Global L^∞ Solutions to System of Isentropic Gas Dynamics in a Divergent Nozzle with Friction

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Abstract

In this paper, we study the global L^∞ entropy solutions for the Cauchy problem of system of isentropic gas dynamics in a divergent nozzle with a friction (1.1) with bounded initial date (1.2). Especially when the adiabatic exponent $\gamma = 3$, we apply for the maximum principle to obtain the L^∞ estimates $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(t)$ and $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(t)$ for the viscosity solutions $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon})$ of the Cauchy problem (1.9) and (1.10), where w and z are the Riemann invariants of (1.1), and $B(t)$ is a nonnegative bounded function for any finite time t. This work, in the special case $\gamma \geq 3$, extends the previous works [Lu, Nonlinear Analysis, Real World Applications, 39: 418-423, 2018], which provided the global entropy solutions for the Cauchy problem (1.1) and (1.2) with the restriction $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq 0$ or $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq 0$.

Key Words: Global L^∞ solution; isentropic gas dynamics; source terms; flux approximation; compensated compactness
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1 Introduction

The following system of isentropic gas dynamics in a divergent nozzle with a friction, whose physical phenomena called "choking or choked flow", occurs in the nozzle (see [Sh, Ts4] for the details),

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u \\ \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 - \alpha \rho u|u|, \end{cases} \tag{1.1}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ is a slowly variable cross section area at x in the nozzle and α denotes a friction constant. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, where $\gamma > 1$ is the adiabatic exponent.

The golobal entropy solutions for the Cauchy problem (1.1) with bounded initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

was first studied in [Ts4] for the usual gases $1 < \gamma \leq \frac{5}{3}$, and later, by the author in [Lu4] for any adiabatic exponent $\gamma > 1$, provided that the initial data are bounded and satisfy the strong restriction condition $z_0(\rho_0(x), u_0(x)) \leq 0$.

It is well-known that after we have a method to obtain the global existence of solutions for the Cauchy problem of the following homogeneous system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \end{cases} \tag{1.3}$$

with the bounded initial data (1.2), the unique difficulty to treat the inhomogeneous system (1.1) is to obtain the a-priori L^∞ estimate of the approximation solutions of (1.1), for instance, the a-priori L^∞ estimate of the viscosity solutions for the Cauchy problem of the parabolic system

$$\begin{cases} \rho_t + +(\rho u)_x = -\frac{a'(x)}{a(x)}\rho u + \varepsilon \rho_{xx} \\ \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 - \alpha \rho u|u| + \varepsilon (\rho u)_{xx} \end{cases} \tag{1.4}$$

with the initial data (1.2).

When $a'(x) = 0$, (1.1) is the river flow equations, a shallow-water model describing the vertical depth ρ and mean velocity u, where $\alpha \rho u|u|$ corresponds physically to a friction term and α is a nonnegative constant. This kind of inhomogeneous systems is simple since the source terms have, in some senses, the symmetric behavior. We may introduce the Riemann invariants (w, z) of system (1.3) to rewrite (1.1) as the following symmetric, coupled system

$$\begin{cases} w_t + \lambda_2 w_x = \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 - \frac{1}{2}\alpha(w - z)|u| \\ \\ z_t + \lambda_1 z_x = \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 - \frac{1}{2}\alpha(z - w)|u|, \end{cases} \tag{1.5}$$

where

$$z(\rho, u) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds - u, \quad w(\rho, u) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds + u \tag{1.6}$$

and c is a constant. We can apply for the maximum principle directly to (1.5) to obtain the necessary a-priori L^∞ estimates $w(\rho^\varepsilon, u^\varepsilon) \le M$ and $z(\rho^\varepsilon, u^\varepsilon) \le N$, for two suitable constants M, N (see ([KL]) for the details).

When $\alpha = 0$, i.e., the nozzle flow without friction, system (1.1) was well studied in ([Ts1, Ts2, Ts4, Lu3, LG]). Roughly speaking, the technique, introduced in these papers, is to control the super-linear source terms $\frac{a'(x)}{a(x)}\rho u$ and $\frac{a'(x)}{a(x)}\rho u^2$ by the flux functions ρu and $\rho u^2 + P(\rho)$ in (1.1) and to deduce a upper bound of w or z by a bounded nonegative function $B(x)$, which depends on the function $a(x)$.

