Non–uniqueness of entropy–conserving solutions to the ideal compressible MHD equations

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Abstract

In this note we consider the ideal compressible magneto–hydrodynamics (MHD) equations in a special two dimensional setting. We show that there exist particular initial data for which one obtains infinitely many entropy–conserving weak solutions by using the convex integration technique. Finally this is extended to the isentropic case.

1 Introduction

We consider the ideal compressible magneto–hydrodynamics (MHD) equations

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\operatorname{curl} \mathbf{B}) \times \mathbf{B} &= 0, \\
\partial_t \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + \frac{1}{2}|\mathbf{B}|^2 \right) + \operatorname{div} \left[\left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho, p) + p + |\mathbf{B}|^2 \right) \mathbf{u} \right] - \operatorname{div}\big((\mathbf{B} \cdot \mathbf{u}) \mathbf{B} \big) &= 0, \\
\partial_t \mathbf{B} + \operatorname{curl}(\mathbf{B} \times \mathbf{u}) &= 0, \\
\operatorname{div} \mathbf{B} &= 0.
\end{aligned} \tag{1}$$

The unknown functions in (1) are the density $\varrho > 0$, the pressure $p > 0$, the velocity $\mathbf{u} \in \mathbb{R}^3$ and the magnetic field $\mathbf{B} \in \mathbb{R}^3$, which are all functions of the time $t \in [0, T)$ and the spatial variable $\mathbf{x} = (x, y, z)^\mathsf{T} \in \mathbb{R}^3$. The internal energy e is a given function of the density ϱ and the pressure p.

In this note we consider a special two dimensional setting. Let $\Omega \subset \mathbb{R}^2$ a bounded two dimensional spacial domain. We consider $\mathbf{u} = (u, v, 0)^\mathsf{T}$ and $\mathbf{B} = (0, 0, b)^\mathsf{T}$ and furthermore we let all the unknowns only depend on $(x, y) \in \Omega$. From now on we write $\mathbf{u} = (u, v)^\mathsf{T} \in \mathbb{R}^2$ and

$\mathbf{x} = (x,y)^\mathsf{T} \in \Omega \subset \mathbb{R}^2$ for the corresponding two dimensional vectors. Then the MHD system (1) turns into

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}\big(\varrho \mathbf{u} \otimes \mathbf{u}\big) + \nabla\Big(p + \frac{1}{2}b^2\Big) &= 0, \\ \partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + \frac{1}{2}b^2\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + p + b^2\right)\mathbf{u}\right] &= 0, \\ \partial_t b + \operatorname{div}(b\mathbf{u}) &= 0. \end{aligned} \tag{2}$$

Note that in (2) $\operatorname{div}, \nabla$ are two dimensional differential operators in contrast to (1), where they are three dimensional differential operators.

We endow system (2) with initial conditions

$$\big(\varrho, p, \mathbf{u}, b\big)(0,\cdot) = \big(\varrho_0, p_0, \mathbf{u}_0, b_0\big) \tag{3}$$

and impermeability boundary conditions

$$\mathbf{u}\cdot\mathbf{n}\big|_{\partial\Omega} = 0. \tag{4}$$

Definition 1.1. A 4-tuple $(\varrho, p, \mathbf{u}, b) \in L^\infty\big([0,T)\times\Omega; (0,\infty)\times(0,\infty)\times\mathbb{R}^2\times\mathbb{R}\big)$ is a weak solution to (2), (3), (4) if the following equations hold for all test functions $\varphi, \phi, \psi \in C_c^\infty\big([0,T)\times\mathbb{R}^2\big)$ and $\boldsymbol{\varphi} \in C_c^\infty\big([0,T)\times\mathbb{R}^2;\mathbb{R}^2\big)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}\big|_{\partial\Omega}$:

$$\int_0^T \int_\Omega \big[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0\varphi(0,\cdot)\,\mathrm{d}\mathbf{x} = 0 \tag{5}$$

$$\int_0^T \int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \big(\varrho\mathbf{u}\otimes\mathbf{u}\big):\nabla\boldsymbol{\varphi} + \Big(p + \frac{1}{2}b^2\Big)\operatorname{div}\boldsymbol{\varphi}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}\mathbf{x} = 0 \tag{6}$$

$$\begin{aligned} \int_0^T \int_\Omega \left[\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + \frac{1}{2}b^2\Big)\partial_t\phi + \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + p + b^2\Big)\mathbf{u}\cdot\nabla\phi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,p_0) + \frac{1}{2}b_0^2\Big)\phi(0,\cdot)\,\mathrm{d}\mathbf{x} = 0 \end{aligned} \tag{7}$$

$$\int_0^T \int_\Omega \big[b\partial_t\psi + b\mathbf{u}\cdot\nabla\psi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega b_0\psi(0,\cdot)\,\mathrm{d}\mathbf{x} = 0 \tag{8}$$

Remark. The impermeability boundary condition is represented by the choice of the test functions.

