $\lambda_o > \lambda > 0$. In addition

$$\begin{aligned} |f(x,y,\zeta,\lambda) - F(x,y,\zeta)| &\le \epsilon \\ |F(x,y_2,\zeta) - F(x,y_1,\zeta)| &\le M|y_2 - y_1|, \\ \frac{F(x,y,\zeta_2) - F(x,y,\zeta_1)}{\zeta_2 - \zeta_1} &\ge L \end{aligned}$$

for some positive constants ϵ, M, L. If $y = y(x,\lambda)$ is a solution of the following ordinary differential equation of second order:

$$\lambda y'' + f(x, y, y', \lambda) = 0$$

with $y(0) = Y(0)$ and $y'(0)$ being arbitrary, then for sufficiently small $\lambda > 0$, $\epsilon > 0$ and $P = |y'(0) - Y'(0)|$, $y(x)$ exists for all $0 \le x \le h$ and satisfies

$$|y(x,\lambda) - Y(x)| < \big[\frac{\epsilon}{M} + \lambda(\frac{P}{L} + \frac{N}{M})\big] e^{\frac{Mx}{L}},$$

where $N = max|Y''(x)|$ for $0 \le x \le h$.

Using (2.9), we have

$$\begin{aligned} q_\rho &= f'(\rho)\eta_u + \eta_\rho, \\ q_u &= \rho\eta_\rho + u\rho_u. \end{aligned}$$

Then a progressing wave of the system (2.9) is provided by

$$\begin{aligned} \eta_k^1 &= e^{kw}(\alpha(\rho) + \frac{\beta(\rho,k)}{k}), \\ q_k^1 &= (u + \sqrt{p'(\rho)})\eta_k^1 + e^{kw}\big(\frac{\rho\alpha' - \alpha}{k} + \frac{\rho\beta' - \beta}{k^2}\big). \end{aligned}$$

In a similar way we can obtain the other entropy-entropy pairs of Lax type as follows:

$$\begin{aligned} \eta_{-k}^2 &= e^{-kw}(\alpha(\rho) + \frac{\beta_1(\rho,k)}{k}), \\ q_{-k}^2 &= (u + \sqrt{p'(\rho)})\eta_{-k}^2 + e^{-kw}\big(\frac{\alpha - \rho\alpha'}{k} + \frac{\beta_1 - \rho\beta_1'}{k^2}\big), \\ \eta_k^3 &= e^{kz}(\alpha(\rho) + \frac{\beta_2(\rho,k)}{k}), \\ q_k^3 &= (u - \sqrt{p'(\rho)})\eta_k^3 + e^{kz}\big(\frac{\rho\alpha' - \alpha}{k} + \frac{\rho\beta_2' - \beta_2}{k^2}\big), \\ \eta_{-k}^4 &= e^{-kz}(\alpha(\rho) + \frac{\beta_3(\rho,k)}{k}), \\ q_{-k}^4 &= (u - \sqrt{p'(\rho)})\eta_{-k}^4 + e^{-kz}\big(\frac{\alpha - \rho\alpha'}{k} + \frac{\beta_3 - \rho\beta_3'}{k^2}\big), \end{aligned}$$

where β_i (i=1,2,3) satisfy respectively

$$\alpha'' - 2g(\rho)\beta_1' - g(\rho)\beta_1 + \frac{\beta_1''}{k} = 0, \tag{2.13}$$

$$\alpha'' - 2g(\rho)\beta_2' - g(\rho)\beta_2 + \frac{\beta_2''}{k} = 0, \tag{2.14}$$

$$\alpha'' + 2g(\rho)\beta_3' + g(\rho)\beta_3 + \frac{\beta_3''}{k} = 0. \tag{2.15}$$

Using Lemma 4, we can get from (2.15) the existence of β_3 and its uniformly bounded estimate w.r.t. k. Differentiating (2.15) with respect to ρ, we can obtain the uniformly bounded estimate of β_3' w.r.t. k. By making the transformation $\rho_1 = \rho - M$ in the Eqs. (2.14) and (2.13), where M is the upper bound of ρ, we also obtain existence of β_1, β_2, β_1' and β_2' by using Lemma 4.

