Viscosity-flux approximation method to inhomogeneous system of isentropic gas dynamics ☆

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Abstract

In this paper, we apply the maximum principle with the viscosity-flux approximation to obtain the a-priori L^∞ estimates on the Riemann invariants of (1.1), $\Delta(\nu^{\xi,\varepsilon}, u^{\xi,\varepsilon}) \leq \omega(x-kt)$ and $\Upsilon(\nu^{\xi,\varepsilon}, u^{\xi,\varepsilon}) \leq \omega(x-kt)$ for the solutions $(\nu^{\xi,\varepsilon}, u^{\xi,\varepsilon})$ of the Cauchy problem (1.6) and (1.7), where $\omega(x-kt)$ is a nonnegative bounded function, and to prove the global existence of the L^∞ entropy solutions for the Cauchy problem of inhomogeneous system of isentropic gas dynamics (1.1) with arbitrary bounded initial data (1.2).

Keywords: Global L^∞ solution, inhomogeneous system of isentropic gas dynamics, flux approximation, compensated compactness
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1. Introduction

In this paper we studied the Cauchy problem of the following system of isentropic gas dynamics with a moving source term

$$\begin{cases} \nu_t + (\nu u)_x = \alpha(x-kt)\nu u \\ \\ (\nu u)_t + (\nu u^2 + \sigma(\nu))_x = \alpha(x-kt)\nu u^2 - \zeta(x,t)\nu u|u|, \end{cases} \tag{1.1}$$

where u is the velocity of gas, ν the density, $\sigma(\nu)$ the pressure, the positive constant k denotes a moving speed [12], and $\zeta(x,t) \geq 0$ is a friction function of the space variable x and the time variable t. In physics, the pressure σ taking the special form $\sigma(\nu) = \frac{1}{\gamma}\nu^\gamma$, is for the polytropic gas, where the adiabatic exponent $\gamma > 1$.

When ζ is a constant, $k = 0$ and $\alpha(x) = -\frac{a'(x)}{a(x)}$, where the function $a(x)$ denotes a slowly variable cross section area at x in the nozzle, the global entropy solutions for the Cauchy problem (1.1) with bounded initial data

$$(\nu(x,0), u(x,0)) = (\nu_0(x), u_0(x)), \quad \nu_0(x) \geq 0 \tag{1.2}$$

were first studied in [3, 20] when the adiabatic exponent $1 < \gamma \leq \frac{5}{3}$, and by the author in [15] for any adiabatic exponent $\gamma > 1$, if the initial data satisfy the strong restriction condition $\Upsilon(\nu_0(x), u_0(x)) \leq 0$, where Υ is a Riemann invariant given in (1.8). The initial-boundary value problem of compressible Euler equations including friction and heating that model the transonic Fanno-Rayleigh flows through symmetric variable area nozzles is studied in [4].

It is well-known that the unique difficulty to deal with the inhomogeneous system (1.1) is to obtain the a-priori L^∞ estimates of the approximation solutions of (1.1), for instance, the a-priori L^∞ estimates of the classical viscosity solutions for the Cauchy problem of the following parabolic system

$$\begin{cases} \nu_t + (\nu u)_x = \alpha(x-kt)\nu u + \varepsilon \nu_{xx} \\ \\ (\nu u)_t + (\nu u^2 + \sigma(\nu))_x = \alpha(x-kt)\nu u^2 - \zeta(x,t)\nu u|u| + \varepsilon(\nu u)_{xx}, \end{cases} \tag{1.3}$$

with the initial data (1.2).

When $\alpha(x-kt) = 0$, (1.1) is the river flow equations, a shallow-water model describing the vertical depth ν and mean velocity u, where $\zeta(x,t)\nu u|u|$ corresponds physically to a friction term and ζ is the friction coefficient. This kind of inhomogeneous systems is simple since the source terms have, in some senses, the symmetric behavior (see [8] for the details).

When $\zeta(x,t)=0$ and $k=0$, i.e., the nozzle flow without friction, system (1.1) was well studied in (cf. [3, 13, 14, 16, 21, 22, 23], and the references cited therein). Roughly speaking, the technique introduced in these papers, is to control the super-linear source terms $\alpha(x)\nu u$ and $\alpha(x)\nu u^2$ by the flux functions νu and $\nu u^2+\sigma(\nu)$ in (1.1) and to deduce a upper bound of Δ or Υ by a bounded nonegative function $\psi(x)$, which depends on the function $\alpha(x)$.

When $\alpha(x)\not\equiv 0$ and $\zeta(x,t)\not\equiv 0$, both the above techniques do not work because the flux functions can not be used to control the super-linear friction source terms $\zeta\nu u|u|$, and the functions $\alpha(x)\nu u$ destroy the symmetry of the Riemann invariants (Δ,Υ) (see [9] for the numerical analysis).

However, we may copy the following steps given in [16] to overcome the above difficulty.

First, to avoid the singularity of the flux function νu^2 near the vacuum $\nu=0$, we still use the technique of the ξ-flux-approximation given in [17] and introduce the sequence of systems

$$\begin{cases} \nu_t+(-2\xi u+\nu u)_x=\alpha(x-kt)(\nu-2\xi)u \\ (\nu u)_t+(\nu u^2-\xi u^2+\sigma_1(\nu,\xi))_x=\alpha(x-kt)(\nu-2\xi)u^2-\zeta(x,t)\nu u|u| \end{cases} \tag{1.4}$$

to approximate system (1.1), where $\xi>0$ denotes a regular perturbation constant and the perturbation pressure

$$\sigma_1(\nu,\xi)=\int_{2\xi}^{\nu}\frac{t-2\xi}{t}\sigma'(t)dt. \tag{1.5}$$

Second, we add the classical viscosity terms to the right-hand side of (1.4) and obtain the following standard parabolic system

$$\begin{cases} \nu_t+((\nu-2\xi)u)_x=\alpha(x-kt)(\nu-2\xi)u+\varepsilon\nu_{xx} \\ (\nu u)_t+(\nu u^2-\xi u^2+\sigma_1(\nu,\xi))_x=\alpha(x-kt)(\nu-2\xi)u^2-\zeta(x,t)\nu u|u|+\varepsilon(\nu u)_{xx} \end{cases} \tag{1.6}$$

with initial data

$$(\nu^{\xi,\varepsilon}(x,0),u^{\xi,\varepsilon}(x,0))=(\nu_0(x)+2\xi,u_0(x)), \tag{1.7}$$

where $(\nu_0(x), u_0(x))$ are given in (1.2). Now multiplying (1.6) by $(\frac{\partial\Delta}{\partial\nu}, \frac{\partial\Delta}{\partial m})$ and $(\frac{\partial\Upsilon}{\partial\nu}, \frac{\partial\Upsilon}{\partial m})$ respectively, where

