On computing compressible Euler equations with gravity

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Abstract We present a well-balanced finite volume solver for the compressible Euler equations with gravity. The Riemann solver used in the finite volume method is approximated by a so called relaxation Riemann solution. Besides the well-balanced property, the scheme is also positivity preserving regarding the density and internal energy. The scheme is able to capture not only isothermal and polytropic stationary solutions of the hydrostatic equilibrium but also to preserve more general steady states up to machine precision. The scheme is tested on numerical examples including the preservation of arbitrary steady states and the evolution of small perturbations of stationary solutions to demonstrate the properties of the designed scheme.

1 Introduction

When solving the two or three dimensional Euler equations with gravity via a finite volume discretization, we are faced with several challenges. Firstly we need a discretization which works well at both low and high Mach numbers for the homogeneous system. Secondly we need a discretization which maintains hydrostatic equilibria to machine precision. Finally, when combining these two methods, we need to find a scheme that is numerically stable in more than one space dimension. Solutions to the first challenge can be found in the literature, see e.g. in [1]. Solutions to the second challenge can be found e.g. in [3, 4].

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We found experimentally, when combining these type of approaches, that typically instabilities arise when computing in more than one space dimensions. In our numerical experiments, we found that one well-balanced method in particular was more stable than others. In this contribution, we shall report on this method. It is based on a relaxation approach leading to a positivity and entropy preserving scheme which is therefore especially useful in applications. In addition it can be extended to higher order of accuracy and to higher dimensions.

Consider the system of compressible Euler equations with gravity in one space dimension given by the following set of equations

$$\begin{aligned} \partial_t \rho + \partial_x \rho u &= 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) &= -\rho \partial_x \Phi, \\ \partial_t E + \partial_x (E + p) u &= -\rho u \partial_x \Phi. \end{aligned} \tag{1}$$

Here, $\rho > 0$ denotes the density, u the velocity, p the pressure and $E = \rho e + \frac{1}{2}\rho u^2$ the total energy, where $e > 0$ is the internal energy. The function Φ which is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ denotes the gravitational potential. The pressure is described by a general pressure law which depends on the internal energy and specific volume $\tau = \frac{1}{\rho}$. We require for the solution $w = (\rho, \rho u, E)$ the density and the internal energy to be positive. That means the state vector w must belong to the set $\left\{ w \in \mathbb{R}^3 \mid \rho > 0,\ e > 0 \right\}$.

The paper is organised as follows. In Section 2, the relaxation method we use is described. The approximate Riemann solver which is designed to have the well balanced property is presented in Section 3. Section 4 is devoted to the associated numerical scheme which is tested in Section 5 to verify the well-balancing property.

2 Relaxation

We consider the following relaxation model derived in [4] where a Suliciu-type relaxation approach is used, see [2],

$$\begin{aligned} \partial_t \rho + \partial_x \rho u &= 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \pi) &= -\rho \partial_x Z, \\ \partial_t E + \partial_x (E + \pi) u &= -\rho u \partial_x Z, \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u &= \frac{\rho}{\varepsilon} (p(\tau, e) - \pi), \\ \partial_t \rho Z + \partial_x \rho Z u &= \frac{\rho}{\varepsilon} (\Phi - Z). \end{aligned} \tag{2}$$

Here, the gravity Φ is approximated by a new variable Z, the pressure p by the new variable π and $a > 0$ denotes the relaxation parameter.

Since we also need the density and internal energy to be positive, we require the state vector of the relaxation system $W = (\rho, \rho u, E, \rho\pi, \rho Z)$ to belong to the set $\{W \in \mathbb{R}^5 \mid \rho > 0,\ e > 0\}$.