By Assumption 3 of Theorem 1,

$$\alpha - \rho\alpha' = g^{-\frac{1}{2}} + \frac{1}{2}\rho g^{-\frac{3}{2}} g' > 0.$$

Thus system (1.2) is genuinely nonlinear. Thus we end the proof just as in Sect. 4 of [17], where the framework of Diperna [3] gets applied to a non-strictly hyperbolic system.

3. Proof of Theorem 2

In this section, we give a proof of Theorem 2. Following the framework given by Diperna [4] and Chen [1], the only work we need to do is to obtain the existence of the viscosity solutions and related estimates for the Cauchy problem

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= \epsilon\rho_{xx} - h(x,t)\rho u \\ (\rho u)_t + \left(\rho u^2 + \tfrac{\rho^\gamma}{\gamma}\right)_x &= \epsilon(\rho u)_{xx} - h(x,t)\rho u^2 \end{aligned}\right\} \tag{3.1}$$

with the initial data

$$\big(\rho(x,0), \rho(x,0)u(x,0)\big) = \big(\rho_0^\delta(x), \rho_0^\delta(x)u_0^\delta(x)\big), \tag{3.2}$$

where

$$\big(\rho_0^\delta(x), u_0^\delta(x)\big) = \Big(\big((\rho_0(x)^{\frac{\gamma-1}{2}} + \delta^{\frac{\gamma-1}{2}}) * G^\delta\big)^{\frac{2}{\gamma-1}}, \big(u_0(x) + f(\delta)\big) * G^\delta\Big) \tag{3.3}$$

with $f(\delta)$ a function of δ given in (3.6) or (3.7) and $f(\delta) \to 0$ as $\delta \to 0$. Here G^δ is a mollifier. Then

$$\big(\rho_0^\delta(x), u_0^\delta(x)\big) \in C^\infty \times C^\infty,$$

and

$$\left.\begin{aligned} \rho_0^\delta(x) &\geq \delta \quad , \quad \rho_0^\delta(x) + |u_0^\delta(x)| \leq M \\ \delta|u_{0,x}^\delta(x)| &\leq M \\ |\rho_{0,x}^\delta(x)| &\leq M(\delta) \quad , \quad |u_{0,x}^\delta| \leq M(\delta) \\ |\rho_{0,xx}^\delta(x)| &\leq M(\delta) \quad , \quad |u_{0,xx}^\delta| \leq M(\delta) \end{aligned}\right\}, \tag{3.4}$$

where M denotes a positive constant, $M(\delta)$ a positive constant depending on δ. The relation between ϵ and δ is given in (3.12).

When $h(x,t) \geq 0$, we assume $z_0(x) \geq 0$ as Assumption 2 in Theorem 2, then

$$\begin{aligned} z_0^\delta &= u_0^\delta(x) - \int_0^{\rho_0^\delta(x)} \frac{\sqrt{p'(s)}}{s} ds = u_0^\delta(x) - \frac{2}{\gamma-1}\left(\rho_0^\delta(x)\right)^{\frac{\gamma-1}{2}} \\ &= u_0(x) * G^\delta + f(\delta) - \frac{2}{\gamma-1}\left(\rho_0(x)^{\frac{\gamma-1}{2}} * G^\delta\right) - \frac{2}{\gamma-1}\delta^{\frac{\gamma-1}{2}} \\ &= z_0(x) * G^\delta + f(\delta) - \frac{2}{\gamma-1}\delta^{\frac{\gamma-1}{2}} \geq 0, \end{aligned} \tag{3.5}$$