$$\Upsilon(\nu, u) = \int_c^\nu \frac{\sqrt{\sigma'(s)}}{s} ds - u, \quad \Delta(\nu, u) = \int_c^\nu \frac{\sqrt{\sigma'(s)}}{s} ds + u \tag{1.8}$$

are the Riemann invariants of (1.1), c is a constant and $m = \nu u$ denotes the momentum, we obtain

$$\begin{aligned} &\Delta_t + \Lambda_2^\xi \Delta_x \\ &= \varepsilon\Delta_{xx} + \tfrac{2\varepsilon}{\nu}\nu_x\Delta_x - \tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma' + \nu\sigma'')\nu_x^2 \\ &+\alpha(x-kt)(\nu-2\xi)u\tfrac{\sqrt{\sigma'(\nu)}}{\nu} - \zeta(x,t)u|u| \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &\Upsilon_t + \Lambda_1^\xi \Upsilon_x \\ &= \varepsilon\Upsilon_{xx} + \tfrac{2\varepsilon}{\nu}\nu_x\Upsilon_x - \tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma' + \nu\sigma'')\nu_x^2 \\ &+\alpha(x-kt)(\nu-2\xi)u\tfrac{\sqrt{\sigma'(\nu)}}{\nu} + \zeta(x,t)u|u|, \end{aligned} \tag{1.10}$$

where

$$\Lambda_1^\xi = \frac{m}{\nu} - \frac{\nu - 2\xi}{\nu}\sqrt{\sigma'(\nu)}, \quad \Lambda_2^\xi = \frac{m}{\nu} + \frac{\nu - 2\xi}{\nu}\sqrt{\sigma'(\nu)} \tag{1.11}$$

are two eigenvalues of the approximation system (1.4).

It is obvious that the terms $\alpha(x-kt)(\nu-2\xi)u\frac{\sqrt{\sigma'(\nu)}}{\nu}$ in (1.9) and (1.10) are not symmetric with respect to the Riemann invariants Δ, Υ. However, with the strong restriction $\Upsilon(\nu_0(x), u_0(x)) \le 0$ on the initial data, we obtained the uniformly upper bounds of Υ and Δ in [20, 15], by using the maximum principle. Unfortunately, without the condition $\Upsilon(\nu_0(x), u_0(x)) \le 0$, we will meet some new technical difficulties when we study the Cauchy problem (1.1) and (1.2).

2. Main Results

In this paper, we will unite the techniques given in [8] and in [20, 15] to obtain the estimates $\Upsilon \le \omega(x-kt)$ and $\Delta \le \omega(x-kt)$, for a suitable uniformly bounded function $\omega(x-kt)$, and to prove the global existence of the entropy solutions for the Cauchy problem (1.1) and (1.2).

The main results of this paper are in the following Theorems 1-3.

Theorem 1. *Let* $\sigma(\nu)=\frac{1}{\gamma}\nu^{\gamma},\gamma>1$, $\zeta(x,t)\geq 0$ *and* $\alpha(x-kt)\leq 0$.

(I). *Let* $1<\gamma\leq 3, k>0$. *Then there exists a function* $\psi(x)\in\mathcal{B}_d^2(-\infty,\infty)$ *satisfying* $\psi(x)\leq M<k$ *and the inequality (2.5) given below such that*

$$\Upsilon(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))=\frac{(\nu^{\xi,\varepsilon}(x,t))^{\theta}}{\theta}-u^{\xi,\varepsilon}(x,t)\leq\psi(x-kt) \tag{2.1}$$

and

$$\Delta(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))=\frac{(\nu^{\xi,\varepsilon}(x,t))^{\theta}}{\theta}+u^{\xi,\varepsilon}(x,t)\leq\psi(x-kt) \tag{2.2}$$

if the initial data $\Upsilon(\nu^{\xi,\varepsilon}(x,0),u^{\xi,\varepsilon}(x,0))\leq\psi(x)$ *and* $\Delta(\nu^{\xi,\varepsilon}(x,0),u^{\xi,\varepsilon}(x,0))\leq\psi(x)\in\mathcal{B}_d^2$, *where* $\psi_x=\frac{\partial\psi(x-kt)}{\partial x}$ *and* $\theta=\frac{\gamma-1}{2}$.

(II). *Let* $\gamma>3, k<0$. *Then there exists a function* $\chi(x)\in\mathcal{C}_d^2(-\infty,\infty)$ *satisfying* $\chi(x)\leq M<-\frac{2}{\gamma+1}k$ *and the inequality (2.6) given below such that* $\Upsilon(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))\leq\chi(x-kt),\Delta(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))\leq\chi(x-kt)$, *which are similar with the estimates (2.1) and (2.2).*

Theorem 2. *Let* $\sigma(\nu)=\frac{1}{\gamma}\nu^{\gamma},\gamma>1$, $\zeta(x,t)\geq 0$ *and* $\alpha(x-kt)=\alpha_-(x-kt)+\alpha_+(x-kt)$, *where* $\alpha_-(x-kt)\leq 0,\alpha_+(x-kt)\geq 0$.

(III). *Let* $1<\gamma\leq 3,k>0$. *Then there exists a function* $\psi(x)\in\mathcal{B}_d^2(-\infty,\infty)$ *satisfying* $\psi(x)\leq M<k$, *the inequalities (2.12) and (2.13) given below, such that the same estimates like (2.1) and (2.2) are true.*

(IV). *Let* $\gamma>3,k<0$. *Then there exists a function* $\chi(x)\in\mathcal{C}_d^2(-\infty,\infty)$ *satisfying* $\chi(x)\leq M<-\frac{2}{\gamma+1}k$, *the inequalities (2.14) and (2.15) given below, such that* $\Upsilon(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))\leq\chi(x-kt),\Delta(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))\leq\chi(x-kt)$ *are true.*

Theorem 3. *If the conditions about the functions* $\alpha(x-kt),\zeta(x,t)$ *and the initial data in Theorem 1 or Theorem 2 are satisfied, then there exists a subsequence of* $(\nu^{\xi,\varepsilon}(x,t),u^{\xi,\varepsilon}(x,t))$, *which converges pointwisely to a pair of bounded functions* $(\nu(x,t),u(x,t))$ *as* ξ,ε *tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2).*

Definition 1. *We call a pair of bounded functions $(\nu(x,t), u(x,t))$ is a weak entropy solution of the Cauchy problem (1.1)-(1.2) if*

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \nu\phi_t + (\nu u)\phi_x + \alpha(x-kt)\nu u\phi dxdt + \int_{-\infty}^\infty \nu_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \nu u\phi_t + (\nu u^2 + \sigma(\nu))\phi_x + (\alpha(x-kt)\nu u^2 - \zeta(x,t)\nu u|u|)\phi dxdt \\ + \int_{-\infty}^\infty \nu_0(x)u_0(x)\phi(x,0)dx = 0 \end{cases} \tag{2.3}$$

holds for all test function $\phi \in \mathcal{C}_0^1(R \times R^+)$ and

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \eta(\nu,m)\phi_t + q(\nu,m)\phi_x + \alpha(x-kt)\nu u\eta(\nu,m)_\nu \\ &+(\alpha(x-kt)\nu u^2 - \zeta(x,t)\nu u|u|)\eta(\nu,m)_m\phi dxdt \geq 0 \end{aligned} \tag{2.4}$$

holds for any non-negative test function $\phi \in \mathcal{C}_0^\infty(R \times R^+ - \{t=0\})$, where $m = \nu u$ and (η, q) is a pair of convex entropy-entropy flux of system (1.1).

