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Cauchy problem for hyperbolic conservation laws with a relaxation term

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This paper considers the Cauchy problem for hyperbolic conservation laws arising in chromatography:

$$(u+v)_t+f(u)_x=0,\quad v_t=\frac{A(u)-v}{g(\delta,u,v)},$$

with bounded measurable initial data, where the relaxation term $g(\delta, u, v)$ converges to zero as the parameter $\delta>0$ tends to zero. We show that a solution of the equilibrium equation

$$(u+A(u))_t+f(u)_x=0$$

is given by the limit of the solutions of the viscous approximation

$$(u+v)_t+f(u)_x=\varepsilon(u+v)_{xx},\quad v_t=\varepsilon v_{xx}+\frac{A(u)-v}{g(\delta,u,v)},$$

of the original system as the dissipation ε and the relaxation δ go to zero related by $\delta=O(\varepsilon)$. The proof of convergence is obtained by a simplified method of compensated compactness [**2**], avoiding Young measures by using the weak continuity theorem (3.3) of two by two determinants.

1. Introduction

In this paper we consider the existence of global weak solutions for an extended model of relaxation

$$(u+v)_t+f(u)_x=0,\quad v_t=\frac{A(u)-v}{g(\delta,u,v)},\tag{1.1}$$

with initial data

$$(u,v)|_{t=0}=(u_0(x),v_0(x)),\tag{1.2}$$

where the positive constant δ is referred to as relaxation time. We make the following assumptions about $f(u)$, $A(u)$, $g(\delta, u, v)$ and the initial data:

(A1) $f(u), A(u)\in C^2$ satisfying $A'(u)\geqq c_1>0$, meas $\{u:(f'(u)/(1+A'(u)))'=0\}=0$;

(A2) $g(\delta,u,v)\in C^1(\mathbf{R}^2)$ for any fixed δ, $c_1(u,v)\delta\leqq g(\delta,u,v)\leqq c_2(u,v)\delta$,

$|g_u| \leqq c_3(u, v)\delta$, $|g_v| \leqq c_3(u,v)\delta$, where $c_i(u, v)$ are positive, continuous functions, $i = 1, 2, 3$;

(A3) $u_0(x)$, $v_0(x)$ are bounded in $L^\infty(\mathbf{R})$ and $\int_G |A(u_0(x)) - v_0(x)|\, dx \leqq \delta M(G)$, for any compact set G in $\mathbf{R}$.

System (1.1) consists of a conservation law and an equation with relaxation term. In [**4**, **7**, **8**] a similar model arising in combustion

$$(u + qz)_t + f(u)_x = 0, \quad z_t = k\varphi(u)z,$$

was studied. Other models with relaxation terms, [**1**, **9**, **10**], have been considered by other authors. For $g(\delta, u, v) = \delta$, the system (1.1) arises in chromatography, see [**9**].

In a recent paper [**10**], Tveito and Winther study vanishing relaxation of (1.1), (1.2) with $g = \delta$ in the framework of BV solutions under the assumption of monotonicity of f. Note that our proof needs no such restriction on the initial data and on f by using a complete different technique.

In this paper, we study the zero relaxation limit $g(\delta, u, v) \to 0$ as $\delta \to 0$ by using compensated compactness. This method has been well used on models for combustion [**4**, **7**], for viscoelasticity and phase transitions [**1**]. We show that the solutions of the viscous equations

$$(u + v)_t + f(u)_x = \varepsilon(u + v)_{xx}, \quad v_t = \varepsilon v_{xx} + \frac{A(u) - v}{g(\delta, u, v)}, \tag{1.3}$$

converge to the solutions of the equilibrium equation

$$(u + A(u))_t + f(u)_x = 0, \tag{1.4}$$

when δ and ε tend to zero related by $\delta = O(\varepsilon)$.

