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The Cauchy Problem for Hyperbolic Conservation Laws with Three Equations

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1. INTRODUCTION

This paper considers the Cauchy problem for the nonlinear system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, s)_x + \alpha u &= 0 \\ s_t + \frac{\beta \cdot \{s - f(v)\}}{\tau} &= 0 \end{aligned} \tag{1.1}$$

with bounded L^2 measureable initial data

$$(v, u, s)_{/t=0} = (v_0(x), u_0(x), s_0(x)), \tag{1.2}$$

where α, β, τ are nonnegative constants. When $\beta = 0$, system (1.1) can be used to model the adiabatic gas flow through porous media [5], where v is specific volume, u denotes velocity, s stands for entropy, and σ denotes pressure. Its form in Euler coordinates is also a model of isothermal unsteady two phase flow in pipelines [1]. In this paper we study the global generalized solution for this case. The case $\beta \neq 0$, when written in Euler

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coordinates, can be used to model the chemically reacting flow [7]. Again v is specific volume, u denotes velocity but s is the mass fraction of one mode of the two-mode gas and $f(v)$ is a given equilibrium distribution in v. In this case, τ denotes a reaction time. We show that the solution of the equilibrium system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, f(v))_x + \alpha u &= 0 \end{aligned} \tag{1.3}$$

is given by the limit of the solutions of the viscous approximation

$$\begin{aligned} v_t - u_x &= \varepsilon v_{xx} \\ u_t - \sigma(v, s)_x + \alpha u &= \varepsilon u_{xx} \\ s_t + \frac{\beta \cdot \{s - f(v)\}}{\tau} &= \varepsilon s_{xx} \end{aligned} \tag{1.4}$$

as the dissipation and the reaction time τ go to zero. Similar results about zero realization systems of two equations and solutions in L^∞ space are discussed in papers [2, 3, 6, 11], and for solutions in L^p space, see the recent paper [4].

In dealing with the Cauchy problem (1.1), (1.2), one basic difficulty is the a priori estimate of the viscosity solution of (1.4), independent of ε in a suitable L^p space ($p > 1$). Since system (1.1) in general cannot be diagonalized by using Riemann invariants, it is not to be expected that viscosity solution $(v^\varepsilon, u^\varepsilon, s^\varepsilon)$ of the Cauchy problem (1.4) will be bounded in L^∞, uniformly in ε, by using the invariant region principle [13]. We have to search for solutions of system (1.1) in L^p space. In some sense, the a priori estimate of the solutions of (1.4) in L^2 is easy to get, if we can find a strictly convex entropy for the system (1.1). However, a new difficulty arises by considering the compactness of the viscosity solutions in L^p space by trying to use compensated compactness. To the author's knowledge compactness of the viscous solutions in L^p space has been shown for scalar equations in [9], for a simple model of combustion very close to a scalar equation in [10], and for a system of two equations by P. X. Lin [8], J. W. Shearer [12], and M. Santos and H. Fried [15]. Since for the viscosity parabolic system (1.4) both the L^∞ estimate and the one sided L^∞ estimate using the Riemann invariants (as given in (1.6) of [8] are not easy to get, here we take the entropy-entropy flux pair as considered in [12] as the base of this paper.

In this paper we make the following assumptions about $\sigma(v, s)$ and the initial data as follows:

Case (I). $\beta > 0$. (1) $f(v) = cv$

(2) $\sigma(v, s) = \sigma(v) - cs$, where $\sigma(v)$ satisfies all the conditions given in paper [12], namely:

(a) $\sigma \in C^3(R)$, $\sigma(0) = 0$, $\sigma' \geq d > c^2$, d is a positive constant