When $a'(x) \not\equiv 0$ and $\alpha \neq 0$, both the above techniques do not work because the flux functions can not be used to control the super-linear friction source terms $\alpha\rho u|u|$, and the functions $\frac{a'(x)}{a(x)}\rho u$ destroyed the symmetry of the Riemann invariants (w, z). In fact, we may copy the method given in ([Ts4, Lu4]) to obtain the following process.

First, to avoid the singularity of the flux function ρu^2 near the vacuum $\rho = 0$, we still use the technique of the δ-flux-approximation given in [Lu2] and introduce the sequence of systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x)(\rho - 2\delta)u \\ \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x)(\rho - 2\delta)u^2 - \alpha\rho u|u| \end{cases} \tag{1.7}$$

to approximate system (1.1), where $A(x) = -\frac{a'(x)}{a(x)}, \delta > 0$ denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^\rho \frac{t - 2\delta}{t} P'(t) dt. \tag{1.8}$$

Second, we add the viscosity terms to the right-hand side of (1.7) to obtain the following parabolic system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = A(x)(\rho - 2\delta)u + \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = A(x)(\rho - 2\delta)u^2 - \alpha\rho u|u| + \varepsilon(\rho u)_{xx} \end{cases} \tag{1.9}$$

with initial data

$$(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) = (\rho_0(x) + 2\delta, u_0(x)), \tag{1.10}$$

where $(\rho_0(x), u_0(x))$ are given in (1.2).

Now we multiply (1.9) by (w_ρ, w_m) and (z_ρ, z_m), respectively, where (w, z) are given in (1.6), to obtain

$$\begin{aligned} & w_t + \lambda_2^\delta w_x \\ & = \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ & + A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha u|u| \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} & z_t + \lambda_1^\delta z_x \\ & = \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ & + A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha u|u|, \end{aligned} \tag{1.12}$$

where

$$\lambda_1^\delta = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2^\delta = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)} \tag{1.13}$$

are two eigenvalues of the approximation system (1.7).

It is obvious that the terms $A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}$ in (1.11) and (1.12) are not symmetric with respect to the Riemann invariants w, z. However, with the strong restriction $z_0(\rho_0(x), u_0(x)) \leq 0$ on the initial data, in ([Ts4, Lu4]), we may obtain the uniformly upper bounds of z and w by using the maximum principle. In some senses, it is similar to obtain the estimate $u \leq 0$ for the following scalar equation

$$u_t + f(u)_x + S(u,x,t)u = \varepsilon u_{xx} \tag{1.14}$$

for any local bounded function $S(u,x,t)$ when the initial data $u_0(x) \leq 0$.

In this paper, we will remove the condition $z_0(\rho_0(x), u_0(x)) \leq 0$, and prove the uniformly upper bound of z and w by a bounded nonnegative function $B(t)$ of t, and obtain the global existence theorem of the entropy solutions for the Cauchy problem (1.1) and (1.2) as follows:

Theorem 1 *(I). Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma \geq 3$, the function $A(x) \leq 0$ (or $A(x) \geq 0$) be bounded. Then there exists a bounded, nonegative function $B(t)$ such that $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(t), z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(t)$, where $B(t)$ depends only on the bound of the initial data.*

(II). Under the conditions in (I), there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as δ, ε tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2)

2 Proof of Theorem 1.

In this section, we shall prove Theorem 1.