This paper is structured as follows: in Section 2 we consider the existence of viscosity solutions of system (1.3) with initial data

$$(u^\varepsilon, v^\varepsilon)|_{t=0} = (u_0^\varepsilon, v_0^\varepsilon), \tag{1.5}$$

where u_0^ε, v_0^ε are smooth functions obtained by smoothing $u_0(x)$, $v_0(x)$ with a mollifier, satisfying

$$u_0^\varepsilon \to u_0(x), \quad v_0^\varepsilon \to v_0(x) \quad \text{when } \varepsilon \to 0,$$

$$|\varepsilon u_{0,x}^\varepsilon(x)| \leqq M, \quad |\varepsilon v_{0,x}^\varepsilon(x)| \leqq M,$$

where M is a positive constant depending only on the bound of $|u_0^\varepsilon(x)|_{L^\infty}$, $|v_0^\varepsilon(x)|_{L^\infty}$ and is independent of ε. We first prove local existence using the contraction mapping principle. The next step is to show an *a priori* estimate in the L^∞ norm of the solutions, which is obtained by using the maximum principle given in [**5**]. Section 3 constitutes the heart of our analysis. There it is proved that for a function pair $\{\eta(u^{\varepsilon,\delta}), q(u^{\varepsilon,\delta})\}$ which satisfies

$$q'(u) = \frac{\eta'(u) f'(u)}{1 + A'(u)}$$

we have

$$\eta(u^{\varepsilon,\delta})_t + q(u^{\varepsilon,\delta})_x \quad \text{is compact in } H^{-1}_{\text{loc}}(\mathbf{R} \times \mathbf{R}^+).$$

This is proved mainly through energy estimates. In Section 4 the method of compensated compactness given in [2, 4] is used to study the convergence of the viscosity solutions $\{u^{\varepsilon,\delta}, v^{\varepsilon,\delta}\}$. First the convergence of $u^{\varepsilon,\delta}$ is shown and then, using Lemma 3.2, the convergence of $v^{\varepsilon,\delta}$. When taking $\delta = O(\varepsilon)$, the global weak solution of the equilibrium (1.4) is obtained as $\delta \to 0$.

2. Viscosity solutions

In this section, we consider the existence of the Cauchy problem for the parabolic system (1.3) with initial data (1.5). The local existence of solutions can be obtained by applying the general contraction mapping principle to an integral representation of (1.3). To extend the local solution to the global one, the *a priori* estimate in the following lemma is essential:

LEMMA 2.1. *If* $|A(u_0^\varepsilon(x))| \leqq M$, $|v_0^\varepsilon(x)| \leqq M$, $A'(u) \geqq c_1 > 0$ *and for any fixed* ε, $\delta > 0$, *the solution* $(u^\varepsilon, v^\varepsilon) \in C^2$ *of the Cauchy problem* (1.3), (1.5) *exists in* $(-\infty, \infty) + [0, T]$. *Then the following estimates hold*:

$$|A(u^\varepsilon(x, t))| \leqq M, \quad |v^\varepsilon(x, t)| \leqq M \quad \text{for } (x, t) \in (-\infty, \infty) \times [0, T]. \tag{2.1}$$

Proof. Substituting the second equation of (1.3) into the first, we get

$$\begin{cases} u_t + f(u)_x + \dfrac{A(u) - v}{g(\delta, u, v)} = \varepsilon u_{xx}, \\ v_t + \dfrac{v - A(u)}{g(\delta, u, v)} = \varepsilon v_{xx}. \end{cases} \tag{2.2}$$

Multiplying the first equation in (2.2) by $A'(u)$, we have

$$(A(u))_t + f'(u)(A(u))_x + A'(u)\frac{A(u) - v}{g(\delta, u, v)} = \varepsilon A(u)_{xx} - \varepsilon \frac{A''(u)}{(A'(u))^2}(A(u)_x)^2. \tag{2.3}$$