(b) $\sigma'' \neq 0$, $\sigma'' \in L^1 \cap L^\infty(R)$

(c) $\sigma''' \in L^\infty(R)$, $|\sigma'''|_{L^1} \leq M$

(3) $v_0(x), u_0(x), s_0(x)$ are all bonded in L^2 and tend to zero as $|x| \to \infty$ sufficiently fast such that

$$\begin{aligned} &\lim_{x\to\pm\infty} \left(v_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x)\right) = (0,0,0), \\ &\lim_{x\to\pm\infty} \left(\frac{dv_0^\varepsilon(x)}{dx}, \frac{du_0^\varepsilon(x)}{dx}, \frac{ds_0^\varepsilon(x)}{dx}\right) = (0,0,0), \end{aligned} \tag{1.5}$$

where $(v_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x))$ are smooth function obtained by smoothing the initial data $(v_0(x), u_0(x), s_0(x))$ with a mollifier, satisfying

$$\left(v_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x)\right) \to \left(v_0(x), u_0(x), s_0(x)\right) \qquad \text{when } \varepsilon \to 0 \tag{1.6}$$

$$\begin{aligned} &|v_0^\varepsilon(x)|_{L^2} \leq |v_0(x)|_{L^2} \leq M, \qquad |u_0^\varepsilon(x)|_{L^2} \leq |u_0(x)|_{L^2} \leq M, \\ &|s_0^\varepsilon(x)|_{L^2} \leq |s_0(x)|_{L^2} \leq M \end{aligned} \tag{1.7}$$

$$|v_0^\varepsilon(x)|_{H^1(R)} \leq M(\varepsilon), \qquad |u_0^\varepsilon(x)|_{H^1(R)} \leq M(\varepsilon), \qquad |s_0^\varepsilon(x)|_{H^1(R)} \leq M(\varepsilon) \tag{1.8}$$

$$\left|\frac{d^i v_0^\varepsilon(x)}{dx^i}\right|, \left|\frac{d^i u_0^\varepsilon(x)}{dx^i}\right|, \left|\frac{d^i s_0^\varepsilon(x)}{dx^i}\right| \leq M(\varepsilon), \qquad i = 0, 1, 2. \tag{1.9}$$

Case (II). $\beta = 0$. (1) $\sigma(v, s) = \sigma(v)g(s) - cs$, $g(s) \in C^3$, and $g(s) \geq d > 0$

(2) $\sigma(v)$ satisfies (a), (b), and (c) in (I)

(3) $v_0(x), u_0(x), s_0(x)$ satisfy the same conditions as in (I). Moreover

$|s_0(x)|_{H^1_{loc}(R)} \leq M$, where M is a positive constant and independent of ε, $M(\varepsilon)$ is a positive constant which depends on ε.

2. VISCOSITY SOLUTIONS

In this section, we consider the existence of the Cauchy problem for the parabolic system (1.4) with initial data

$$(v, u, s)_{|t=0} = \big(v_0(x), u_0(x), s_0(x)\big). \tag{2.1}$$

We only give the proof of case (I), the proof for case (II) is similar. The local existence of solutions can be obtained by applying the contracting mapping principle to an integral representation of (1.4). The local time depends on the L^∞ norm $M(\varepsilon)$ of $((v_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x))$ given in (1.9) (cf. [13]).

LEMMA 2.1. (Local Existence Theorem). *If the initial data satisfy the condition* (1.9), *then for any fixed* ε, $\tau > 0$, *the Cauchy problem* (1.4) *and* (2.1) *admits a unique smooth local solution* (v, u, s) *which satisfies*

$$\left|\frac{\partial^i v}{\partial x^i}\right| + \left|\frac{\partial^i u}{\partial x^i}\right| + \left|\frac{\partial^i s}{\partial x^i}\right| \leq M(t_1, \varepsilon) < +\infty \qquad \text{on } R \times \{0, t_1\}, \tag{2.2}$$

where $M(t_1, \varepsilon)$ *is a positive constant that depends only on* t_1 *and* t_1 *on* $|v_0^\varepsilon(x)|_{L^\infty}, |u_0^\varepsilon(x)|_{L^\infty}, |s_0^\varepsilon(x)|_{L^\infty}$, *for nonnegative integers* $i = 0, 1, 2$. *Moreover if we assume further that the initial data satisfy* (1.5), *then*

$$\lim_{x \to \pm\infty} (v, u, s) = (0, 0, 0), \qquad \lim_{x \to \pm\infty} (v_x, u_x, s_x) = (0, 0, 0) \tag{2.3}$$

uniformly in $t \in [0, t_1]$.

