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SINGULAR LIMITS FOR INHOMOGENEOUS EQUATIONS OF ELASTICITY*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract Based on the framework introduced in [4] or [5], the singular limits of stiff relaxation and dominant diffusion for the Cauchy problem of inhomogeneous equations of elasticity is studied. We are able to reach equilibrium even though the nonlinear stress term is not strictly increasing.

Key words relaxation limit; equations of elasticity; compensated compactness; invariant regions theory

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1 Introduction

In this paper, we will study the singular limits of stiff relaxation and dominant diffusion for the Cauchy problem of inhomogeneous equations of elasticity with relaxation and diffusion

$$\begin{cases} v_t - u_x + g(v,u) = \varepsilon v_{xx}, \\ u_t - s(v)_x + f(v,u) + \dfrac{u-h(v)}{\tau} = \varepsilon u_{xx}, \end{cases} \tag{1.1}$$

with bounded initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)), \tag{1.2}$$

where v denotes the strain, the nonlinear function $s(v)$ is the stress and u the velocity. The second equation in (1.1) contains a relaxation mechanism with $h(v)$ as the equilibrium value for u, τ the relaxation time and ε is the diffusion coefficient.

In the literature relaxation limits without coupled viscosity have been studied. Noteworthy references are Chen, Levermore and Liu [2] and Natalini [6] where the relaxation limit for (1.1)

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(with $g = f = 0$ and assuming $s'(v) \geq c > 0, vs''(v) > 0$ for all $v \in R \setminus \{0\}$) was first studied without viscosity. When you have no viscosity a so called stability condition is needed, in our case it is $|h'(v)| \leq \sqrt{s'(v)}$. When one couples viscosity with the relaxation limit it acts as the stabilizing mechanism.

2 Main Theorems

In this paper, we assume that the functions $s(v), h(v), f(v,u), g(v,u)$ satisfy the following conditions:

(C_1) $s'(v) = a^2(v), a(v) \geq 0$, but $\text{meas}\{v : a(v) = 0\} = 0; a'(v)$ is continuous on $[v_-, v_+]$ and $a'(v) < 0$ when $-\infty < v \leq v_-, a'(v) > 0$ when $v_+ \leq v < \infty$.

(C_2) $a'(v)$ changes sign on $v_i, i = 1, 2, \cdots; m : v_- = v_1 < v_2 < \cdots < v_m = v_+$.

Clearly m is odd. So let $m = 2n + 1$ and

(C_3) $a'(v)(v - v_i) \geq 0$ for $v \in [v_{i-1}, v_{i+1}]$ when i is odd; $a'(v)(v - v_i) \leq 0$ for $v \in [v_{i-1}, v_{i+1}]$ when i is even.

Moreover,

(C_4) $a(v)g(v,u)\text{sgn}(v-v_i)+f(v,u)\text{sgn}(u) \geq 0$ for $v \in [v_{i-1}, v_{i+1}]$ when i is odd; $a(v)g(v,u)\text{sgn}(v - v_i) + f(v,u)\text{sgn}(u) \leq 0$ for $v \in [v_{i-1}, v_{i+1}]$ when i is even.

Let w_i and z_i be two families of Riemann invariants:

$$w_i(v,u) = \int_{v_i}^{v} a(s)\mathrm{d}s - u, \tag{2.1}$$

$$z_i(v,u) = \int_{v_i}^{v} a(s)\mathrm{d}s + u, \tag{2.2}$$

and v^-, v^+ two numbers determined by

$$\int_{v_{2n+1}}^{v^+} a(s)\mathrm{d}s = N, \quad \int_{v_1}^{v^-} a(s)\mathrm{d}s = -N + \sum_{j=1}^{2n}(-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s,$$

for a sufficiently large constant N.