We are going to use the maximum principle [5] for the system

$$\begin{cases} (A(u))_t + f'(u)(A(u))_x + A'(u)\dfrac{A(u) - v}{g(\delta, u, v)} = \varepsilon A(u)_{xx} - \varepsilon \dfrac{A''(u)}{(A'(u))^2}(A(u)_x)^2, \\ v_t + \dfrac{v - A(u)}{g(\delta, u, v)} = \varepsilon v_{xx}, \end{cases} \tag{2.4}$$

with initial data

$$(A(u), v)|_{t=0} = (A(u_0^\varepsilon(x)), v_0^\varepsilon(x)). \tag{2.5}$$

Make the transformation

$$A(u) = w + M + \frac{N(x^2 + cLe^t)}{L^2}, \quad v = z + M + \frac{N(x^2 + cLe^t)}{L^2}, \tag{2.6}$$

where c, N, and L are positive constants, N is the upper bound of $A(u)$, v on $\mathbf{R} \times [0, T]$. The functions w and z, as easily seen, satisfy the equations

$$\begin{cases} w_t + \left(f'(u) + \dfrac{\varepsilon A'(u)}{(A'(u))^2} A(u)_x\right) w_x + \left(cLe^t + f'(u) + \varepsilon \dfrac{A''(u)}{(A'(u))^2} A(u)_x - 2\varepsilon\right) \dfrac{N}{L^2} \\ \qquad\qquad + \dfrac{A'(u)}{g}(w - z) = \varepsilon w_{xx}, \\ z_t + (cLe^t - 2\varepsilon) \dfrac{N}{L^2} + \dfrac{z - w}{g} = \varepsilon z_{xx}, \end{cases} \tag{2.7}$$

resulting from (2.4). Moreover,

$$\begin{cases} w_0(x) = A(u_0(x)) - M - \dfrac{N(x^2 + cL)}{L^2} < 0, \\ z_0(x) = v_0(x) - M - \dfrac{N(x^2 + cL)}{L^2} < 0, \\ w(L, t) < 0, \quad w(-L, t) < 0, \quad z(L, t) < 0, \quad z(-L, t) < 0. \end{cases} \tag{2.8}$$

Then, similar to the proof of [**5**, Lemma 2.2], we can obtain from (2.7), (2.8)

$$w(x, t) < 0, \quad z(x, t) < 0 \quad \text{on } (-L, L) \times [0, T].$$

Letting $L \to \infty$ in (2.6), we have $A(u) \leqq M$, $v \leqq M$ on $(-\infty, \infty) \times [0, T]$. Similarly $A(u) \geqq -M$, $v \geqq -M$. □

From Lemma 2.1, the solutions of the Cauchy problem (1.3), (1.5) have an *a priori* estimate

$$|u^\varepsilon(x, t)| \leqq M, \quad |v^\varepsilon(x, t)| \leqq M, \tag{2.9}$$

where M is a positive constant which depends only on the initial data. Therefore the following global existence theorem is obtained:

THEOREM 2.2. *Let $u_0(x)$, $v_0(x)$ be bounded measurable, $A(u) \in C^1$, $A'(u) \geqq c_1 > 0$, $g(\delta, u, v)$ satisfy the condition (A_2). Then for any fixed δ, $\varepsilon > 0$, the Cauchy problem* (1.3), (1.5) *has a unique classical solution $(u^{\varepsilon,\delta}(x, t), v^{\varepsilon,\delta}(x, t))$ satisfying* (2.9).

3. Compactness in $H^{-1}_{\text{loc}}(R \times R^+)$

In this section, we mainly obtain the following lemma:

LEMMA 3.1. *For any C^2 pair of functions $(\eta(u^{\varepsilon,\delta}), q(u^{\varepsilon,\delta}))$, let $\delta = O(\varepsilon)$, then*

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ are compact in } H^{-1}_{\text{loc}}(R \times R^+), \tag{3.1}$$

where $(\eta(u), q(u))$ satisfies

$$q'(u) = \frac{\eta'(u) f'(u)}{1 + A'(u)}. \tag{3.2}$$

Lemma 3.1 is the core of this paper, which guarantees the div-curl lemma of the theory of compensated compactness to be true, namely:

$$\overline{\eta_1(u^\varepsilon) q_2(u^\varepsilon) - \eta_2(u^\varepsilon) q_1 u^\varepsilon)} = \overline{\eta_1(u^\varepsilon)}\, \overline{q_2(u^\varepsilon)} - \overline{\eta_2(u^\varepsilon)}\, \overline{q_1(u^\varepsilon)}, \tag{3.3}$$

for any C^2 pair of functions (η_i, q_i) $(i=1, 2)$ satisfying (3.2), where '$\overline{\cdot}$' means weak limit.