Remark 1: The definitions of $\mathcal{B}_d^2(R)$ and $\mathcal{C}_d^2(R)$ are given in [15]. For a given $\psi(x) \in \mathcal{B}_d^2(R)$, if we let $\chi(x) = \psi(-x)$, then $\chi(x) \in \mathcal{C}_d^2(R)$.

Before we prove Theorems 1-3 in the next sections, we first construct some necessary special functions in $\mathcal{B}_d^2(R)$ and $\mathcal{C}_d^2(R)$.

Example 1: For a given $\alpha(x)$, we can find many functions $\psi(x)$ in the set $\mathcal{B}_d^2(R)$, which satisfy

$$\begin{aligned} &\theta^2\alpha^2(x-kt)\psi^2(x-kt) + (1-\theta)^2\psi_x^2 - 2\theta(1+\theta)\alpha(x-kt)\psi(x-kt)\psi_x \\ &+4\varepsilon_1\theta\alpha(x-kt)\psi(x-kt)\psi_x \leq 0 \end{aligned} \tag{2.5}$$

or $\chi(x)$ in $\mathcal{C}_d^2(R)$ satisfy

$$\begin{aligned} &\theta^2\alpha^2(x-kt)\chi^2(x-kt) + (1-\theta)^2\chi_x^2 + 2\theta(1+\theta)\alpha(x-kt)\chi(x-kt)\chi_x \\ &-4\varepsilon_1\theta\alpha(x-kt)\chi(x-kt)\chi_x \leq 0, \end{aligned} \tag{2.6}$$

for a small $\varepsilon_1 > 0$

For instance, we choose $a(x) = x^2, \alpha(x) = -\frac{a'(x)}{a(x)} = -\frac{2}{x}$ as the author studied in [21] for the spherically, symmetric solutions in $x > 1$. We now extend it to the whole space $x \in (-\infty, \infty)$ in the following way:

$$\alpha(x) = \begin{cases} -\frac{2}{x}, & \text{for} \quad x > \varepsilon_0, \\ -\frac{2x}{\varepsilon_0^2}, & \text{for} \quad 0 \le x \le \varepsilon_0, \\ 0, & \text{for} \quad x < 0, \end{cases} \tag{2.7}$$

where $\varepsilon_0 > 0$ is a constant. Then we can easily check that the following function

$$\psi(x) = \begin{cases} qx^\beta, & \text{for} \quad x > \varepsilon_0, \\ q_1 e^{\frac{\beta x^2}{2\varepsilon_0^2}}, & \text{for} \quad 0 \le x \le \varepsilon_0, \\ q_1, & \text{for} \quad x < 0, \end{cases} \tag{2.8}$$

satisfies

$$\psi'(x) = -\frac{\beta}{2}\alpha(x)\psi(x) \text{ if } q\varepsilon_0^\beta = q_1 e^{\frac{\beta}{2}}, \tag{2.9}$$

where q, q_1 are two positive constants and β is a negative constant. Then clearly $-M\psi(x) \le \psi'(x) \le 0$.

Moreover,

$$\psi''(x) = \begin{cases} q\beta(\beta-1)x^{\beta-2}, & \text{for} \quad x > \varepsilon_0, \\ q_1 \frac{\beta}{\varepsilon_0^2} e^{\frac{\beta x^2}{2\varepsilon_0^2}} + q_1 (\frac{\beta x}{\varepsilon_0^2})^2 e^{\frac{\beta x^2}{2\varepsilon_0^2}}, & \text{for} \quad 0 \le x \le \varepsilon_0, \\ 0, & \text{for} \quad x < 0, \end{cases} \tag{2.10}$$

and $\psi''(x) = \psi_1(x) + \psi_2(x)$, where $\psi_1(x), \psi_2(x)$ satisfy the conditions in the definition of $\mathcal{B}_d^2(R)$, thus $\psi(x) \in \mathcal{B}_d^2(R)$.

Let $-\frac{\beta}{2} = \frac{\theta}{1-\theta}$. Then we can easily check that $\psi(x-kt)$ satisfies (2.5), where $\psi(x)$ is given by (2.9).

In fact, let $\psi_x = \frac{\partial \psi(x-kt)}{\partial x} = \frac{\theta}{1-\theta}\alpha(x-kt)\psi(x-kt)$. Then $\psi_x \le 0$ and

(2.5) is equivalent to

$$\theta^2\alpha^2(x-kt)\psi^2(x-kt)+(1-\theta)^2\psi_x^2-2\theta(1+\theta)\alpha(x-kt)\psi(x-kt)\psi_x$$

$$+4\varepsilon_1\theta\alpha(x-kt)\psi(x-kt)\psi_x$$

$$=2\theta(1-\theta)\alpha(x-kt)\psi(x-kt)\psi_x-2\theta(1+\theta)\alpha(x-kt)\psi(x-kt)\psi_x$$

$$+4\varepsilon_1\theta\alpha(x-kt)\psi(x-kt)\psi_x$$

$$=(-4\theta+4\varepsilon_1)\theta\alpha(x-kt)\psi(x-kt)\psi_x\le 0. \tag{2.11}$$

Example 2: We may choose suitable functions $\alpha_-(x-kt)$ and $\alpha_+(x-kt)$ to obtain the functions $\psi(x-kt)$ satisfying

$$\theta^2\alpha_-^2(x-kt)\psi^2(x-kt)+(1-\theta)^2\psi_x^2-2\theta(1+\theta)\alpha_-(x-kt)\psi(x-kt)\psi_x$$

$$+4\varepsilon_1\theta\alpha_-(x-kt)\psi(x-kt)\psi_x\le 0 \tag{2.12}$$

and

$$4(-k+(1+\varepsilon_1)\psi(x-kt))\psi_x(x-kt)-\alpha_+(x-kt)\psi^2(x-kt)\ge 0 \tag{2.13}$$

or $\chi(x-kt)$ satisfying

$$\theta^2\alpha_-^2(x-kt)\chi^2(x-kt)+(1-\theta)^2\chi_x^2+2\theta(1+\theta)\alpha_-(x-kt)\chi(x-kt)\chi_x$$

$$-4\varepsilon_1\theta\alpha_-(x-kt)\chi(x-kt)\chi_x\le 0 \tag{2.14}$$

and

$$4(-k+(1+\varepsilon_1)\chi(x-kt))\chi_x(x-kt)-\alpha_+(x-kt)\chi^2(x-kt)\ge 0 \tag{2.15}$$

for a small $\varepsilon_1>0$.