if we let

$$f(\delta) = \frac{2}{\gamma-1}\delta^{\frac{\gamma-1}{2}}. \tag{3.6}$$

When $h(x,t) \leq 0$, we assume $w_0(x) \leq 0$, then

$$w_0^\delta = w_0(x) * G^\delta + f(\delta) + \frac{2}{\gamma-1} 2\delta^{\frac{\gamma-1}{2}} \leq 0,$$

if we let

$$f(\delta) = -\frac{2}{\gamma-1}\delta^{\frac{\gamma-1}{2}}. \tag{3.7}$$

Lemma 5 (Local Existence of (3.1) , (3.2)). *Let the initial data (3.2) satisfy the conditions (3.3), (3.4), then for any fixed ϵ and δ, there exists a smooth solution for the Cauchy problem (3.1), (3.2) in some $R_s = R \times [0,s]$, which satisfies*

$$\begin{gathered} \rho(x,t) \geq \frac{\delta}{2} \quad , \quad \rho(x,t) + |u(x,t)| \leq 2M \\ |\rho_x|, |\rho_{xx}|, |u_x|, |u_{xx}| \leq 2M(\epsilon, \delta). \end{gathered} \tag{3.8}$$

The proof of the lemma can be obtained by applying the general contraction mapping to an integral representation of (3.1). The global existence is based on the local existence and an apriori L^∞ estimate of (ρ, u) and a lower bound estimate of ρ by using the maximum principle (Lemma 7). These estimates are carefully given in [20].

Lemma 6. *Let $p(\rho) = \dfrac{\rho^\gamma}{\gamma}$, $\gamma > 1$. If $h(x,t) \geq 0$, $z_0(x) \geq 0$ and $w_0(x) \leq M$, (or $h(x,t) \leq 0$, $z_0 \geq -M$, $w_0 \leq 0$), then $z(\rho,m) \geq 0$ and $w(\rho,m) \leq M$ $(m = \rho u)$ (or $z \geq -M$, $w \leq 0$).*

Proof. Multiplying (w_ρ, w_m) and (z_ρ, z_m) respectively to the system (3.1), where w, z are given by (1.4), we have

$$\begin{aligned} & w_t + \lambda_2 w_x \\ = \;& \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x - \frac{\epsilon(\gamma+1)}{2}\rho^{\frac{\gamma-5}{2}}\rho_x^2 - h(x,t)\rho u\Big(-\frac{m}{\rho^2} + \rho^{\frac{\gamma-3}{2}}\Big) - h(x,t)\rho u^2 \frac{1}{\rho} \\ \leq \;& \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x - h(x,t)\rho^{\frac{\gamma-1}{2}} z \end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
& z_t + \lambda_1 z_x \\
= \quad & \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x + \frac{\epsilon(\gamma+1)}{2}\rho^{\frac{\gamma-5}{2}}\rho_x^2 - h(x,t)\rho u\Big(- \frac{m}{\rho^2} - \rho^{\frac{\gamma-3}{2}}\Big) - h(x,t)\rho u^2 \frac{1}{\rho} \\
\geq \quad & \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x + h(x,t)\rho^{\frac{\gamma-1}{2}} z, \qquad (3.10)
\end{aligned}
$$

where $\lambda_2 = u + \sqrt{p'(\rho)}$, $\lambda_1 = u - \sqrt{p'(\rho)}$ are two eigenvalues of the systems (1.1), (1.2), (1.3).

Since $z_0^\delta(x) \geq 0$ from (3.5), we have $z \geq 0$ by applying the maximum principle to (3.10). So we have from (3.9),

$$
w_t + \lambda_2 w_x \leq \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x,
$$

and thus $w \leq M$. We can similarily get the estimates $z \geq -M$, $w \leq 0$ if $h(x,t) \leq 0$. Lemma 6 is proven. $\square$

Lemma 6 gives us the uniform boundedness of (ρ, u),

$$
0 \leq \frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \leq u \leq M \quad or \quad -M \leq u \leq -\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}. \qquad (3.11)
$$

To get the global existence of (3.1), we still need a lower, positive bound of ρ. We follow the method given in [20] to prove it.