Before giving the proof of Lemma 3.1, we first study some estimates of the solutions $(u^\varepsilon, v^\varepsilon)$.

LEMMA 3.2. *If $A'(u) \geqq c_1 > 0$, then*

$$\varepsilon((u_x^\varepsilon)^2, \varepsilon(v_x^\varepsilon)^2), \quad \frac{(A(u^\varepsilon)-v^\varepsilon)^2}{\delta}$$

are bounded in $L^1_{\mathrm{loc}}(R \times R^+)$.

Proof. Multiplying the first equation in (2.2) by $A(u)$ and the second one by v and adding the result, we obtain

$$\left(\int^u A(u)\,du\right)_t + \left(\int^u A(u)f'(u)\,du\right)_x + \left(\frac{v^2}{2}\right)_t + \frac{(A(u)-v)^2}{g(\delta, u, v)}$$
$$= \varepsilon\left(\int^u A(u)\,du + \frac{v^2}{2}\right)_{xx} - \varepsilon A'(u)u_x^2 - \varepsilon v_x^2. \tag{3.4}$$

Since

$$g(\delta, u, v) = O(\delta),$$

We can end the proof of Lemma 3.2 by multiplying a suitable test function to (3.4). □

LEMMA 3.3. *If the assumptions* (A_1), (A_2), (A_3) *are satisfied, then* $\delta\varepsilon^2(u_{xx}^\varepsilon)^2$, $\delta(u_t^\varepsilon)^2$ *are bounded in* $L^1_{\mathrm{loc}}(R \times R^+)$ *if* $\delta = O(\varepsilon)$.

Proof. Differentiating the first equation in (2.2) with respect to x, we have

$$(u_x)_t + (f'(u)u_x)_x + \left(\frac{A(u)-v}{g(\delta, u, v)}\right)_x = \varepsilon(u_{xx})_x. \tag{3.5}$$

Multiplying (3.5) by u_x, we have

$$\left(\frac{u_x^2}{2}\right)_t + (f'(u)u_x^2)_x - f'(u)u_xu_{xx} + \left(\frac{A(u)-v}{g(\delta, u, v)}u_x\right)_x - \frac{A(u)-v}{g(\delta, u, v)}u_{xx} = \varepsilon\left(\frac{u_x^2}{2}\right)_{xx} - \varepsilon(u_{xx})^2. \tag{3.6}$$

Multiplying (3.6) by any test function $\varphi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$, $\varphi \geqq 0$, we have

$$\int_R dx \int_{R^+} \varepsilon u_{xx}^2 \varphi\,dt \leqq c\left(\int_R dx \int_{R^+} \varepsilon u_x^2|\varphi_{xx}| + u_x^2(|\varphi_t| + |\varphi_x|) + \left(\frac{u_x^2}{\varepsilon} + \frac{A(u)-v}{\varepsilon\delta}\right)\varphi\right.$$
$$\left. + \frac{|A(u)-v|}{\delta}|u_x|\,|\varphi_x|\,dt + \int_R (u_{0,x}^\varepsilon)^2\varphi(x, 0)\,dx\right). \tag{3.7}$$

So the estimates in Lemma 3.2 give us the boundedness of $\delta\varepsilon^2u_{xx}^2$ in $L^1_{\mathrm{loc}}(R \times R^+)$ if $\delta = O(\varepsilon)$.