For instance, if we let $\alpha_-(x),\psi(x)$ satisfy (2.9) or

$$\psi_x(x-kt)=-\frac{\beta}{2}\alpha_-(x-kt)\psi(x-kt), \tag{2.16}$$

then (2.12) is true. Clearly (2.13) is also true if

$$4(-k+(1+\varepsilon_1)M)\psi_x(x-kt)-M\alpha_+(x-kt)\psi(x-kt)\ge 0 \tag{2.17}$$

or

$$\psi_x(x-kt) \geq -l_0\alpha_+(x-kt)\psi(x-kt) \tag{2.18}$$

for a suitable positive constant l_0. If we choose $\alpha_-(x-kt)$ and $\alpha_+(x-kt)$ such that

$$-\frac{\beta}{2}\alpha_-(x-kt) \geq -l_0\alpha_+(x-kt), \tag{2.19}$$

then $\psi(x-kt)$ satisfies (2.13).

The proofs of Theorems 1-3 are given in the following several sections.

3. Proof of Theorem 1.

By using the transformation $\Upsilon = \psi(x,t)+\pi$, for a suitable function $\psi(x,t)$ in (1.10), we have

$$\begin{aligned}
&\pi_t+\psi_t+(u-\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})(\pi_x+\psi_x)\\
&=\varepsilon\pi_{xx}+\varepsilon\psi_{xx}+\tfrac{2\varepsilon}{\nu}\nu_x\pi_x+\tfrac{2\varepsilon}{\nu}\nu_x\psi_x-\tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma'+\nu\sigma'')\nu_x^2\\
&-\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}(\psi(x,t)+\pi-\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu)+\zeta(x,t)u|u|,
\end{aligned} \tag{3.1}$$

which is

$$\begin{aligned}
&\pi_t+\psi_t+(u-\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})\pi_x-\psi_x(\psi(x,t)+\pi-\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu)-\psi_x\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}\\
&=\varepsilon\pi_{xx}-\tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma'+\nu\sigma'')[\nu_x^2-\tfrac{4\nu\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\nu_x\psi_x+(\tfrac{2\nu\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x)^2]\\
&+\varepsilon\psi_{xx}+\tfrac{2\varepsilon}{\nu}\nu_x\pi_x+\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2-\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}\psi(x,t)\\
&-\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}\pi+\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu+\zeta(x,t)u|u|
\end{aligned} \tag{3.2}$$

or

$$\begin{aligned}&\pi_t+\psi_t+a(x,t)\pi_x+b(x,t)\pi+[-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2-\varepsilon\psi_{xx}-\varepsilon_1\psi(x,t)\psi_x]\\&+\int_c^\nu\tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\psi_x-(1-\varepsilon_1)\psi(x,t)\psi_x-\zeta(x,t)u|u|\\&+[\alpha(x-kt)\psi(x,t)-\psi_x](\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\\&-\alpha(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\int_c^\nu\tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\le\varepsilon\pi_{xx},\end{aligned}\tag{3.3}$$

for a suitable small constant $\varepsilon_1>0$, where $a(x,t)=u-\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}-\frac{2\varepsilon}{\nu}\nu_x$ and $b(x,t)=-\psi_x+\alpha(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}$.

Similarly, if we make the transformation $\Delta=\psi(x,t)+s$ in (1.9), we obtain

$$\begin{aligned}&s_t+\psi_t+(u+\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})(s_x+\psi_x)\\&=\varepsilon s_{xx}+\varepsilon\psi_{xx}+\tfrac{2\varepsilon}{\nu}\nu_xs_x+\tfrac{2\varepsilon}{\nu}\nu_x\psi_x-\tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma'+\nu\sigma'')\nu_x^2\\&+\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}u-\zeta(x,t)u|u|,\end{aligned}\tag{3.4}$$

which is

$$\begin{aligned}&s_t+\psi_t+(u+\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})s_x+\psi_x(\psi(x,t)+s-\int_c^\nu\tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu)+\psi_x\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}\\&=\varepsilon s_{xx}-\tfrac{\varepsilon}{2\nu^2\sqrt{\sigma'(\nu)}}(2\sigma'+\nu\sigma'')[\nu_x^2+\tfrac{4\nu\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\nu_x\psi_x+(\tfrac{2\nu\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x)^2]\\&+\varepsilon\psi_{xx}+\tfrac{2\varepsilon}{\nu}\nu_xs_x+\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2+\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}u-\zeta(x,t)u|u|\end{aligned}\tag{3.5}$$

or

$$\begin{aligned}&s_t+\psi_t+c(x,t)s_x+d(x,t)s+[-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2-\varepsilon\psi_{xx}-\varepsilon_1\psi(x,t)\psi_x]\\&+\psi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}-\int_c^\nu\tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu)+(1+\varepsilon_1)\psi(x,t)\psi_x\\&-\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}u+\zeta(x,t)u|u|\le\varepsilon s_{xx},\end{aligned}\tag{3.6}$$

where $c(x,t) = u + \frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \frac{2\varepsilon}{\nu}\nu_x$ and $d(x,t) = \psi_x$.