The global existence is obtained by extending the local solution step by step, which is based on the following a priori L^∞ estimate, depending on ε, τ by using the energy method.

LEMMA 2.2. *If the initial data satisfies* (1.5), (1.8), (1.9) *and for any fixed* ε, $\tau > 0$ *the solutions* $(v, u, s) \in C^2$ *of the Cauchy problem* (1.4), (2.1) *exist in* $(-\infty, \infty) \times [0, T]$. *Then the following estimates*

$$|v(x,t)| \leq M(\varepsilon, \tau, T), \qquad |u(x,t)| \leq M(\varepsilon, \tau, T),$$
$$|s(x,t)| \leq M(\varepsilon, \tau, T) \tag{2.4}$$

hold.

Proof. Multiplying $\sigma(v) - cs$ to the first equation in (1.4), u to the second, $s - cv$ to the third, and adding the result, we have

$$\left(\frac{u^2}{2} + \int_0^v \sigma(v)\,dv + \frac{s^2}{2} - cvs\right)_t - \alpha u^2 + \frac{\beta(s-cv)^2}{\tau} + (csu - u\alpha(x))_x$$
$$= \varepsilon\left(\frac{u^2}{2} + \int_0^v \sigma(v)\,dv + \frac{s^2}{2} - cvs\right)_{xx} \tag{2.5}$$
$$- \varepsilon u_x^2 - \varepsilon\sigma'(v)v_x^2 + 2c\varepsilon s_x v_x - \varepsilon s_x^2.$$

Using assumptions (a), (b) in (I), we have $dv \le \sigma \le |\sigma'|_{L^\infty} \cdot v = Mv$ for a positive constant M. So using the estimates (2.2), (2.3) we have

$$|u^2(\cdot,t)|_{L^1(R\times R^+)}, |v^2(\cdot,t)|_{L^1(R\times R^+)}, |s^2(\cdot,t)|_{L^1(R\times R^+)} \le M, \tag{2.6}$$

$$\left|\frac{(s-cv)^2}{\tau}\right|_{L^1(R\times R^+)} \le M, \tag{2.7}$$

$$|\varepsilon u_x^2|_{L^1(R\times R^+)}, |\varepsilon v_x^2|_{L^1(R\times R^+)}, |\varepsilon s_x^2|_{L^1(R\times R^+)} \le M. \tag{2.8}$$

Using the estimates (2.2), (2.3), (2.6), (2.7), (2.8), and condition (1.8) with the method given in [13], we can immediately get the energy inequality

$$\int_{-\infty}^{\infty} (v_x)^2 + (u_x)^2 + (s_x)^2\,dx + \varepsilon\int_0^T \int_{-\infty}^{\infty} (v_{xx})^2$$
$$+ (u_{xx})^2 + (s_{xx})^2\,dxdt$$
$$\le M(\tau, \varepsilon, |v_0^\varepsilon|_{H^1}, |u_0^\varepsilon|_{H^1}, |s_0^\varepsilon|_{H^1}). \tag{2.9}$$

So the estimates (2.4) follow from the estimates (2.6) and (2.9).

Now we give the main theorem of this section:

THEOREM 2.3. *Let the condition in* (I) *or* (II) *hold. Then there is a global solution* $\{v^{\varepsilon,\tau}, u^{\varepsilon,\tau}, s^{\varepsilon,\tau}\}$ *of the Cauchy problem* (1.4), (2.1) *such that the estimates in* (2.6) *hold.*

3. ZERO RELAXATION LIMIT FOR THE CASE (I)

In this section, we are going to study the strong convergence of solutions $v^{\varepsilon,\tau}, u^{\varepsilon,\tau}, s^{\varepsilon,\tau}$ for the Cauchy problem (1.4), (2.1) when all the conditions in this case of (I) are satisfied and the dissipation ε and the relaxation τ

go to zero. Our technique is to apply the method of compensated compactness and use the result on two by two nonlinear elastodynamics in [12].