To estimate δu_t^2, we have from the first equation of (2.2)

$$u_t^2 \leqq c\left(\varepsilon^2 u_{xx}^2 + u_x^2 + \frac{(A(u)-v)^2}{\delta^2}\right). \tag{3.8}$$

So the boundedness of δu_t^2 in $L^1_{\text{loc}}(R \times R^+)$ is proved if $\delta = O(\varepsilon)$. □

Proof of Lemma 3.1. We rewrite the first equation of (1.3) as

$$u_t + \frac{(v-A(u))_t}{1+A'(u)} + \frac{f'(u)}{1+A'(u)} u_x = \varepsilon \frac{(u+v)_{xx}}{1+A'(u)}. \tag{3.9}$$

Then, for any pair of C^2 functions (η, q) satisfying (3.2), we have

$$\eta_t + q_x + \left(\frac{v-A(u)}{1+A'(u)}\eta'(u)\right)_t - (v-A(u))\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_t$$
$$= \varepsilon\left((u+v)_x \frac{\eta'(u)}{1+A'(u)}\right)_x - \varepsilon(u+v)_x\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_x. \tag{3.10}$$

We write

$$\eta_t + q_x = I_1 + I_2,$$

where

$$I_1 = \varepsilon\left((u+v)_x \frac{\eta'(u)}{1+A'(u)}\right)_x - \left(\frac{v-A(u)}{1+A'(u)}\eta'(u)\right)_t,$$

$$I_2 = (v-A(u))\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_t - \varepsilon(u+v)_x\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_x.$$

Since, for any $\varphi \in C_0^1(R \times R^+)$,

$$\left|\iint_{R\times R^+} \varepsilon\left((u+v)_x \frac{\eta'(u)}{1+A'(u)}\right)_x \varphi - \left(\frac{v-A(u)}{1+A'(u)}\eta'(u)\right)_t \varphi \, dx\, dt\right|$$
$$\leqq c\sqrt{\varepsilon}\left(\iint_G \varepsilon(u_x^2+v_x^2)\right)^{\frac{1}{2}} \iint_{R\times R^+}\left(\frac{\eta'(u)}{1+A'(u)}\varphi_x\right)^2 dx\, dt$$
$$+ c\sqrt{\delta}\left(\iint_G \frac{(v-A(u))^2}{\delta}\, dx\, dt\right)^{\frac{1}{2}}\left(\iint_{R\times R^+}\left(\frac{\eta'(u)}{1+(A'(u))}\varphi_t\right)^2 dx\, dt\right)^{1/2}$$
$$+ c\int_{G|_{t=0}} |v_0(x) - A(u_0(x))|\, dx,$$

where G is the compactly supported set of φ. Thus I_1 is compact in $H^{-1}_{\text{loc}}(R \times R^+)$.

Since for any $\varphi \in C_0(R \times R^+)$

$$\left|\iint_{R\times R^+}\left\{(v-A(u))\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_t - \varepsilon(u+v)_x\left(\frac{\eta'(u)}{1+A'(u)}\right)' u_x\right\}\varphi\, dx\, dt\right|$$
$$\leqq c\left\{\left(\iint_G \frac{(v-A(u))^2}{\delta}\, dx\, dt\right)^{\frac{1}{2}}\left(\iint_G \delta u_t^2\, dx\, dt\right)^{\frac{1}{2}} + \iint_G \varepsilon u_x^2 + \varepsilon v_x^2\, dx\, dt\right\} \leqq M,$$

I_2 is bounded in $C_0^*(R \times R^+)$ if $\delta = O(\varepsilon)$. So I_2 is compact in $W_{\mathrm{loc}}^{1,q}(R \times R^+)$ for some $1 < q < 2$. Therefore $\eta_t + q_x$ is compact in $W_{\mathrm{loc}}^{1,q}(R \times R^+)$. Moreover, the boundedness of u^ε gives us that $\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$ is bounded in $W^{-1,r}(R \times R^+)$ for any $r > 2$. By an embedding theorem ([**11**, Lemma 28], also used in [**3**, Theorem 3]) compactness of $\eta_t + q_x$ in $W_{\mathrm{loc}}^{-1,2}(R \times R^+)$ follows. □

4. Zero relaxation limit

In this section, we are going to consider the convergence of the solutions for the Cauchy problem (1.3), (1.5) as the dissipation ε and the relaxation δ tends to zero at the same rate. Our technique is to apply the method of compensated compactness. Our analysis is based on first showing the convergence of $u^{\varepsilon,\delta}$ and, through coupling given by the third term in Lemma 3.2, the convergence of $v^{\varepsilon,\delta}$.