Remark. Note that we exclude vacuum for our consideration, i.e. in this note $\varrho > 0$, $p > 0$.

It is a well–known fact that there may exist physically non–relevant weak solutions to conservation laws. Hence one has to introduce additional selection criteria in order to single out the physically relevant weak solutions. A common approach is to impose an entropy inequality. However for the MHD system (1) there is no known entropy.

Note that for the Euler system the functions

$$\eta = -\varrho s(\varrho, p) \quad \text{and} \quad q = -\varrho s(\varrho, p)\mathbf{u}$$

form an entropy pair. Here the specific entropy $s = s(\varrho, p)$ is a given function as well as the internal energy e and note that these functions are interrelated by Gibbs' relation.

It is a straightforward computation to show that a strong solution to the MHD system (1) fulfills

$$\partial_t\big(\varrho s(\varrho, p)\big) + \operatorname{div}\big(\varrho s(\varrho, p)\mathbf{u}\big) = 0. \tag{9}$$

Although this suggests that (η, q) is an entropy pair for the MHD system, too, (η, q) is *not* an entropy pair for MHD, cf. [2]. However (η, q) is still used as a selection criterion in the literature for example if Riemann problems are considered and one wants to find out whether or not a shock is physical, see e. g. [9]. We misuse terminology and call η and q still entropy, entropy flux respectively.

The weak solutions, whose existence we will prove in this note, fulfill the entropy equation (9) in the weak sense. We call such solutions entropy–conserving.

Definition 1.2. A weak solution $(\varrho, p, \mathbf{u}, b)$ to (2), (3), (4) is called *entropy–conserving*, if for all test functions $\varphi \in C_c^\infty\big([0,T) \times \mathbb{R}^2\big)$ the entropy equation

$$\int_0^T \int_\Omega \big[\varrho s(\varrho, p)\partial_t\varphi + \varrho s(\varrho, p)\mathbf{u} \cdot \nabla\varphi\big] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0 s(\varrho_0, p_0)\varphi(0, \cdot)\,\mathrm{d}\mathbf{x} = 0 \tag{10}$$

holds.

The following theorem is our main result:

Theorem 1.3. *Let $\varrho_0, p_0 \in L^\infty(\Omega; (0,\infty))$ and $b_0 \in L^\infty(\Omega)$ be arbitrary piecewise constant functions. Then there exists $\mathbf{u}_0 \in L^\infty(\Omega; \mathbb{R}^2)$ such that there are infinitely many entropy–conserving weak solutions to (2) with initial data $\varrho_0, p_0, \mathbf{u}_0, b_0$ and impermeability boundary condition. These solutions have the property that ϱ, p and b do not depend on time; in other words $\varrho \equiv \varrho_0$, $p \equiv p_0$ and $b \equiv b_0$.*

The proof of Theorem 1.3 relies on the non–uniqueness proof for the full Euler system provided in [5] and consists of two main ideas. The first one is to make use of a result (see Proposition 2.1 below) which was proved by Feireisl [4] and also by Chiodaroli [3]. This result is based on the convex integration method, that was developed by De Lellis and Székelyhidi [7, 6] in the context of the pressureless incompressible Euler equations. The second idea is the fact that ϱ, p and b can be chosen *piecewise* constant, what was observed originally by Luo, Xie and Xin [8].

Note that non–uniqueness of weak solutions fulfilling an entropy inequality (even in one space dimension) is well–known: There exist Riemann initial data for which one has more than one solutions, see e. g. Torrilhon [9] and references therein.

Note furthermore that there is also a convex integration result to incompressible ideal MHD by Bronzi et al. [1]. There the same two dimensional setting is considered as in the present note. In contrast to this note, where a convex integration result for Euler is used, Bronzi et al. apply the convex integration techique directly to an incompressible version of (2).