When $A(x) \leq 0$, we rewrite systems (1.11) and (1.12) as follows:

$$
\begin{aligned}
&w_t + \lambda_2^\delta w_x + \tfrac{1}{2}(-A(x)(\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha|u|)(w - z) \\
&= \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
&z_t + \lambda_1^\delta z_x + \tfrac{1}{2}\alpha|u|(z - w) - A(x)(\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}(\textstyle\int_c^\rho \tfrac{\sqrt{P'(s)}}{s}ds - z) \\
&= \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2.
\end{aligned} \tag{2.2}
$$

Letting $v_1 = w - B(t), v_2 = z - B(t)$, we have from (2.1) and (2.2) that

$$
\begin{aligned}
&v_{1t} + B'(t) + \lambda_2^\delta v_{1x} + \tfrac{1}{2}(-A(x)(\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha|u|)(v_1 - v_2) \\
&= \varepsilon v_{1xx} + \tfrac{2\varepsilon}{\rho}\rho_x v_{1x} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2
\end{aligned} \tag{2.3}
$$

and

$$v_{2t}+B'(t)+\lambda_1^\delta v_{2x}+\tfrac{1}{2}\alpha|u|(v_2-v_1)+A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}v_2$$

$$-A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_c^\rho \tfrac{\sqrt{P'(s)}}{s}ds+B(t)A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} \tag{2.4}$$

$$=\varepsilon v_{2xx}+\tfrac{2\varepsilon}{\rho}\rho_x v_{2x}-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2.$$

Choose $B(t)$ such that $B'(t)\ge 0, 0\le B(t)\le\beta$, where β be a positive constant. Then when $\int_{2\delta}^\rho \frac{\sqrt{P'(s)}}{s}ds\ge\beta$ or ρ is large since $\gamma\ge 3$, we have

$$B'(t)-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_c^\rho\frac{\sqrt{P'(s)}}{s}ds+B(t)A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\ge 0. \tag{2.5}$$

When $\int_c^\rho \frac{\sqrt{P'(s)}}{s}ds<\beta$ or ρ is small, we can always choose a $B(t)$, where $B'(t)$ is suitable large, such that (2.5) is also true. Therefore, we have the following inequalities from (2.3) and (2.4)

$$v_{1t}+\lambda_2^\delta v_{1x}+\tfrac{1}{2}(-A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}+\alpha|u|)(v_1-v_2)\le\varepsilon v_{1xx}+\tfrac{2\varepsilon}{\rho}\rho_x v_{1x} \tag{2.6}$$

and

$$v_{2t}+\lambda_1^\delta v_{2x}+\tfrac{1}{2}\alpha|u|(v_2-v_1)+A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}v_2\le\varepsilon v_{2xx}+\tfrac{2\varepsilon}{\rho}\rho_x v_{2x}. \tag{2.7}$$

If the initial data satisfy $z_0(\rho_0(x),u_0(x))\le B(0), w_0(\rho_0(x),u_0(x))\le B(0)$, then $v_{i0}(\rho_0(x),u_0(x))\le 0, i=1,2$, we can apply for the maximum principle given in ([Lu5]) to (2.6) and (2.7) to obtain $v_i(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))\le 0, i=1,2$, where the coefficient $A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$ before v_2 is only necessary to be local bounded.

In fact, we rewrite inequalities (2.6) and (2.7) as follows:

$$v_{1t}+\lambda_2^\delta v_{1x}+c_1(x,t)(v_1-v_2)\le\varepsilon v_{1xx}+\tfrac{2\varepsilon}{\rho}\rho_x v_{1x} \tag{2.8}$$

and

$$v_{2t}+\lambda_1^\delta v_{2x}+c_2(x,t)(v_2-v_1)+c_3(x,t)v_2\le\varepsilon v_{2xx}+\tfrac{2\varepsilon}{\rho}\rho_x v_{2x}. \tag{2.9}$$

where $c_1(x,t)=\frac{1}{2}(-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+\alpha|u|)\ge 0, c_2(x,t)=\frac{1}{2}\alpha|u|\ge 0, c_3(x,t)=A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$.