To prove the convergence of viscosity solutions $(v^\varepsilon, u^\varepsilon)$ of the system

$$\begin{aligned} v_t - u_x &= \varepsilon\, v_{xx} \\ u_t - p(v)_x &= \varepsilon\, u_{xx}, \end{aligned} \tag{3.1}$$

the author in [12] constructed two classes of entropy-entropy flux pairs $\{\eta(v,u), q(v,u)\}$ for the system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - p(v)_x &= 0, \end{aligned} \tag{3.2}$$

where $p(v) = \sigma(v) - c^2 v$. One class of entropy pairs is the Fourier entropy, the other the half plane supported entropy. Both classes of entropy-entropy flux pairs $\{\eta, q\}$ satisfy the following estimates:

(i) $\eta = a^{-1/2}\, O(1), q = a^{1/2}\, O(1)$

(ii) $\eta_u = a^{-1/2}\, O(1), \eta_v = a^{1/2}\, O(1)$

(iii) $\eta_{uu} = a^{-1/2}\, O(1), \eta_{uv} = a^{1/2}\, O(1), \eta_{vv} = a^{3/2}\, O(1),$

where $a^2 = p'(v) = \sigma'(v) - c^2$ and O(1) denotes an L^∞ function. Based on the compactness of $\eta_t + q_x$ in H^{-1}_{loc} for these entropy-entropy flux pairs and the compensated compactness, the strong convergence of the functions $v^\varepsilon, u^\varepsilon$, as ε goes to zero is proven. Therefore we only need to check the compactness of $\eta(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})_t + q(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})_x$ in H^{-1}_{loc} (with respect to the approximate solutions given in (1.4)). After that, we can get the strong convergence of the functions $v^{\varepsilon,\tau}, u^{\varepsilon,\tau}$ as ε and τ go to zero and, through the estimate (2.7), then the convergence of $s^{\varepsilon,\tau}$.

Multiply $\{\eta, q\}$ with the first and second equation in (1.4), where $\eta(v,u)$ is an entropy of (3.2) and satisfies the estimates (i), (ii), (iii), and we get

$$\begin{aligned} \eta_t + q_x = &-c\big(\eta_u\,(s^{\varepsilon,\tau} - c\cdot v^{\varepsilon,\tau})\big)_x + c\eta_{uu}\,u_x\big(s^{\varepsilon,\tau} - c\cdot v^{\varepsilon,\tau}\big) \\ &+ c\eta_{uv}\,v_x\,(s^{\varepsilon,\tau} - c\cdot v^{\varepsilon,\tau}) \\ &- \alpha\eta_u\,u^{\varepsilon,\tau} + \varepsilon\eta_{xx} - \varepsilon\,(H\eta)\big(u_x^{\varepsilon,\tau}, v_x^{\varepsilon,\tau}\big) \\ = &\; I_1 + I_2, \end{aligned}$$

where $H\eta$ is the Hessian and

$$I_1 = -c(\eta_u(s^{\varepsilon,\tau} - c \cdot v^{\varepsilon,\tau})_x + \varepsilon\eta_{xx},$$

$$I_2 = c\,\eta_{uu}\,u_x(s^{\varepsilon,\tau} - c \cdot v^{\varepsilon,\tau}) + c\eta_{uv}\,v_x\,(s^{\varepsilon,\tau} - c \cdot v^{\varepsilon,\tau}) - \alpha\eta_u\,u^{\varepsilon,\tau} - \varepsilon\,(H\eta)(u_x^{\varepsilon,\tau}, v_x^{\varepsilon,\tau}).$$