For a given gravity function Φ, an equilibrium state for the relaxation model is defined by

$$W^{eq} = (\rho, \rho u, E, \rho p(\tau, e), \rho\Phi)^T. \tag{3}$$

The eigenvalues of the system are $\lambda^{\pm} = u \pm \frac{a}{\rho}$ and $\lambda^u = u$ where the eigenvalue λ^u has multiplicity three. Following citeLeVeque2002 one finds the fields associated to the eigenvalues are linearly degenerate and the Riemann invariants with respect to $\lambda^{\pm}$ are

$$I_1^{\pm} = u \pm \frac{a}{\rho},\ I_2^{\pm} = \pi \mp au,\ I_3^{\pm} = e - \frac{\pi^2}{2a^2},\ I_4^{\pm} = Z \tag{4}$$

and with respect to λ^u are

$$I_1^u = u. \tag{5}$$

In the following, let us consider a Riemann Problem as initial data with two constant values separated by a discontinuity at $x = 0$

$$W_0(x) = \begin{cases} W_L & x < 0 \\ W_R & x > 0. \end{cases} \tag{6}$$

The solution, if it exists, consists of four constant states separated by contact discontinuities and has the following structure

$$W_R\left(\frac{x}{t}; W_L, W_R\right) = \begin{cases} W_L & \frac{x}{t} < \lambda^- \\ W_L^* & \lambda^- < \frac{x}{t} < \lambda^u \\ W_R^* & \lambda^u < \frac{x}{t} < \lambda^+ \\ W_R & \lambda^+ < \frac{x}{t} \end{cases}, \tag{7}$$

where W_L^*, W_R^* denote the intermediate states. This leads to 10 unknowns in the Riemann problem, five unknowns each for the intermediate states $W_{L,R}^*$ but one obtains only nine relations from the Riemann invariants (4) and (5), for the computations see [9]. This leaves us with one degree of freedom to choose the 10th relation such that the resulting scheme has the well-balanced property.

How to obtain this 10th relation will be described in the following section.

3 Well-balanced property

In the following, we will focus on steady states at rest, which are solutions of

$$\begin{aligned} u &= 0, \\ \partial_x p &= -\rho\,\partial_x\Phi. \end{aligned} \tag{8}$$

Following [5], we write the hydrostatic solution as $\bar{\rho} = \rho_c \alpha(x),\ \bar{p} = p_c \beta(x)$, where the constants p_c, ρ_c are reference values at some location $x = x_c$ and $\alpha(x), \beta(x)$ are non-dimensional functions. Since the density and the pressure are strictly positive, we require $\alpha, \beta > 0$. These functions must satisfy the hydrostatic condition (8) which leads to an expression for the derivative of the potential given by

$$\partial_x \Phi(x) = -\frac{p_c}{\rho_c} \frac{\partial_x \beta(x)}{\alpha(x)}. \tag{9}$$

A well-balanced scheme must satisfy the discretized form of the hydrostatic equation. Since the discretized flux derivative must exactly balance the discretized source term a, we choose the following symmetrical discretization

$$\pi_R - \pi_L = \frac{\pi_c}{2\rho_c} (\beta_R - \beta_L) \left(\frac{\rho_L}{\alpha_L} + \frac{\rho_R}{\alpha_R} \right). \tag{10}$$

Using this relation in addition to the relations gained from the Riemann invariants, the intermediate states $W^*_{L,R}$ can be determined, for details see [9]. Thus, the Riemann problem of the Relaxation system completed by relation (10) has a unique solution which is given by (7).

Using this Riemann solution one obtains an approximate Riemann solver for the original system (1) by projecting the solution of the Relaxation system on its first three components

$$w^{eq}\left(\frac{x}{t}; w_L, w_R\right) = W_R^{(\rho,\rho u,E)}\left(\frac{x}{t}; W_L, W_R\right). \tag{11}$$

The following result shows the well-balanced property of the approximative Riemann solver.

Theorem 1. *The approximate Riemann solver stated by* (11) *is well-balanced in the sense that the initial condition on each cell i given by*

$$u_i = 0,\ \frac{\rho_i}{\alpha_i} = const.,\ \frac{p_i}{\beta_i} = const., \tag{12}$$

is preserved.

Proof. For the proof, we refer the reader to [9]. □

We want conclude this section by mentioning some additional properties of the above defined Riemann solver, for detailed proof see [4].

- The approximative Riemann solver ensures the positivity of the density ρ and the pressure p for a sufficiently large relaxation parameter a. That means starting with data belonging to $\Omega := \left\{ w = (\rho, \rho u, E) \in \mathbb{R}^3,\ \rho > 0,\ e > 0 \right\}$ then the solution $w^{eq}(\frac{x}{t}; w_L, w_R)$ also belongs to Ω.
- If one considers an entropy inequality $\partial_t \rho F(\eta) + \partial_x F(\eta) u \leq 0$ for the Euler equations with gravity where $\eta(\tau, e)$ denotes a specific entropy then the approximative Riemann solver is consistent with the entropy inequality.