By using the maximum principle to the first equation in system (1.6), we may obtain the a-priori estimate $\nu \geq 2\xi$. Then letting $\varepsilon = o(\xi)$ and $\varepsilon = o(\varepsilon_1)$, we have the following estimates on the three terms of the left-hand side of (3.3) and (3.6)

$$-\frac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma' + \nu\sigma''}\psi_x^2 - \varepsilon\psi_{xx} - \varepsilon_1\psi(x,t)\psi_x > 0. \tag{3.7}$$

Now we rewrite (3.3) and (3.6) as follows:

$$\begin{aligned} &\pi_t + \psi_t + a(x,t)\pi_x + b(x,t)\pi \\ &+\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\psi_x - (1-\varepsilon_1)\psi(x,t)\psi_x + \frac{1}{2}\zeta(x,t)(\pi - s)|u| \\ &+[\alpha(x-kt)\psi(x,t) - \psi_x](\nu - 2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu} \\ &-\alpha(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \leq \varepsilon\pi_{xx}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &s_t + \psi_t + c(x,t)s_x + d(x,t)s \\ &+\psi_x(\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1+\varepsilon_1)\psi(x,t)\psi_x \\ &+\frac{1}{2}(\zeta(x,t)|u| - \alpha(x-kt)\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})(s-\pi) \leq \varepsilon s_{xx}. \end{aligned} \tag{3.9}$$

If we may choose a suitable bounded function $\psi(x,t)$ such that the following inequalities hold

$$\begin{aligned} &\psi_t + \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\psi_x - (1-\varepsilon_1)\psi(x,t)\psi_x \\ &+[\alpha(x-kt)\psi(x,t) - \psi_x](\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu} \\ &-\alpha(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \geq 0 \end{aligned} \tag{3.10}$$

and

$$\psi_t + \psi_x(\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1+\varepsilon_1)\psi(x,t)\psi_x \geq 0, \tag{3.11}$$

then we have from (3.8) and (3.9) that

$$\pi_t + a(x,t)\pi_x + b(x,t)\pi + \tfrac{1}{2}\zeta(x,t)|u|(\pi - s) \leq \varepsilon\pi_{xx} \tag{3.12}$$

and

$$s_t + c(x,t)s_x + d(x,t)s + \tfrac{1}{2}(\zeta(x,t)|u| - \alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})(s-\pi) \leq \varepsilon s_{xx}. \tag{3.13}$$

Before we check the possibility of (3.10) and (3.11), we apply the inequalities (3.12) and (3.13) to prove the following Lemma 4 about the a priori estimates of π and s:

Lemma 4. *If at the time $t = 0$, $\pi(x,0) \leq 0$ and $s(x,0) \leq 0$, then the maximum principle is true to the functions $\pi(x,t)$ and $s(x,t)$, namely, $\pi(x,t) \leq 0, s(x,t) \leq 0$ for all $t > 0$.*

Proof of Lemma 4: Make a transformation

$$\pi = e^{\beta t}(\bar{\pi} + \frac{N(x^2 + qLe^t)}{L^2}), \quad s = e^{\beta t}(\bar{s} + \frac{N(x^2 + qLe^t)}{L^2}), \tag{3.14}$$

where L, q, β are suitable positive constants and N is the upper bound of π, s on $R \times [0,T]$ (N can be obtained by the local existence). The functions $\bar{\pi}, \bar{s}$, as are easily seen, satisfy the equations

$$\begin{cases} \bar{\pi}_t + a(x,t)\bar{\pi}_x - \varepsilon\bar{\pi}_{xx} + (\beta + b(x,t) + \frac{1}{2}\zeta(x,t)|u|)\bar{\pi} - \frac{1}{2}\zeta(x,t)|u|\bar{s} \\ \qquad \leq -(qLe^t + 2xa(x,t) - 2\varepsilon)\frac{N}{L^2} - (\beta + b(x,t))\frac{N(x^2+qLe^t)}{L^2}, \\ \\ \bar{s}_t + c(x,t)\bar{s}_x - \varepsilon\bar{s}_{xx} + (\beta + d(x,t) + \frac{1}{2}\zeta(x,t)|u| - \alpha(x-kt)\frac{\nu-2\xi}{\nu}\sqrt{P'(\nu)})\bar{s} \\ -(\frac{1}{2}\zeta(x,t)|u| - \alpha(x-kt)\frac{\nu-2\xi}{\nu}\sqrt{P'(\nu)})\bar{\pi} \\ \qquad \leq -(qLe^t + 2xc(x,t) - 2\varepsilon)\frac{N}{L^2} - (\beta + d(x,t))\frac{N(x^2+qLe^t)}{L^2}, \end{cases} \tag{3.15}$$

resulting from (3.12) and (3.13). Moreover

$$\bar{\pi}(x,0) = \pi(x,0) - \frac{N(x^2+qL)}{L^2} < 0, \;\; \bar{s}(x,0) = s(x,0) - \frac{N(x^2+qL)}{L^2} < 0, \tag{3.16}$$

$$\bar{\pi}(+L,t)<0,\ \bar{\pi}(-L,t)<0,\ \bar{s}(+L,t)<0,\ \bar{s}(-L,t)<0. \tag{3.17}$$

From (3.15),(3.16) and (3.17), we have

$$\bar{\pi}(x,t)<0, \quad \bar{s}(x,t)<0, \quad \text{on} \quad (-L,L)\times(0,T). \tag{3.18}$$

If (3.18) is violated at a point $(x,t)\in(-L,L)\times(0,T)$, let $\bar{t}$ be the least upper bound of values of t at which $\bar{\pi}<0$ (or $\bar{s}<0$); then by the continuity we see that $\bar{\pi}=0, \bar{s}\leq 0$ at some points $(\bar{x},\bar{t})\in(-L,L)\times(0,T)$. So

$$\bar{\pi}_t\geq 0, \quad \bar{\pi}_x=0, \quad -\varepsilon\bar{\pi}_{xx}\geq 0, \quad \text{at} \quad (\bar{x},\bar{t}). \tag{3.19}$$

If we choose sufficiently large constants q,β (which may depend on the bound of the local existence) such that

$$qL+2xa(x,t)-2\varepsilon>0, \quad \beta+b(x,t)>0 \quad \text{on} \quad (-L,L)\times(0,T). \tag{3.20}$$

(3.19) and (3.20) give a conclusion contradicting the first inequality in (3.15). So (3.18) is proved. Therefore, for any point $(x_0,t_0)\in(-L,L)\times(0,T)$,

$$\pi(x_0,t_0)<(\frac{N(x_0^2+qLe_0^t)}{L^2})e^{\beta t_0}, \quad s(x_0,t_0)<(\frac{N(x_0^2+qLe_0^t)}{L^2})e^{\beta t_0}, \tag{3.21}$$

which gives the desired estimates $\pi\leq 0, s\leq 0$ if we let L go to infinity. So Lemma 4 is proved.

From $v\leq 0, s\leq 0$, we may immediately obtain the estimates $\Delta\leq\psi(x,t)$ and $\Upsilon\leq\psi(x,t)$ given in Part (I) of Theorem 1, if we may choose $\psi(x,t)$ such that (3.10) and (3.11) are true.