Make a transformation

$$v_1=(\bar{v_1}+\frac{N(x^2+cLe^t)}{L^2})e^{\beta t},\quad v_2=(\bar{v_2}+\frac{N(x^2+cLe^t)}{L^2})e^{\beta t}, \tag{2.10}$$

where L, c, N, β are positive constants and N is the upper bound (locally) of v_1, v_2, and β the upper bound (locally) of $c_3(x,t)$ on $R \times [0,T]$ (N, β can be obtained by the local existence). The function $\bar{v_1}, \bar{v_1}$, as are easily seen, satisfy the inequalities

$$\begin{cases} \bar{v}_{1t} + \lambda_2^\delta \bar{v}_{1x} + c_1(x,t)(\bar{v_1} - \bar{v_2}) + \beta(\bar{v_1} + \dfrac{N(x^2 + cLe^t)}{L^2}) \\ \qquad +(cLe^t + 2\lambda_2^\delta x - 2\varepsilon)\dfrac{N}{L^2} \leq \varepsilon \bar{v}_{1xx} + \dfrac{2\varepsilon}{\rho}\rho_x \bar{v}_{1x}, \\ \\ \bar{v}_{2t} + \lambda_1^\delta \bar{v}_{2x} + c_2(x,t)(\bar{v_2} - \bar{v_1}) + (\beta + c_3(x,t))(\bar{v_2} + \dfrac{N(x^2 + cLe^t)}{L^2}) \\ \qquad +(cLe^t + 2\lambda_1^\delta x - 2\varepsilon)\dfrac{N}{L^2} \leq \varepsilon \bar{v}_{2xx} + \dfrac{2\varepsilon}{\rho}\rho_x \bar{v}_{2x} \end{cases} \tag{2.11}$$

resulting from (2.8) and (2.9). Moreover

$$\begin{cases} \bar{v_1}(x,0) = w(x,0) - B(0) - \dfrac{N(x^2 + cL)}{L^2} < 0, \\ \bar{v_2}(x,0) = z(x,0) - B(0) - \dfrac{N(x^2 + cL)}{L^2} < 0, \end{cases} \tag{2.12}$$

$$\bar{v_1}(+L,t) < 0,\ \bar{v_1}(-L,t) < 0,\ \bar{v_2}(+L,t) < 0,\ \bar{v_2}(-L,t) < 0. \tag{2.13}$$

From (2.11),(2.12) and (2.13), we have

$$\bar{v_1}(x,t) < 0, \quad \bar{v_1}(x,t) < 0, \quad \text{on} \quad (-L,L) \times (0,T). \tag{2.14}$$

If (2.14) is violated at a point $(x,t) \in (-L,L) \times (0,T)$, let $\bar{t}$ be the least upper bound of values of t at which $\bar{v_1} < 0$ (or $\bar{v_2} < 0$); then by the continuity we see that $\bar{v_1} = 0, \bar{v_2} \leq 0$ at some points $(\bar{x}, \bar{t}) \in (-L,L) \times (0,T)$. So

$$\bar{v}_{1t} \geq 0, \quad \bar{v}_{1x} = 0, \quad \varepsilon \bar{v}_{1xx} \geq 0, \quad \text{at} \quad (\bar{x}, \bar{t}). \tag{2.15}$$

If we choose sufficiently large constant c, which may depend on $,\varepsilon\delta$, such that

$$cLe^t + \lambda_2^\delta x - 2\varepsilon > 0 \quad \text{on} \quad (-L,L) \times (0,T), \tag{2.16}$$

the first equation in (2.11) gives a conclusion contradicting (2.15). So (2.14) is proved. Therefore, for any point $(x_0,t_0) \in (-L,L) \times (0,T)$,

$$\bar{v_1}(x_0,t_0) < \frac{N(x_0^2 + cLe_0^t)}{L^2}, \quad \bar{v_2}(x_0,t_0) < \frac{N(x_0^2 + cLe_0^t)}{L^2}, \tag{2.17}$$

which gives the desired estimates

$$w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(t), \quad z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(t), \tag{2.18}$$

if we let L go to infinity. Similiarly, we can obtain the same result for the case of $A(x) \ge 0$, so Part (I) in Theorem 1 is proved.

From the upper estimates given in (2.18), we deduce the estimates $2\delta\rho^{\delta,\varepsilon} \le C(t)$ and $|u^{\delta,\varepsilon}| \le C(t)$ for a suitable nonnegative, bound function $C(t)$. Since $\gamma \ge 3$, the pointwise convergence of a subsequence of $\rho^{\delta,\varepsilon}$ and $u^{\delta,\varepsilon}$, as δ, ε tend to zero, follows directly by using the compact results given in ([LPT]). Theorem 1 is proved.

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