Let the region R_1 be encircled by the lines (see Fig. 1):

$$w_{2n+1}(v,u) = N, \quad z_{2n+1}(v,u) = N, \tag{2.3}$$

when $v \geq v_{2n+1}$,

$$\begin{cases} w_{2i} = -N + \sum\limits_{j=2i}^{2n}(-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s \\ z_{2i} = -N + \sum\limits_{j=2i}^{2n}(-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s, \end{cases} \tag{2.4}$$

when $v \in [v_{2i}, v_{2i+1}], i = 1, 2, \cdots, n$, and

$$\begin{cases} w_{2i-1} = N - \sum\limits_{j=2i-1}^{2n} (-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s, \\ z_{2i-1} = N - \sum\limits_{j=2i-1}^{2n} (-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s, \end{cases} \tag{2.5}$$

when $v \in [v_{2i-1}, v_{2i}], i = 1, 2, \cdots, n$,

$$\begin{cases} w_1 = -N + \sum_{j=1}^{2n} (-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s, \\ z_1 = -N + \sum_{j=1}^{2n} (-1)^j \int_{v_j}^{v_{j+1}} a(s)\mathrm{d}s, \end{cases} \tag{2.6}$$

when $v \leq v_1$.

Based on the invariant regions theory introduced in [1], we have the following theorem.

Theorem 1 If the conditions (C_1)–(C_4) are satisfied and the curve $u = h(v)$ passes through the points $(v^-, 0), (v^+, 0)$ and is inside the region R_1 for $v \in [v^-, v^+]$, then R_1 is an invariant regions (see Fig. 1).

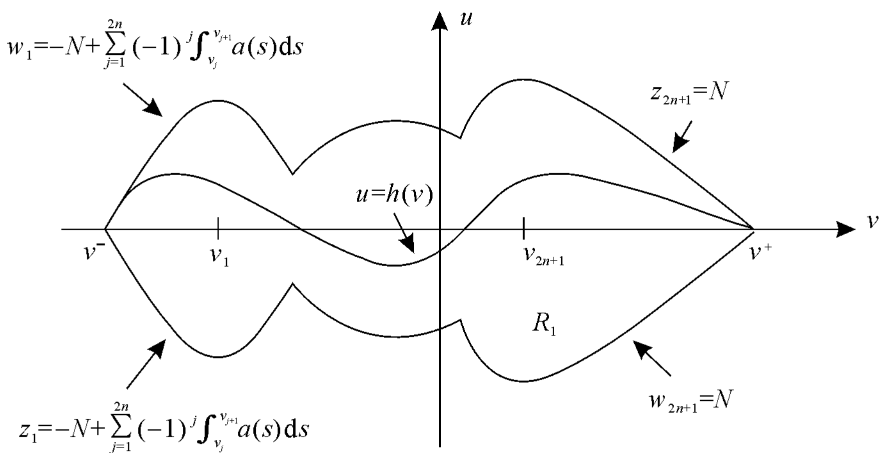

Fig. 1

Using the general compact framework given in [4] or [5], we have the following theorem.

Theorem 2 Let all conditions in Theorem 1 be satisfied and the initial data $(v_0(x), u_0(x)) \in R_1$, then for any fixed ε and τ, there exists a unique, global solution $(v^{\varepsilon,\tau}(x,t), u^{\varepsilon,\tau}(x,t))$ of the Cauchy problem (1.1)–(1.2) satisfying the uniformly bounded estimate

$$|(v^{\varepsilon,\tau}(x,t), u^{\varepsilon,\tau}(x,t))| \leq M, \qquad (x,t) \in R \times [0,T], \tag{2.7}$$

for any given time T, where M is a positive constant. Furthermore, if $\tau = o(\varepsilon)$ as $\varepsilon \to 0$, then there exists a subsequence $(v^{\varepsilon_k}, u^{\varepsilon_k})$ of $(v^{\varepsilon,\tau}(x,t), u^{\varepsilon,\tau}(x,t))$ converging strongly to functions (v, u) as $\varepsilon_k \to 0$, which are the equilibrium states uniquely determined by (E_1)–(E_2):