Lemma 5. *Let $\sigma(\nu)=\frac{1}{\gamma}\nu^\gamma, 1<\gamma\leq 3, c=0$. For a given function $\psi(x)\in\mathcal{B}_d^2(R)$, if $\psi(x)\leq M<k$ and satisfies the inequality (2.5) in Theorem 1 , then (3.10) and (3.11) are true if we choose $\psi(x,t)=\psi(x-kt)$.*

Proof of Lemma 5: We first prove (3.11). Let $\psi(x,t)=\psi(x-kt)$, then $\psi_t=-k\psi_x$. When $\sigma(\nu)=\frac{1}{\gamma}\nu^\gamma, 1<\gamma\leq 3$ and $c=0$, we have

$$\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}-\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu=(\nu-2\xi)\nu^{\frac{\gamma-3}{2}}-\int_{2\xi}^\nu s^{\frac{\gamma-3}{2}}ds-\int_0^{2\xi}s^{\frac{\gamma-3}{2}}ds\leq 0, \tag{3.22}$$

and

$$\psi_t + (1+\varepsilon_1)\psi(x,t)\psi_x = (-k + (1+\varepsilon_1)\psi(x-kt))\psi_x \geq 0. \tag{3.23}$$

Thus (3.11) is proved.

To prove (3.10), we let the left side of (3.10) be L, then

$$\begin{aligned}
L &= [\textstyle\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu - \frac{2\xi}{\nu} \int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu]\psi_x \\
&+(\tfrac{2\xi}{\nu}\psi_x \textstyle\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu - k\psi_x) - (1-\varepsilon_1)\psi(x-kt)\psi_x \\
&+[\alpha(x-kt)\psi(x-kt) - \psi_x](\nu - 2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu} \\
&-\alpha(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu} \textstyle\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu \\
&= -\tfrac{1}{\theta}\alpha(x-kt)(\nu-2\xi)^2\nu^{2\theta-2} + (\alpha(x-kt)\psi(x-kt) - \psi_x + \tfrac{\psi_x}{\theta})(\nu-2\xi)\nu^{\theta-1} \\
&+(\tfrac{2\xi}{\nu}\psi_x \textstyle\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu - k\psi_x) - (1-\varepsilon_1)\psi(x-kt)\psi_x - \frac{2\xi}{\theta}\alpha(x-kt)(\nu-2\xi)\nu^{2\theta-2}.
\end{aligned} \tag{3.24}$$

Since

$$-\frac{2\xi}{\theta}\alpha(x)(\nu-2\xi)\nu^{2\theta-2} \geq 0 \tag{3.25}$$

and

$$(\frac{2\xi}{\nu}\psi_x \int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu - k\psi_x) = (\frac{2\xi}{\theta}\nu^{\frac{\gamma-3}{2}} - k)\psi_x \geq (\frac{(2\xi)^\theta}{\theta} - k)\psi_x \geq 0, \tag{3.26}$$

we have

$$\begin{aligned}
L &\geq -\tfrac{1}{\theta}\alpha(x-kt)(\nu-2\xi)^2\nu^{2\theta-2} - (1-\varepsilon_1)\psi(x-kt)\psi_x \\
&+(\alpha(x-kt)\psi(x-kt) - \psi_x + \tfrac{\psi_x}{\theta})(\nu-2\xi)\nu^{\theta-1} \\
&= -\tfrac{1}{\theta}\alpha(x-kt)[(\nu-2\xi)^2\nu^{2\theta-2} + \tfrac{\theta\psi_x - \psi_x - \theta\alpha(x-kt)\psi(x-kt)}{\alpha(x-kt)}(\nu-2\xi)\nu^{\theta-1} \\
&+(\tfrac{\theta\psi_x - \psi_x - \theta\alpha(x-kt)\psi(x-kt)}{2\alpha(x-kt)})^2] - (1-\varepsilon_1)\psi(x-kt)\psi_x \\
&+\tfrac{(\theta\psi_x - \psi_x - \theta\alpha(x-kt)\psi(x-kt))^2}{4\theta\alpha(x-kt)}.
\end{aligned} \tag{3.27}$$

Since $\alpha(x-kt)\le 0$ and $\psi(x-kt)>0$, then $L\ge 0$ if we let $\psi_x\le 0$ satisfy

$$-(1-\varepsilon_1)\psi(x-kt)\psi_x+\frac{(\theta\psi_x-\psi_x-\theta\alpha(x-kt)\psi(x-kt))^2}{4\theta\alpha(x-kt)}\ge 0, \quad (3.28)$$

which is equivalent to

$$\begin{aligned}&(\theta\psi_x-\psi_x-\theta\alpha(x-kt)\psi(x-kt))^2\\ &\le 4\theta\alpha(x-kt)\psi(x-kt)\psi_x-4\theta\varepsilon_1\alpha(x-kt)\psi(x-kt)\psi_x\end{aligned} \quad (3.29)$$

or

$$\begin{aligned}&(\theta-1)^2\psi_x^2-2\theta(\theta+1)\alpha(x-kt)\psi(x-kt)\psi_x\\ &+4\theta\varepsilon_1\alpha(x-kt)\psi(x-kt)\psi_x+\theta^2(\alpha(x-kt)\psi(x-kt))^2\le 0.\end{aligned} \quad (3.30)$$

(3.30) is true for a suitably small $\varepsilon_1>0$ since the inequality (2.5) given in Theorem 1. **Part (I) of Theorem 1 is proved.**

To prove Part (II), we let $\Upsilon=\chi(x,t)+\pi, \Delta=\chi(x,t)+s$ for a suitable function $\chi(x,t)$ in (1.10) and (1.9), we may repeat the process in the proof of (3.3) and (3.6) to obtain

$$\begin{aligned}&\pi_t+\chi_t+a(x,t)\pi_x+b(x,t)\pi+[-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\chi_x^2-\varepsilon\chi_{xx}+\varepsilon_1\chi(x,t)\chi_x]\\ &+\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\chi_x-(1+\varepsilon_1)\chi(x,t)\chi_x-\zeta(x,t)u|u|\\ &+[\alpha(x-kt)\chi(x,t)-\chi_x](\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\\ &-\alpha(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\le\varepsilon\pi_{xx},\end{aligned} \quad (3.31)$$

and

$$\begin{aligned}&s_t+\chi_t+c(x,t)s_x+d(x,t)s+[-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\chi_x^2-\varepsilon\chi_{xx}+\varepsilon_1\chi(x,t)\chi_x]\\ &+\chi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}-\textstyle\int_c^\nu\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu)+(1-\varepsilon_1)\chi(x,t)\chi_x\\ &-\alpha(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)}u+\zeta(x,t)u|u|\le\varepsilon s_{xx},\end{aligned} \quad (3.32)$$

where $\varepsilon_1 > 0$ is a suitable small constant.