Lemma 7. *Let the conditions in Lemma 6 be satisfied. Let $p(\rho) = \frac{\rho^\gamma}{\gamma}$, $1 < \gamma \leq \frac{5}{3}$. Let $|h(x,t)| \leq \dfrac{\alpha}{1+t}$, where $\alpha < \dfrac{2-\gamma}{2(\gamma-1)M}$, here $\bar{M}$ denotes the bound of u(x,t). Let ϵ and δ be related as*

$$
2M\epsilon = \delta^\gamma, \qquad (3.12)
$$

where M is the bound of $|\delta u_{o,x}^\delta|$ given in (3.4), then

$$
u_x \leq \frac{\rho^{\gamma-1}}{2\epsilon} \quad , \quad \rho \geq \epsilon^{\frac{1}{\gamma-1}}(1+t)^{-\frac{1}{\gamma-1}} \quad , \quad 1 < \gamma \leq \frac{5}{3} \quad . \qquad (3.13)
$$

Proof. Substituting the first equation in (3.1) into the second, we get

$$
u_t + uu_x + \rho^{\gamma-2}\rho_x = \epsilon u_{xx} + 2\epsilon(ln\rho)_x u_x.
$$

Differentiating this with respect to x, we have

$$
\begin{aligned}
(u_x)_t + u_x^2 + uu_{xx} + \rho^{\gamma-1}(ln\rho)_{xx} + (\gamma-1)\rho^{\gamma-2}(ln\rho)_x = \\
\epsilon u_{xxx} + 2\epsilon(ln\rho)_{xx}u_x + 2\epsilon(ln\rho)_x u_{xx}. \qquad (3.14)
\end{aligned}
$$

Let

$$
v = u_x - \frac{\rho^{\gamma-1}}{2\epsilon},
$$

then

$$\begin{aligned} v_t + \big(u - 2\epsilon(ln\rho)_x\big)v_x + \big(v + \frac{3-\gamma}{2\epsilon}\rho^{\gamma-1} - 2\epsilon(ln\rho)_{xx}\big)v + \qquad\qquad (3.15) \\ + \frac{2-\gamma}{4\epsilon^2}\rho^{2(\gamma-1} + \frac{(\gamma-1)(2-\gamma)}{2}\rho^{\gamma-3}\rho_x^2 - \frac{\gamma-1}{2\epsilon}(h(x,t))\rho^{\gamma-1}u \quad = \quad \epsilon v_{xx}, \end{aligned}$$

resulting from (3.14) with the aid of the first equation in (3.1). Moreover

$$v|_{t=0} = u^\delta_{0,x} - \frac{(\rho^\delta_0)^{\gamma-1}}{2\epsilon} \le 0, \tag{3.16}$$

since $\delta|u^\delta_{0,x}| \le M$ and the relation (3.12) between ϵ and δ.

First, we have from the local solution (Lemma 5), $\rho \ge \frac{\delta}{2}$ and the boundedness of $h(x,t)$,

$$\begin{aligned} \frac{2-\gamma}{4\epsilon^2}\rho^{2(\gamma-1)} - \frac{\gamma-1}{2\epsilon}h(x,t)\rho^{\gamma-1}u \quad &\ge \quad \frac{\rho^{\gamma-1}}{2\epsilon}\big(\frac{2-\gamma}{2\epsilon}(\frac{\delta}{2})^{\gamma-1} - (\gamma-1)\alpha M\big) \\ &\ge \quad \frac{\rho^{\gamma-1}}{2\epsilon}\big((\frac{2-\gamma}{2\epsilon})\frac{(2M\epsilon)^{\frac{1}{\gamma}}}{2^{\gamma-1}} - (\gamma-1)\alpha M\big) \\ &\ge \quad 0 \qquad\qquad (3.17) \end{aligned}$$

when ϵ is sufficiently small and $\gamma \in (1, \frac{5}{3})$. Thus we have from (3.15),

$$v_t + \big(u - 2\epsilon(ln\rho)_x\big)v_x + \big(v + \frac{3-\gamma}{2\epsilon}\rho^{\gamma-1} - 2\epsilon(ln\rho)_{xx}\big)v \le \epsilon v_{xx}. \tag{3.18}$$