By the estimates (2.7), (2.8), we know that I_1 is compact in $H_{loc}^{-1}(R \times R^+)$ and I_2 is bounded in $L_{loc}^1(R \times R^+)$ if we let $\tau = O(\varepsilon)$. So $I_1 + I_2$ is compact in $W_{loc}^{-1,\alpha}(R \times R^+)$ for some α between 1 and 2 $(1 < \alpha < 2)$. Moreover $\eta_t + q_x$ is bounded in $W_{loc}^{-1,p}(R \times R^+)$ for some $p > 2$ by using estimate (i). Therefore $\eta_t + q_x$ is compact in $H_{loc}^{-1}(R \times R^+)$. So the proof in the paper [12] gives us the strong convergence of the functions $(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ as $\tau = O(\varepsilon)$ and ε goes to zero. Moreover the estimate (2.7) gives us the strong convergence of the function $s^{\varepsilon,\tau}$ in any compact set of $\Omega \subset R \times R^+$.

THEOREM 3.1. *The solutions* $\{v^{\varepsilon,\tau}, u^{\varepsilon,\tau}, s^{\varepsilon,\tau}\}$ *of the Cauchy problem* (1.4), (2.1) *converge to measurable functions* (v, u, s) *as* $\tau = O(\varepsilon)$ *and* ε *tends to* 0 *under the assumption given in* (I). *Moveover* (u, v) *is a global weak solution of the system* (1.3).

4. THE GLOBAL WEAK SOLUTION FOR THE CASE (II)

In this section, we consider the existence of the global weak solution for the Cauchy problem (1.1), (1.2) in the case (II).

We first give the following L^2 estimate of the viscosity solutions to the Cauchy problem (1.4) ($\beta = 0$), (2.1), which is also the crux to get a uniform bound given (2.4) when we follow the method given in Section 2.

Multiplying $\sigma(v)g(s) - cs$ to the first equation in (1.4), u to the second, and $g'(s) \cdot \int_0^v \sigma(v)dv - cv + \gamma s$ to the third equation, finally adding the result, we obtain

$$\begin{aligned}
&\left(\frac{u^2}{2} + \int_0^v \sigma(v)dv \cdot g(s) - csv + \frac{\gamma s^2}{2}\right)_t + (csu - \sigma(v)g(s) \cdot u)_x + \alpha u^2 \\
&\quad = \varepsilon\left(\frac{u^2}{2} + \int_0^v \sigma(v)dv \cdot g(s) - csv + \frac{\gamma s^2}{2}\right)_{xx} \\
&\qquad - \varepsilon\Bigg(\sigma'(v)g(s)v_x^2 + 2(\sigma(v)g'(s) - c)s_x v_x + u_x^2 \\
&\qquad\qquad + \int_0^v \sigma(v)dv \cdot g''(s)s_x^2 + \gamma s_x^2\Bigg),
\end{aligned} \tag{4.1}$$

where γ is a large positive constant.

By the condition $|s_0(x)|_{H^1_{loc}(R)} \le M$ and $\lim_{x\to\pm\infty} s_0(x) = 0$, we get the uniform boundedness of $s_0(x)$ in $L^\infty(R)$. So the functions $s_0^\varepsilon(x)$ satisfy $|s_0^\varepsilon(x)|_{L^\infty} \le M$, $|\varepsilon^{1/2}(s_0^\varepsilon(x))_x|_{L^\infty} \le M, |\varepsilon(s_0^\varepsilon(x))_{xx}|_{L^\infty} \le M$. The third equation in (1.4)

$$s_t = \varepsilon\, s_{xx}$$

gives us the same estimates

$$|s^\varepsilon(x,t)|_{L^\infty} \le M, |\varepsilon^{1/2}(s^\varepsilon(x,t))_x|_{L^\infty} \le M, |\varepsilon(s^\varepsilon(x,t))_{xx}|_{L^\infty} \le M,$$