(E_1) $u(x,t) = h(v(x,t))$, for almost all $(x,t) \in R \times (0,T]$;

(E_2) $v(x,t)$ is the unique L^∞ entropy solution of the Cauchy problem:

$$v_t - h(v)_x + g(v, h(v)) = 0, \quad v|_{t=0} = v_0(x). \tag{2.8}$$

Proof The estimate (2.7) can be obtained directly by Theorem 1. The existence and uniqueness of the solution of (1.1)–(1.2) are based on the estimate (2.7) and the general parabolic equation theory.

To prove the compactness in the second part in Theorem 2, we choose a large constant C_1 such that the function

$$p(v,u)=\frac{u^2}{2}-h(v)u+\frac{C_1v^2}{2}$$

satisfies

$$p_{vv}(v,u)v_x^2+2p_{vu}(v,u)v_xu_x+p_{uu}(v,u)u_x^2\geq C_2(v_x^2+u_x^2), \tag{2.9}$$

for some constant $C_2>0$ since (v,u) is bounded.

Multiplying system (1.1) by (p_v,p_u), we have from (2.9) that

$$\begin{aligned}
&p(v,u)_t-p_v(v,u)u_x-p_u(v,u)s(v)_x+p_v(v,u)g(v,u)\\
&+p_u(v,u)f(v,u)+\frac{(u-h(v))^2}{\tau}\\
=\;&\varepsilon p_{xx}(v,u)-\varepsilon p_{vv}v_x^2-2\varepsilon p_{vu}v_xu_x-\varepsilon p_{uu}u_x^2
\end{aligned} \tag{2.10}$$

or

$$\begin{aligned}
&p(v,u)_t-p_v(v,u)u_x-p_u(v,u)s(v)_x+p_v(v,u)g(v,u)+p_u(v,u)f(v,u)\\
&+\frac{(u-h(v))^2}{\tau}+\varepsilon C_2(v_x^2+u_x^2)\leq\varepsilon p_{xx}(v,u).
\end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}
-p_v(v,u)u_x&=(h'(v)u-C_1v)u_x\\
&=h'(v)u(u-h(v))_x+(h'(v))^2(u-h(v))v_x\\
&\quad+(h'(v))^2h(v)v_x-C_1v(u-h(v))_x-C_1vh'(v)v_x\\
&=(h'(v)u(u-h(v)))_x-h''(v)uv_x(u-h(v))-h'(v)u_x(u-h(v))\\
&\quad+(h'(v))^2(u-h(v))v_x+(h'(v))^2h(v)v_x-C_1(v(u-h(v)))_x\\
&\quad+C_1v_x(u-h(v))-C_1vh'(v)v_x,
\end{aligned} \tag{2.12}$$

and

$$-p_u(v,u)s(v)_x=-(u-h(v))s(v)_x=-s'(v)(u-h(v))v_x, \tag{2.13}$$

it follows from (2.11) that

$$p(v,u)_t+q(v,u)_x+\frac{(u-h(v))^2}{2\tau}+\varepsilon C_2(v_x^2+u_x^2)-\tau C_3(v_x^2+u_x^2)\leq\varepsilon p_{xx}(v,u), \tag{2.14}$$

for a function q and a positive constant C_3 depending on the bounds of u,v.

Multiplying (2.14) by a suitable nonnegative test function and then integrating by parts on R_+^2, we get the estimates

$$\left\|\left(\varepsilon v_x^2,\varepsilon u_x^2,\frac{(u-h(v))^2}{\tau}\right)\right\|_{L^1_{\rm loc}(R_+^2)}\leq M, \tag{2.15}$$

provided $2\tau C_3\leq\varepsilon C_2$.

Using the estimates in (2.15) and the general compact framework in [4], we can end the proof of Theorem 2.

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