4 Numerical scheme

In this section, we describe the numerical scheme associated with the approximative Riemann solver developed above.

The computational domain is divided in N cells $C_i = (x_{i-1/2}, x_{i+1/2})$ with fixed step-size Δx. The time discretization on the interval $[0,T]$ is given by $t^{n+1} = t^n + \Delta t$ where $\Delta t > 0$ denotes the time step restricted by a CFL condition. Define the approximative solution at time t^n as $w^n(x,t^n) = w_i^n$ for $x \in \left(x_{i-1/2}, x_{i+1/2}\right)$ and the updated state at time t^{n+1} as

$$w_i^{n+1} = \frac{1}{\Delta x}\int_{C_i} w^n(x,t^n+\Delta t). \tag{13}$$

Thereby $w^n(x,t^n+t)$ is a sequence of the approximative Riemann solver (11) at each interface $x_{i+/2}$ given by

$$w^n(x,t^n+t) = w^{eq}(\frac{x-x_{i+1/2}}{t}, w_i^n, w_{i+1}^n) \tag{14}$$

for $x \in (x_i, x_{i+1})$ and $t \in (0,\Delta t)$.

Following the computations in [6, 7, 10], we obtain for the updated state

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2}) + \frac{\Delta t}{2}(S_{i-1/2} + S_{i+1/2}). \tag{15}$$

The approximated source term is given by

$$S_{i+1/2} = \left(0, \frac{p_c}{\rho_c}\frac{(\beta_{i+1}-\beta_i)}{\Delta x}\frac{1}{2}\left(\frac{\rho_i}{\alpha_i}+\frac{\rho_{i+1}}{\alpha_{i+1}}\right), u^*_{i+1/2}\frac{\pi_c}{\rho_c}\frac{(\beta_{i+1}-\beta_i)}{\Delta x}\frac{1}{2}\left(\frac{\rho_i}{\alpha_i}+\frac{\rho_{i+1}}{\alpha_{i+1}}\right)\right) \tag{16}$$

Defining $s_{LR} = \frac{p_c}{\rho_c}\frac{(\beta_R-\beta_L)}{\Delta x}\frac{1}{2}\left(\frac{\rho_L}{\alpha_L}+\frac{\rho_R}{\alpha_R}\right)$ and using the formulas for the intermediate states, the numerical flux function reads

$$f_{i+1/2} = \begin{cases} (\rho_L u_L, \rho_L u_L^2 + \pi_L + s_{LR}, (E_L+\pi_L)u_L + u^* s_{LR})^T & u_L - \frac{a}{\rho_L} > 0, \\ (\rho_L^* u^*, \rho_L^* u^{*2} + \pi_L^* + s_{LR}, (E_L^*+\pi_L^*)u^* + u^* s_{LR})^T & u_L - \frac{a}{\rho_L} < 0 < u^*, \\ (\rho_R^* u^*, \rho_R^* u^{*2} + \pi_R^* - s_{LR}, (E_R^*+\pi_R^*)u^* - u^* s_{LR})^T & u* < 0 < u_R - \frac{a}{\rho_R}, \\ (\rho_R u_R, \rho_R u_R^2 + \pi_R - s_{LR}, (E_R+\pi_R)u_R - u^* s_{LR})^T & u_R - \frac{a}{\rho_R} < 0. \end{cases} \tag{17}$$

5 Numerical results

In the following section two types of test cases are presented. First a well-balanced test is performed, to verify that the initial condition, if satisfying the condition (8),

is preserved on machine precision. The second test addresses the evolution of small perturbations of a hydrostatic atmosphere.

Well-balanced tests

For the well-balanced tests, we consider stationary solutions for three different potential functions $\Phi(x) = x$, $\Phi(x) = \frac{1}{2}x^2$ and $\Phi(x) = \sin(2\pi x)$ to demonstrate that the scheme can deal with more complex gravitational fields.