$$\text{and } |s^\varepsilon(\cdot,t)|_{H^1_{loc}(R)} \le M. \tag{4.2}$$

So $g(s)$, $g'(s)$, $g''(s)$ are all bounded. Moreover $|\int_0^v \sigma(v)dv| \le Mv^2$, therefore integrating (4.1) in $R\times R^+$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{u^2}{2} + \int_0^v \sigma(v)dv\cdot g(s) - csv + \frac{\gamma s^2}{2}\right)dx \\
&\quad + \int_0^T\int_{-\infty}^{\infty} c_1 v_x^2 + u_x^2 + c_2 s_x^2\, dxdt + \alpha\int_0^T\int_{-\infty}^{\infty} u^2\, dxdt \\
&\quad \le \int_{-\infty}^{\infty}\left(\frac{u_0^2}{2} + \int_0^{v_0}\sigma(v_0)dv\cdot g(s_0) - cs_0v_0 + \frac{\gamma s_0^2}{2}\right)dx \\
&\qquad + M\int_0^T\int_{-\infty}^{\infty} v^2\, dxdt \\
&\quad \le M_1 + M_2\int_0^T\int_{-\infty}^{\infty} v^2\, dxdt,
\end{aligned} \tag{4.3}$$

for some positive constants c_1, c_2 and M.

By applying Bellman's inequality to (4.3), we get

$$\int_{-\infty}^{\infty} v^2\, dx \le M(T), \qquad \int_{-\infty}^{\infty} u^2\, dx \le M(T), \qquad \int_{-\infty}^{\infty} s^2\, dx \le M(T), \tag{4.4}$$

and

$$\varepsilon\int_0^T\int_{-\infty}^{\infty} v_x^2 + u_x^2 + s_x^2\, dxdt \le M(T). \tag{4.5}$$

To get a weak solution for the Cauchy problem (1.1), (1.2), since we already know the strong convergence of $s^\varepsilon(x,t)$, we only consider the strong convergence of $(v^\varepsilon, u^\varepsilon)$, following the framework given in [12].

We first let s = constant and consider the entropy-entropy flux pairs for the system

$$\begin{aligned} &v_t - u_x = 0 \\ &u_t - \sigma_x(x)g(s) = 0. \end{aligned} \tag{4.6}$$

We make the transformation

$$x = \sqrt{g(s)}\,\bar{x}, \qquad t = t, \qquad v = v, \qquad u = \sqrt{g(s)}\,\bar{u} \tag{4.7}$$

and so (4.6) is rewritten as

$$\begin{aligned} v_t - \bar{u}_{\bar{x}} &= 0 \\ \bar{u}_t - \sigma(v)_{\bar{x}} &= 0. \end{aligned} \tag{4.8}$$

We construct the same two classes of entropy-entropy flux pairs for the system (4.8) as done in [12], one being the Fourier entropy and the other being the half plane supported entropy, which both satisfy the estimates (i), (ii), and (iii) given in Section 3. Thus we can get the entropy-entropy flux pair (η, q) for system (4.6) satisfying the estimates

$$\begin{aligned} &\text{(i)} \quad \eta(v,\bar{u}) = a^{-1/2}(v)\,O(1), q = a^{1/2}O(1) \\ &\text{(ii)} \quad \eta_{\bar{u}} = a^{-1/2}\,O(1), \eta_v = a^{1/2}\,O(1) \\ &\text{(iii)} \quad \eta_{\bar{u}\bar{u}} = a^{-1/2}\,O(1), \eta_{\bar{u}v} = a^{1/2}\,O(1), \eta_{vv} = a^{3/2}\,O(1) \end{aligned} \tag{4.9}$$

and

$$(q_v, q_{\bar{u}}) = (\eta_v, \eta_{\bar{u}}) \begin{pmatrix} 0 & -1 \\ -\sigma'(v) & 0 \end{pmatrix}, \tag{4.10}$$

where $a^2 = \sigma'(v)$ is bounded for $v \in R$ by the conditions on σ given in assumptions (2)(a), (b), (c) in the Introduction.