With the help of Lemma 4, we only need to choose a suitable function $\chi(x,t) \in \mathcal{C}_d^2(R)$ and a constant c so that the following inequalities (which are similar with (3.10) and (3.11))

$$\begin{aligned} &\chi_t + \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu} d\nu \chi_x - (1+\varepsilon_1)\chi(x,t)\chi_x \\ &+[\alpha(x-kt)\chi(x,t) - \chi_x](\nu - 2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu} \\ &-\alpha(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \geq 0 \end{aligned} \tag{3.33}$$

and

$$\chi_t + \chi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1-\varepsilon_1)\chi(x,t)\chi_x \geq 0 \tag{3.34}$$

are correct.

The proof of the inequality (3.34) is simple because when $\gamma > 3$ and $k < 0$ satisfies the condition in (II) of Theorem 1, we may choose $\chi(x,t) = \chi(x-kt), c = 0$ so that

$$\begin{aligned} &\chi_t + \chi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1-\varepsilon_1)\chi(x,t)\chi_x \\ &= (-k + (1-\varepsilon_1)\chi(x-kt) + (\nu-2\xi)\nu^{\frac{\gamma-3}{2}} - \int_{2\xi}^\nu s^{\frac{\gamma-3}{2}}ds - \int_0^{2\xi} s^{\frac{\gamma-3}{2}}ds)\chi_x \\ &\geq (-k + (1-\varepsilon_1)\chi(x-kt) - \tfrac{1}{\theta}(2\xi)^\theta)\chi_x \geq 0. \end{aligned} \tag{3.35}$$

To prove (3.33), we let the left side of (3.33) be L_1, then

$$\begin{aligned} L_1 &= [\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu - \tfrac{2\xi}{\nu}\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu]\chi_x \\ &+\tfrac{2\xi}{\nu}\chi_x\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu - k\chi_x - (1+\varepsilon_1)\chi(x-kt)\chi_x \\ &+[\alpha(x-kt)\chi(x-kt) - \chi_x](\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu} - \alpha(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \\ &= -\tfrac{1}{\theta}\alpha(x-kt)(\nu-2\xi)^2\nu^{2\theta-2} + (\alpha(x-kt)\chi(x-kt) - \chi_x + \tfrac{\chi_x}{\theta})(\nu-2\xi)\nu^{\theta-1} \\ &+\tfrac{2\xi}{\nu}\chi_x\int_0^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu - k\chi_x - (1+\varepsilon_1)\chi(x-kt)\chi_x - \tfrac{2\xi}{\theta}\alpha(x-kt)(\nu-2\xi)\nu^{2\theta-2}. \end{aligned} \tag{3.36}$$

Since

$$\frac{2\xi}{\nu}\chi_x\int_0^{\nu}\frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \geq 0, \quad -\frac{2\xi}{\theta}\alpha(x)(\nu-2\xi)\nu^{2\theta-2} \geq 0, \tag{3.37}$$

we have

$$\begin{aligned}
&L_1 \geq -\tfrac{1}{\theta}\alpha(x-kt)(\nu-2\xi)^2\nu^{2\theta-2} + (\alpha(x-kt)\chi(x-kt) - \chi_x + \tfrac{\chi_x}{\theta})(\nu-2\xi)\nu^{\theta-1} \\
&-k\chi_x - (1+\varepsilon_1)\chi(x-kt)\chi_x \\
&= -\tfrac{1}{\theta}\alpha(x-kt)[(\nu-2\xi)^2\nu^{2\theta-2} + \tfrac{\theta\chi_x-\chi_x-\theta\alpha(x-kt)\chi(x-kt)}{\alpha(x-kt)}(\nu-2\xi)\nu^{\theta-1} \\
&+(\tfrac{\theta\chi_x-\chi_x-\theta\alpha\psi}{2\alpha})^2] - k\chi_x - (1+\varepsilon_1)\chi\chi_x + \tfrac{(\theta\chi_x-\chi_x-\theta\alpha\psi)^2}{4\theta\alpha}.
\end{aligned} \tag{3.38}$$

Since $\alpha(x-kt) \leq 0$ and $\chi(x-kt) > 0$, then $L_1 \geq 0$ if we let $\chi_x \geq 0$ satisfy

$$-k\chi_x - (1+\varepsilon_1)\chi\chi_x + \frac{(\theta\chi_x - \chi_x - \theta\alpha\chi)^2}{4\theta\alpha} \geq 0, \tag{3.39}$$

which is equivalent to

$$(\theta-1)^2\chi_x^2 + \theta^2(\alpha\chi)^2 - 2\theta(\theta-1)\alpha\chi\chi_x - (4\theta k + 4\theta(1+\varepsilon_1)\chi)\alpha\chi_x \leq 0 \tag{3.40}$$

or

$$\begin{aligned}
&(\theta-1)^2\chi_x^2 + \theta^2(\alpha\chi)^2 + 2\theta(\theta+1)\alpha\chi\chi_x - 4\theta\varepsilon_1\alpha\chi\chi_x \\
&-4\theta\alpha\chi(k+(1+\theta)\chi) \leq 0.
\end{aligned} \tag{3.41}$$

Since $k + (1+\theta)\chi \leq k + \frac{\gamma+1}{2}M < 0$ and the inequality (2.6) in (II) of Theorem 1, (3.41) is correct and so **Part (II) of Theorem 1 is proved.**

4. Proof of Theorem 2.

When $\alpha(x-kt) = \alpha_-(x-kt) + \alpha_+(x-kt)$, where $\alpha_-(x-kt) \le 0, \alpha_+(x-kt) \ge 0$, we may rewrite (3.3) as

$$\begin{aligned}
&\pi_t + \psi_t + a(x,t)\pi_x + b(x,t)\pi + [-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2 - \varepsilon\psi_{xx} - \varepsilon_1\psi(x,t)\psi_x] \\
&+\textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\psi_x - (1-\varepsilon_1)\psi(x,t)\psi_x - (\zeta(x,t)|u| + \alpha_+(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu})u \\
&+[\alpha_-(x-kt)\psi(x,t) - \psi_x](\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu} \\
&-\alpha_-(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \le \varepsilon\pi_{xx}.
\end{aligned} \tag{4.1}$$

In a similar way to obtain (3.6), we may obtain the following inequality

$$\begin{aligned}
&s_t + \psi_t + c_1(x,t)s_x + d_1(x,t)s + [-\tfrac{2\varepsilon\sqrt{\sigma'(\nu)}}{2\sigma'+\nu\sigma''}\psi_x^2 - \varepsilon\psi_{xx} - \varepsilon_1\psi(x,t)\psi_x] \\
&+\psi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1+\varepsilon_1)\psi(x,t)\psi_x \\
&-\alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\psi(x,t) + \alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \\
&+(-\alpha_-(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} + \zeta(x,t)|u|)u \le \varepsilon s_{xx},
\end{aligned} \tag{4.2}$$

where $c_1(x,t) = c(x,t) = u + \frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \frac{2\varepsilon}{\nu}\nu_x$ and $d_1(x,t) = \psi_x - \alpha_+(x-kt)(\nu-2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}$.