2 Proof of the main result

In order to prove Theorem 1.3 we will make use of the following proposition whose proof is based on convex integration.

Proposition 2.1. *Let $Q \subset \mathbb{R}^2$ a bounded domain, $\varrho > 0$ and $C > 0$ positive constants. Then there exists $\mathbf{m}_0 \in L^\infty(Q;\mathbb{R}^2)$ such that there are infinitely many functions*

$$\mathbf{m} \in L^\infty\big((0,T)\times Q;\mathbb{R}^2\big) \cap C_{\mathrm{weak}}\big([0,T];L^2(Q;\mathbb{R}^2)\big)$$

satisfying

$$\int_0^T \int_Q \mathbf{m}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0 \tag{11}$$

$$\int_0^T \int_Q \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right) : \nabla\boldsymbol{\varphi}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_Q \mathbf{m}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}\mathbf{x} = 0 \tag{12}$$

for all test functions $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^2)$ and $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times\mathbb{R}^2;\mathbb{R}^2)$, and additionally

$$E_{\mathrm{kin}} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} = C \quad \text{a.e. in } (0,T)\times Q, \qquad E_{0,\mathrm{kin}} = \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho} = C \quad \text{a.e. in } Q.$$

For the proof of Proposition 2.1 we refer to [4, Theorem 13.6.1].

Now we are able to prove Theorem 1.3.

Proof. Let $\varrho_0, p_0 \in L^\infty(\Omega;(0,\infty))$ and $b_0 \in L^\infty(\Omega)$ given piecewise constant functions. Then there exist finitely many $Q_i \subset \Omega$ open and pairwise disjoint, such that $\Omega = \bigcup_i \overline{Q_i}$ and $\varrho_0\big|_{Q_i} = \varrho_i$, $p_0\big|_{Q_i} = p_i$ and $b_0\big|_{Q_i} = b_i$ with constants $\varrho_i, p_i > 0$ and $b_i \in \mathbb{R}$. We apply Proposition 2.1 on each Q_i to $\varrho = \varrho_i$ and $C = \Lambda - p_i - \frac{1}{2}b_i^2$, where Λ is a constant with $\Lambda > \max_i\big(p_i + \frac{1}{2}b_i^2\big)$. This yields $\mathbf{m}_{0,i} \in L^\infty(Q_i;\mathbb{R}^2)$ and infinitely many $\mathbf{m}_i \in L^\infty((0,T)\times Q_i;\mathbb{R}^2)$ with the properties given in Proposition 2.1. We then piece together the $\mathbf{m}_{0,i} \in L^\infty(Q_i;\mathbb{R}^2)$ to $\mathbf{m}_0 \in L^\infty(\Omega;\mathbb{R}^2)$ and the $\mathbf{m}_i \in L^\infty((0,T)\times Q_i;\mathbb{R}^2)$ to $\mathbf{m} \in L^\infty((0,T)\times\Omega;\mathbb{R}^2)$.

We define $\mathbf{u}_0 := \frac{\mathbf{m}_0}{\varrho_0} \in L^\infty(\Omega;\mathbb{R}^2)$ and for each $\mathbf{m}$ we define a function $\mathbf{u} := \frac{\mathbf{m}}{\varrho_0} \in L^\infty\big((0,T)\times\Omega;\mathbb{R}^2\big)$. Furthermore we define $(\varrho,p,b) \in L^\infty\big([0,T)\times\Omega;(0,\infty)\times(0,\infty)\times\mathbb{R}\big)$ by $\varrho\equiv\varrho_0$, $p\equiv p_0$ and $b\equiv b_0$. We claim that $(\varrho,p,\mathbf{u},b)$ is an entropy–conserving weak solution to (2) with initial data $\varrho_0, p_0, \mathbf{u}_0, b_0$.