So we have from (4.10) (consider here s as a variable)

$$q_v(v,u,s) = -\sigma'(v)\sqrt{g(s)}\;\eta_u, \qquad q_u(v,u,s) = -\eta_v \cdot \frac{1}{\sqrt{g(s)}} \tag{4.11}$$

$$\eta_s = \eta_{\bar{u}} \cdot u \cdot \left(\frac{1}{\sqrt{g(s)}}\right)' = -\frac{1}{2}\eta_u \frac{ug'(s)}{g(s)}, \qquad q_s = -\frac{1}{2}q_u \frac{ug'(s)}{g(s)}. \tag{4.12}$$

Multiplying (η_v, η_u, η_s) with the system (1.4), we have

$$\begin{aligned} \eta_t &+ \left(\sqrt{g(s)}\,q\right)_x - \left(\sqrt{g(s)}\right)_x q - \sqrt{g(s)}\,q_s\,s_x - \sigma(v)g'(s)\eta_u s_x \\ &+ c\eta_u s_x + \alpha u \eta_u \\ &= \varepsilon(\eta_v v_x)_x + \varepsilon(\eta_u u_x)_x \\ &\qquad - \varepsilon\left(\eta_{vv}v_x^2 + \eta_{uu}u_x^2 + 2\eta_{vu}v_x u_x + \eta_{vs}v_x s_x + \eta_{su}s_x u_x\right) + \varepsilon\eta_x s_{xx}. \end{aligned}$$

Since from the estimates (4.5) and (4.2)

$$|\sigma(v) g'(s) \eta_u s_x|_{L^1_{loc}(R\times R^+)} \le M|v|_{L^2_{loc}(R\times R^+)} |S_x|_{L^2_{loc}(R\times R^+)} \le M(T)$$

$$\varepsilon|\eta_{ss} s_x^2|_{L^1_{loc}(R\times R^+)} \le \varepsilon M|u^2 s_x^2|_{L^1_{loc}(R\times R^+)} \le |M_1|u^2|_{L^1_{loc}(R\times R^+)} \le M_2(T)$$

$$\varepsilon(\eta_{vs} v_x s_x + \eta_{su} s_x u_x) \le \varepsilon M\big(|u v_x s_x| + |u s_x u_x|\big)$$

$$\le M_1\Big(\varepsilon|v_x^2|_{L^1_{loc}(R\times R^+)} + |u^2|_{L^1_{loc}(R\times R^+)}\Big) \le M_2(T)$$

for some positive constants M, M_1, $M_2(T)$, we have

$$\eta_t + \left(\sqrt{g(s)}\, q\right)_x = I_1 + I_2,$$

where

$$I_1 = \varepsilon(\eta_v v_x)_x + \varepsilon(\eta_u u_x)_x \text{ is compact in } H^{-1}_{loc}(R\times R^+)$$

and

$$I_2 = \left(\sqrt{g(s)}\right)_x q + \sqrt{g(s)}\, q_s s_x + \sigma(v) g'(s)\eta_u s_x - c\eta_u s_x - \alpha u \eta_u$$

$$+ \varepsilon \eta_s s_{xx} - \varepsilon\Big(\eta_{vv} v_x^2 + \eta_{uu} u_x^2 + 2\eta_{vu} v_x u_x + \eta_{vs} v_x s_x + \eta_{su} s_x u_x\Big)$$

is bounded in $L^1_{loc}(R\times R^+)$.

Therefore $I_1 + I_2$ is compact in $W^{-1,k}_{loc}(R\times R^+)$ for k between one and two ($1 < k < 2$). However, $\eta_t + (\sqrt{g(s)}\, q)_x$ is bounded in $W^{-1,p}_{loc}(R\times R^+)$ for $p > 2$ by the estimates in (4.9). So $\eta_t + (\sqrt{g(s)}\, q)_x$ is compact in $H^{-1}_{loc}(R\times R^+)$. By the convergence result in [12, Theorem 11], $(v^\varepsilon, u^\varepsilon)$ converges almost everywhere in a compact set $\Omega \subset R\times R^+$ to a L^2 bounded function pair (u, v). So we have proved the following theorem:

THEOREM 4.1. *If the conditions in case* (II) *are satisfied, then there exists a global solution for the Cauchy problem* (1.1), (1.2).

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