Therefore, if we may choose a suitable bounded function $\psi(x,t)$ such that the following inequalities hold

$$\begin{aligned}
&\psi_t + \textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu\psi_x - (1-\varepsilon_1)\psi(x,t)\psi_x \\
&+[\alpha_-(x-kt)\psi(x,t) - \psi_x](\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu} \\
&-\alpha_-(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \ge 0
\end{aligned} \tag{4.3}$$

and

$$\psi_t + \psi_x(\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \textstyle\int_c^\nu \tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) + (1+\varepsilon_1)\psi(x,t)\psi_x$$

$$-\alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\psi(x,t) \tag{4.4}$$

$$+\alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu \tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \geq 0,$$

then we have from (4.1) and (4.2) that

$$\pi_t + a(x,t)\pi_x + b(x,t)\pi$$

$$+\tfrac{1}{2}(\zeta(x,t)|u| + \alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu})(\pi - s) \leq \varepsilon\pi_{xx} \tag{4.5}$$

and

$$s_t + c_1(x,t)s_x + d_1(x,t)s$$

$$+\tfrac{1}{2}(\zeta(x,t)|u| - \alpha_-(x-kt)\tfrac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)})(s-\pi) \leq \varepsilon s_{xx}. \tag{4.6}$$

If we let $\psi(x,t) = \psi(x-kt)$, using the inequality (2.12), we may prove (4.3) in a similar way like the proof of (3.10).

Under the conditions in (III) of Theorem 2, (4.4) is true because

$$\psi_x(\frac{\nu-2\xi}{\nu}\sqrt{\sigma'(\nu)} - \int_c^\nu \frac{\sqrt{\sigma'(\nu)}}{\nu}d\nu) \geq 0 \tag{4.7}$$

and

$$\psi_t + (1+\varepsilon_1)\psi(x,t)\psi_x$$

$$-\alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\psi(x,t)$$

$$+\alpha_+(x-kt)(\nu-2\xi)\tfrac{\sqrt{\sigma'(\nu)}}{\nu}\textstyle\int_c^\nu \tfrac{\sqrt{\sigma'(\nu)}}{\nu}d\nu \geq (-k + (1+\varepsilon_1)\psi(x-kt))\psi_x$$

$$-\alpha_+(x-kt)\psi(x-kt)f(\nu) + \alpha_+(x-kt)f^2(\nu)$$

$$= (-k + (1+\varepsilon_1)\psi(x-kt))\psi_x - \tfrac{1}{4}\alpha_+(x-kt)\psi^2(x-kt)$$

$$+\alpha_+(f(\nu) - \tfrac{1}{2}\psi(x-kt))^2$$

$$\geq (-k + (1+\varepsilon_1)\psi(x-kt))\psi_x - \tfrac{1}{4}\alpha_+(x-kt)\psi^2(x-kt) \geq 0 \tag{4.8}$$

due to the condition (2.13), where $f(\nu) = (\nu - 2\xi)\frac{\sqrt{\sigma'(\nu)}}{\nu}$. Thus **Part (III) of Theorem 2 is proved.** Similarly we may prove Part (IV) of Theorem 2 and complete the proof of Theorem 2.

5. Proof of Theorem 3.

From the upper estimates on the Riemann invariants given in Theorems 1-2, we can easily obtain the following estimates on $(\nu^{\xi,\varepsilon}, u^{\xi,\varepsilon})$,

$$2\xi \le \nu^{\xi,\varepsilon}(x,t) \le N(x,t), \quad |u^{\xi,\varepsilon}(x,t)| \le N(x,t), \tag{5.1}$$

where $N(x,t)$ is a positive, bounded function, which depending on the bound of the initial data, but independent of ε, ξ.

Following the standard theory of semilinear parabolic systems, we can apply the contraction mapping principle to an integral representation of a solution to obtain the local existence result of the Cauchy problem (1.6)-(1.7). With the L^∞ estimate (5.1) of the local solution, we can extend the local time step by step to an arbitrary time T , since the step time depends only on the L^∞ norm.

As proved in [17], we know that the original system (1.1) and the approximated system (1.4) have the same entropy equation or the same entropies, and for any weak entropy-entropy flux pair $(\eta(\nu,u), q(\nu,u))$ of system (1.1)

$$\eta_t(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t)) + q_x(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t)) \tag{5.2}$$

are compact in $H^{-1}_{loc}(R \times R^+)$, then the compactness framework given in [6, 10] for $1 < \gamma < 3$ and in [11] for $\gamma \ge 3$ to ensure that there exists a subsequence of $(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\nu(x,t), u(x,t))$ as ξ, ε tend to zero, and the limit $(\nu(x,t), u(x,t))$

satisfies (2.3). Moreover, we multiply (1.6) by (η_ν, η_m) to obtain

$$\begin{aligned}
&\eta_t(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t)) + q_x(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t)) + \xi q_{1x}(\nu^{\xi,\varepsilon}(x,t), u^{\xi,\varepsilon}(x,t)) \\
&= \varepsilon\eta(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon})_{xx} - \varepsilon(\nu_x^{\xi,\varepsilon}, m_x^{\xi,\varepsilon}) \cdot \nabla^2\eta(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon}) \cdot (\nu_x^{\xi,\varepsilon}, m_x^{\xi,\varepsilon})^T \\
&+\alpha(x-kt)u^{\xi,\varepsilon}m^{\xi,\varepsilon}\eta_\nu(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon}) \\
&+(\alpha(x-kt)u^{\xi,\varepsilon}m^{\xi,\varepsilon} - \zeta(x,t)m^{\xi,\varepsilon}|u^{\xi,\varepsilon}|)\eta_m(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon}) \\
&\le \varepsilon\eta(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon})_{xx} + \alpha(x-kt)u^{\xi,\varepsilon}m^{\xi,\varepsilon}\eta_\nu(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon}) \\
&+(\alpha(x-kt)u^{\xi,\varepsilon}m^{\xi,\varepsilon} - \zeta(x,t)m^{\xi,\varepsilon}|u^{\xi,\varepsilon}|)\eta_m(\nu^{\xi,\varepsilon}, m^{\xi,\varepsilon})
\end{aligned} \tag{5.3}$$

for any weak convex entropy-entropy flux pair $(\eta(\nu,u), q(\nu,u))$ of system (1.1), where $q + \xi q_1$ is the entropy flux of system (1.4) corresponding to the entropy η. Thus the entropy inequality (2.4) is proved if we multiply a test function to (5.3) and let ε, ξ go to zero. Thus we obtain the proof of Theorem 3.

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