Since (3.1) in Lemma 3.1 holds and thus the div-curl lemma (3.1) holds, we chose particular entropy pairs

$$(\eta_1(\lambda), q_1(\lambda)) = (\lambda - k, h(\lambda) - h(k)), \quad (\eta_2(\lambda), q_2(\lambda)) = \left(h(\lambda) - h(k), \int_k^\lambda h'^2(s)\, ds\right), \tag{4.1}$$

where k is an arbitrary constant,

$$h(\lambda) = \int^\lambda \frac{f'(s)}{1 + A'(s)}\, ds$$

and get from (3.3),

$$\overline{(u^\varepsilon - k) \int_k^{u^\varepsilon} h'^2(s)\, ds - (h(u^\varepsilon) - h(k))^2} = \overline{(u^\varepsilon - k)}\; \overline{\int_k^{u^\varepsilon} h'(s)\, ds} - (\overline{h(u^\varepsilon)} - h(k))^2. \tag{4.2}$$

Let $u^\varepsilon \overset{*}{\rightharpoonup} u(L^\infty(R))$ ($\overset{*}{\rightharpoonup}$ represents weak star convergence). It is shown in [**2**, Theorem 1] and also in [**4**, Lemma 3.3] that

$$\overline{(u^\varepsilon - u) \int_u^{u^\varepsilon} h'^2(s)\, ds - (h(u^\varepsilon) - h(u))^2} + (\overline{h(u^\varepsilon)} - h(u))^2 = 0. \tag{4.3}$$

Since

$$(u^\varepsilon - u) \int_u^{u^\varepsilon} h'^2(s)\, ds - (h(u^\varepsilon) - h(u))^2 \geqq 0. \tag{4.4}$$

then from (4.3)

$$\overline{h(u^\varepsilon)} - h(u) = 0$$

and

$$\overline{(u^\varepsilon - u) \int_u^{u^\varepsilon} h'^2(s)\, ds - (h(u^\varepsilon) - h(u))^2} = 0. \tag{4.5}$$

Therefore

$$\lim_{\varepsilon\to 0}\int_\Omega\left\{(u^\varepsilon-u)\int_u^{u^\varepsilon}h'^2(s)\,ds-(h(u^\varepsilon)-h(u))^2\right\}dx\,dt=0, \tag{4.6}$$

Ω being any bounded region in $R\times R^+$. From this and the condition

$$\text{meas}\left\{u: h''(u)=\left(\frac{f'(u)}{1+A'(u)}\right)'=0\right\}=0,$$

we may get ([**2**], or [**4**, Lemma 3.3]) the strong convergence of $u^\varepsilon\to u$ a.e. on Ω. Moreover, the boundedness of

$$\frac{(A(u^\varepsilon)-v^\varepsilon)^2}{\delta}\quad\text{in } L^1_{\text{loc}}(R\times R^+)$$

gives us the strong convergence of $v^\varepsilon\to v$ a.e. on Ω. So we end the paper with the following main theorem:

THEOREM 4.1. *The solutions of* $(u^{\varepsilon,\delta},v^{\varepsilon,\delta})(x,t)$ *of* (1.3), (1.5) *with the assumptions* (A_1)–(A_3) *converge to bounded measurable functions* $(u,v)(x,t)$ *as* ε *and* δ *tend to zero related by* $\delta=O(\varepsilon)$. *Moreover,* $u=u(x,t)$ *is a weak solution of the Cauchy problem*

$$(u+A(u))_t+f(u)_x=0,\quad u|_{t=0}=u_0(x)$$

and $v(x,t)=A(u(x,t))$.

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