Let $\varphi,\phi,\psi \in C_c^\infty\big([0,T)\times\mathbb{R}^2\big)$ and $\boldsymbol{\varphi}\in C_c^\infty\big([0,T)\times\mathbb{R}^2;\mathbb{R}^2\big)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}\big|_{\partial\Omega}$ arbitrary test functions. Using (11) and (12), we obtain the following.

$$\begin{aligned}
&\int_0^T\int_\Omega \big[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0\varphi(0,\cdot)\,\mathrm{d}\mathbf{x}\\
&= \sum_i \varrho_i \int_{Q_i}\left(\int_0^T \partial_t\varphi\,\mathrm{d}t + \varphi(0,\cdot)\right)\mathrm{d}\mathbf{x} + \sum_i\int_0^T\int_{Q_i}\mathbf{m}_i\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\int_\Omega\left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \big(\varrho\mathbf{u}\otimes\mathbf{u}\big):\nabla\boldsymbol{\varphi} + \Big(p+\frac12 b^2\Big)\operatorname{div}\boldsymbol{\varphi}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}\mathbf{x}\\
&= \sum_i\left(\int_0^T\int_{Q_i}\left[\mathbf{m}_i\cdot\partial_t\boldsymbol{\varphi} + \left(\frac{\mathbf{m}_i\otimes\mathbf{m}_i}{\varrho_i} - \frac12\frac{|\mathbf{m}_i|^2}{\varrho_i}\mathbb{I}\right):\nabla\boldsymbol{\varphi}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_i}\mathbf{m}_{0,i}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}\mathbf{x}\right)\\
&\quad + \sum_i\int_0^T\int_{Q_i}\left[\frac12\frac{|\mathbf{m}_i|^2}{\varrho_i} + \Big(p_i + \frac12 b_i^2\Big)\right]\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&= \Lambda\int_0^T\int_\Omega \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\int_\Omega\left[\Big(\frac12\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + \frac12 b^2\Big)\partial_t\phi + \Big(\frac12\varrho|\mathbf{u}|^2 + \varrho e(\varrho,p) + p + b^2\Big)\mathbf{u}\cdot\nabla\phi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad + \int_\Omega\Big(\frac12\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,p_0) + \frac12 b_0^2\Big)\phi(0,\cdot)\,\mathrm{d}\mathbf{x}\\
&= \sum_i\Big(\Lambda + \varrho_i e(\varrho_i,p_i) - p_i\Big)\int_{Q_i}\left(\int_0^T\partial_t\phi\,\mathrm{d}t + \phi(0,\cdot)\right)\mathrm{d}\mathbf{x}\\
&\quad + \sum_i\frac{\Lambda + \varrho_i e(\varrho_i,p_i) + \frac12 b_i}{\varrho_i}\int_0^T\int_{Q_i}\mathbf{m}_i\cdot\nabla\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\int_\Omega\big[b\partial_t\psi + b\mathbf{u}\cdot\nabla\psi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega b_0\psi(0,\cdot)\,\mathrm{d}\mathbf{x}\\
&= \sum_i b_i\int_{Q_i}\left(\int_0^T\partial_t\psi\,\mathrm{d}t + \psi(0,\cdot)\right)\mathrm{d}\mathbf{x} + \sum_i\frac{b_i}{\varrho_i}\int_0^T\int_{Q_i}\mathbf{m}_i\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad = \quad 0
\end{aligned}$$

We have shown that the equations (5) - (8) hold. Hence $(\varrho, p, \mathbf{u}, b)$ is indeed a weak solution. It remains to show that this solution is entropy–conserving. In other words we have to show that (10) holds. Let $\varphi \in C_c^\infty\big([0,T) \times \mathbb{R}^2\big)$ be an arbitrary test function. We obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \left[\varrho s(\varrho,p)\partial_t \varphi + \varrho s(\varrho,p)\mathbf{u} \cdot \nabla \varphi\right] \mathrm{d}\mathbf{x}\ \mathrm{d}t + \int_\Omega \varrho_0 s(\varrho_0,p_0)\varphi(0,\cdot)\,\mathrm{d}\mathbf{x} \\
&= \sum_i \varrho_i s(\varrho_i,p_i) \int_{Q_i} \left(\int_0^T \partial_t \varphi \,\mathrm{d}t + \varphi(0,\cdot) \right) \mathrm{d}\mathbf{x} + \sum_i s(\varrho_i,p_i) \int_0^T \int_{Q_i} \mathbf{m}_i \cdot \nabla \psi \,\mathrm{d}\mathbf{x}\ \mathrm{d}t \quad = \quad 0.
\end{aligned}$$

Thus $(\varrho, p, \mathbf{u}, b)$ is an entropy–conserving weak solution. Since there are infinitely many $\mathbf{m}$ from Prop. 2.1, there are infinitely many entropy–conserving solutions $(\varrho, p, \mathbf{u}, b)$. □