For all examples, the computational domain is $[0,1]$ and the initial velocity is zero. All errors are given in the L^1- norm and computations are performed in double precision.

As a first example, we consider a isothermal hydrostatic atmosphere given by

$$\rho_0(x) = \exp(-\Phi(x)),\ p_0(x) = \exp(-\Phi(x)). \tag{18}$$

In Table 1 the error in density, velocity and pressure with respect to the initial condition are given. The calculations are performed on a grid with 100 and 1000 cells respectively up to a final time $T_f = 2.0$. As can be seen from Table 1, the error is of the order of machine precision and thus the hydrostatic atmosphere is preserved.

To show that the scheme can also preserve more general steady states, we consider as a second test the stationary solution from [3]. For the quadratic potential $\Phi(x) = \frac{1}{2}x^2$ a stationary solution is given by

$$\bar{\rho}(x) = \exp(-x),\ \ \bar{p}(x) = (1+x)\exp(-x) \tag{19}$$

which corresponds to a non-uniform temperature profile given by $T(x) = 1+x$. Thus the steady state is not isothermal. The number of cells used for the calculations are doubled for each calculation starting with 100 cells. The error in density, velocity and pressure with respect to the initial condition are reported in Table 2. One can see, that the initial steady state is preserved with machine precision.

Evolution of small perturbations

As a last example, taken from [8], the evolution of a small perturbation added to an initial isothermal hydrostatic equilibrium is investigated. The initial condition on the domain $[0,1]$ is given by

$$\begin{aligned}\Phi(x) &= x,\\ \rho(x) &= \exp(-\Phi(x)),\\ p(x) &= \exp(-\Phi(x)) + 0.01\exp(-100(x-0.5)^2),\end{aligned}$$

where the pressure is perturbed by a Gauß function centered in $x = 0.5$. The solution is computed at time $T = 0.2$ with 100 cells and a reference solution using 30000 cells. In Figure 1, the pressure perturbation $p(x) - p_0(x)$ and the resulting velocity perturbation are plotted in comparison with the initial perturbation.

Table 1 Error in density, velocity and pressure for isothermal example using different potentials.

$\Phi(x)$	Cells	density	velocity	pressure
x	100	1.88738E-017	3.67483E-017	2.05391E-017
	1000	3.47499E-017	8.74191E-017	4.32431E-017
$\frac{1}{2}x^2$	100	3.94129E-016	3.19565E-016	6.11732E-016
	1000	1.10456E-015	4.84117E-016	1.84618E-015
$\sin(2\pi x)$	100	8.60422E-017	7.39687E-017	1.73749E-016
	1000	1.07663E-015	5.38106E-015	1.11399E-015

Table 2 Error in density, velocity and pressure for a non-hydrostatic steady state.

Cells	density	velocity	pressure
100	7.04991E-017	4.84102E-016	7.54951E-017
200	8.21565E-017	3.17104E-016	8.38218E-017
400	2.28983E-016	6.08430E-016	5.95357E-016
800	3.49997E-016	1.40357E-015	5.23331E-016
1600	6.12600E-016	1.22546E-015	5.05290E-016

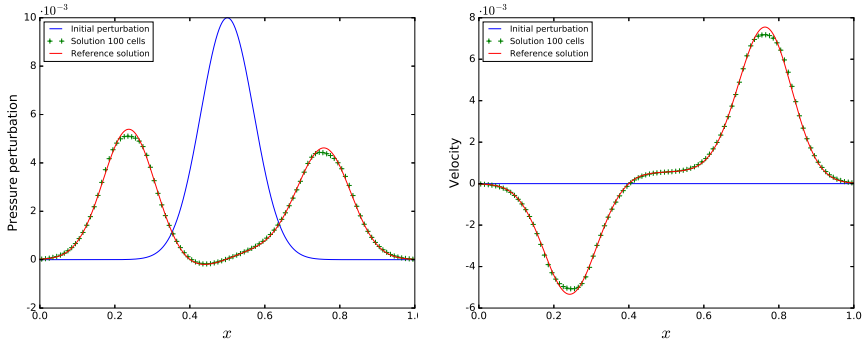

Fig. 1 Perturbation in pressure (left) and velocity (right).

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