Applying the maximum principle to (3.16), (3.18) we have $v \le 0$ and so $u_x \le \frac{\rho^{\gamma-1}}{2\epsilon}$. Using the first equation in (2.2), we get

$$\rho_t + \rho u_x + \rho_x u + h(x,t)\rho u = \epsilon\rho_{xx}.$$

Then

$$\rho_t + \frac{\rho^\gamma}{2\epsilon} + \rho_x u + \alpha\bar{M}(1+t)^{-1}\rho \ge \epsilon\rho_{xx}.$$

Let

$$\chi = \rho^{1-\gamma},$$

then

$$\chi_t + \frac{1-\gamma}{2\epsilon} + \chi_x u + (1-\gamma)\alpha\bar{M}(1+t)^{-1}\chi \le \epsilon\chi_{xx}.$$

Let

$$\Upsilon = \chi(1+t)^{\alpha(1-\gamma)\bar{M}},$$

then

$$\Upsilon_t + \frac{1-\gamma}{2\epsilon}(1+t)^{\alpha(1-\gamma)\bar{M}} + \Upsilon_x u \le \epsilon\Upsilon_{xx}.$$

Let

$$\Omega = \Upsilon + \frac{1-\gamma}{2\epsilon(1-\alpha(\gamma-1)\bar{M})}(1+t)^{1-\alpha(\gamma-1)\bar{M}},$$

then

$$\Omega_t + u\Omega_x \le \epsilon\Omega_{xx}.$$

Thus

$$\begin{aligned}\Omega &\leq sup \quad \Omega_0(x) = sup \quad \Upsilon(0,x) + \frac{1-\gamma}{2\epsilon(1-\alpha(\gamma-1)\bar{M})} \\ &= \delta^{1-\gamma} + \frac{1-\gamma}{2\epsilon(1-\alpha(\gamma-1)\bar{M})} = (2M\epsilon)^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{2\epsilon(1-\alpha(\gamma-1)\bar{M})} \\ &= \frac{1}{\epsilon}\big((2M)^{\frac{1-\gamma}{\gamma}}\epsilon^{\frac{1}{\gamma}} - \frac{\gamma-1}{2(1-\alpha(\gamma-1)\bar{M})}\big) \\ &\leq 0,\end{aligned}$$

if $1-\alpha(\gamma-1)\bar{M} > \frac{\gamma}{2}$ (for $1<\gamma\leq\frac{5}{3}$) and ϵ is small enough. Therefore

$$\Upsilon \leq \frac{\gamma-1}{2\epsilon(1-\alpha(\gamma-1)\bar{M})}(1+t)^{1-\alpha(\gamma-1)\bar{M}} \leq \frac{1}{\epsilon}(1+t)^{1-\alpha(\gamma-1)\bar{M}}.$$

Then

$$\chi \leq \frac{1+t}{\epsilon},$$

and so

$$\rho \geq \epsilon^{\frac{1}{\gamma-1}}(1+t)^{-\frac{1}{\gamma-1}}.$$

Noticing from above

$$\begin{aligned}\frac{2-\gamma}{2\epsilon}\rho^{\gamma-1} &- (\gamma-1)h(x,t)u \\ &\geq \frac{2-\gamma}{2\epsilon}\frac{\epsilon}{(1+t)} - (\gamma-1)\alpha(1+t)^{-1}\bar{M} \\ &\geq 0,\end{aligned}$$

if $1-\alpha(\gamma-1)\bar{M} > \frac{\gamma}{2}$. So (3.17) is still true. Therefore we obtain the proof of Lemma 7. □

Lemma 6 and Lemma 7 give us the following existence lemma

Lemma 8 (Global Existence of 3.1, 3.2). *If the conditions in Theorem 2 are satisfied, then for any fixed ϵ and δ related as in (3.12), there exists a smooth solution for the Cauchy problem (3.1), (3.2) in $R_T = R \times [0.T]$ (T is an arbitrary positive constant) which satisfies the estimates (3.11) and (3.13).*

Now using the framework given in [1] and [4] ends the proof of Theorem 2.

4. Proof of Theorem 3

In this section, we consider the extended river flow equation (1.3) and give a proof of Theorem 3. Adding the artificial viscosity to (1.3) we have

$$\left.\begin{aligned}\rho_t + (\rho u)_x &= \epsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + a(x)\rho + c\rho u|u| &= \epsilon(\rho u)_{xx}\end{aligned}\right\}. \tag{4.1}$$

We are going to study the Cauchy problem (4.1) with the initial data

$$\big(\rho(x,0),\rho(x,0)u(x,0)\big) = \big(\rho_0^\delta(x),\rho_0^\delta(x)u_0^\delta(x)\big), \tag{4.2}$$

where

$$\big(\rho_0^\delta(x),u_0^\delta(x)\big) = \big(\rho_0(x)+\delta, u_0(x)*G^\delta\big)$$

and G^δ is a mollifier. Then

$$\big(\rho_0^\delta(x),u_0^\delta(x)\big) \in C^\infty \times C^\infty$$

and

$$\delta \le \rho_0^\delta(x) \le M \tag{4.3}$$

for some positive M from Assumptions 1 and 2 in Theorem 3. In addition

$$\left.\begin{aligned} \delta|u^\delta_{0,x}(x)| &\le M \\ |\rho^\delta_{0,x}(x)| &\le M(\delta) \quad , \quad |u^\delta_{0,x}| \le M(\delta) \\ |\rho^\delta_{0,xx}(x)| &\le M(\delta) \quad , \quad |u^\delta_{0,xx}| \le M(\delta) \end{aligned}\right\} \tag{4.4}$$

for a suitable positive constant $M(\delta)$ depending on δ. The relation between ϵ and δ is given by (3.12). As stated in the beginning of Sect. 2, we only need to obtain the existence of viscous solutions and related estimates for the Cauchy problem (4.1) , (4.2). Similar to Lemma 5 in Sect. 3, we have

Lemma 9 (Local Existence of (4.1) , (4.2)). *Let the initial data (4.2) satisfy the conditions (4.3), (4.4), then for any fixed ϵ and δ, there exists a smooth solution for the Cauchy problem (4.1) , (4.2) in some $R_s = R \times [0,s]$, which satisfies*

$$\left.\begin{aligned} \rho(x,t) \ge \tfrac{\delta}{2} \quad , \quad \rho(x,t)+|u(x,t)| &\le 3M \\ |\rho_x|,|\rho_{xx}|,|u_x|,|u_{xx}| &\le 2M(\epsilon,\delta) \end{aligned}\right\}. \tag{4.5}$$

Lemma 10 (A Priori Upper Bound Estimate). *Let $p(\rho) = \frac{\rho^\gamma}{\gamma}$, $\gamma > 1$. If $|a(x)| \le M$, $c \ge 0$, $w_0(x) \le M_1$, $\bar{z_0}(x) \le M_1$ ($\bar{z} = -z$), then*

$$w(x,t) \le M_1 + Mt \quad , \quad \bar{z}(x,t) \le M_1 + Mt \tag{4.6}$$

Proof. Multiplying (w_ρ, w_m) and $(\bar{z}_\rho, \bar{z}_m)$ respectively to the system (4.1), where w, z ($\bar{z} = -z$) are given in (1.4), we have

$$\left.\begin{aligned} w_t + \lambda_2 w_x + a(x) + \tfrac{c|u|}{2}(w-\bar{z}) &= \varepsilon w_{xx} - \tfrac{(\gamma+1)\epsilon}{2}\rho^{\frac{\gamma-5}{2}}\rho_x^2 \\ &\le \epsilon w_{xx} \\ \bar{z}_t + \lambda_1 \bar{z}_x - a(x) + \tfrac{c|u|}{2}(\bar{z}-w) &= \epsilon \bar{z}_{xx} - \tfrac{(\gamma+1)\epsilon}{2}\rho^{\frac{\gamma-5}{2}}\rho_x^2 \\ &\le \epsilon \bar{z}_{xx} \end{aligned}\right\}, \tag{4.7}$$

where λ_1 and λ_2 are the eigenvalues of system (1.3) just like in (3.10).

Making a transformation

$$w = X - Mt \quad , \quad \overline{z} = Y - Mt,$$

where M is the bound of $a(x)$, we have from (4.7),

$$\left.\begin{aligned} X_t + \lambda_2 X_x + \tfrac{c|u|}{2}(X - Y) &\leq \epsilon X_{xx} \\ Y_t + \lambda_1 Y_x + \tfrac{c|u|}{2}(Y - X) &\leq \epsilon Y_{xx} \end{aligned}\right\} \tag{4.8}$$

with

$$X|_{t=0} = w|_{t=0} \leq M_1 \quad , \quad Y|_{t=0} = \overline{z}|_{t=0} \leq M_1. \tag{4.9}$$

Thus the Lemma 2.4 in [18] applied to (4.8), (4.9) gives the proof of Lemma 10. □

Lemma 10 gives us the upper bound of (ρ, u) :

$$0 \leq \rho \leq M(T) \quad , \quad |u| \leq M(T). \tag{4.10}$$

Lemma 11 (A Priori Lower Bound Estimate). *Let the conditions in Lemma 10 be satisfied. Let $\gamma \in (1, \frac{5}{3}]$, $a'(x) \geq 0$. Then*

$$u_x \leq \frac{\rho^{\gamma-1}}{2\epsilon} \quad , \quad \rho \geq \Big(\frac{\epsilon}{c_1 + c_2 t}\Big)^{\frac{1}{\gamma-1}} \tag{4.11}$$

for the positive constant c_1, c_2 being independent of ϵ, δ.

Proof. We need only to prove the first part of (4.11), the second inequality follows from the proof of Lemma 7 directly (or see [20]).

Substituting the first equation in (4.1) into the second and letting $v = u_x - \frac{\rho^{\gamma-1}}{2\epsilon}$, we have

$$v_t + \big(u - 2\epsilon(ln\rho)_x\big)v_x + \big(v + \frac{3-\gamma}{2\epsilon}\rho^{\gamma-1} - 2\epsilon(ln\rho)_{xx}\big)v \leq \epsilon v_{xx}.$$

The proof ends by following the proof in Lemma 7. □

Lemma 10 and Lemma 11 give us the following existence lemma.

Lemma 12 (Global Existence of (4.1), (4.2)). *If the conditions of Theorem 2 are satisfied, then for any fixed ϵ and δ related as in (3.12), there exists a smooth solution for the Cauchy problem (4.1), (4.2) in $R_T = R \times (0, T]$ (T is an arbitrary positive constant) which satisfies the estimates (4.10), (4.11).*

Now we have set ourselves up to finish by using the framework as given in [4] or [1] which ends the proof of Theorem 3.

Acknowledgement. Y.-g. L. expresses his gratitude to Prof. E. Tadmor for his hospitality and his helpful discussions on Theorem 1 while being supported in part by the Raymond and Beverley Sackler Institute of Scientific Computing in Tel Aviv.

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Communicated by J.L. Lebowitz