3 Isentropic MHD

In this section we extend our result to isentropic MHD equations. The isentropic MHD system reads

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) - (\operatorname{curl} \mathbf{B}) \times \mathbf{B} &= 0, \\
\partial_t \mathbf{B} + \operatorname{curl}(\mathbf{B} \times \mathbf{u}) &= 0, \\
\operatorname{div} \mathbf{B} &= 0.
\end{aligned} \tag{13}$$

The unknown functions are the density $\varrho > 0$, the velocity $\mathbf{u} \in \mathbb{R}^3$ and the magnetic field $\mathbf{B} \in \mathbb{R}^3$. In contrast to the MHD system (1) the pressure p in (13) is not an unknown but a given function of the density, where $p(\varrho) > 0$ for all $\varrho > 0$.

Again we consider a two dimensional setting. Let $\Omega \subset \mathbb{R}^2$ a bounded two dimensional spacial domain. We consider $\mathbf{u} = (u, v, 0)^\mathsf{T}$ and $\mathbf{B} = (0, 0, b)^\mathsf{T}$ and furthermore we let all the unknowns only depend on $(x, y) \in \Omega$. Then the isentropic MHD system (13) turns into

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}\big(\varrho \mathbf{u} \otimes \mathbf{u}\big) + \nabla\Big(p(\varrho) + \frac{1}{2} b^2\Big) &= 0, \\
\partial_t b + \operatorname{div}(b \mathbf{u}) &= 0.
\end{aligned} \tag{14}$$

For the isentropic *Euler* system, the energy

$$\eta = \frac{1}{2}\varrho |\mathbf{u}|^2 + P(\varrho) + \frac{1}{2}|\mathbf{B}|^2$$

is an entropy. Here $P(\varrho)$ is called pressure potential and is given by

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(r)}{r}\,\mathrm{d}r.$$

Similar to the full MHD system considered above, one can show that the energy is *not* an entropy for (13) but strong solutions fulfill the corresponding energy equation

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+\frac{1}{2}|\mathbf{B}|^2\right)+\operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+p(\varrho)+|\mathbf{B}|^2\right)\mathbf{u}\right]-\operatorname{div}\big((\mathbf{B}\cdot\mathbf{u})\mathbf{B}\big)=0. \quad (15)$$

Hence we will look for energy–conserving weak solutions. In the considered setting the energy equation (15) turns into

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+\frac{1}{2}b^2\right)+\operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+p(\varrho)+b^2\right)\mathbf{u}\right]=0.$$

Definition 3.1. A triple $(\varrho,\mathbf{u},b)\in L^\infty\big([0,T)\times\Omega;(0,\infty)\times\mathbb{R}^2\times\mathbb{R}\big)$ is a weak solution to (14) with initial data $\varrho_0,\mathbf{u}_0,b_0$ and impermeability boundary condition if the following equations hold for all test functions $\varphi,\psi\in C_c^\infty\big([0,T)\times\mathbb{R}^2\big)$ and $\boldsymbol{\varphi}\in C_c^\infty\big([0,T)\times\mathbb{R}^2;\mathbb{R}^2\big)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}\big|_{\partial\Omega}$:

$$\int_0^T\int_\Omega\big[\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_\Omega\varrho_0\varphi(0,\cdot)\,\mathrm{d}\mathbf{x}=0 \quad (16)$$

$$\int_0^T\int_\Omega\left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+\big(\varrho\mathbf{u}\otimes\mathbf{u}\big):\nabla\boldsymbol{\varphi}+\Big(p(\varrho)+\frac{1}{2}b^2\Big)\operatorname{div}\boldsymbol{\varphi}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_\Omega\varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}\mathbf{x}=0 \quad (17)$$

$$\int_0^T\int_\Omega\big[b\partial_t\psi+b\mathbf{u}\cdot\nabla\psi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_\Omega b_0\psi(0,\cdot)\,\mathrm{d}\mathbf{x}=0 \quad (18)$$

A weak solution is called energy–conserving if in addtion for all test functions $\phi\in C_c^\infty\big([0,T)\times\mathbb{R}^2\big)$ the energy equation

$$\begin{aligned}\int_0^T\int_\Omega\left[\Big(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+\frac{1}{2}b^2\Big)\partial_t\phi+\Big(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+p(\varrho)+b^2\Big)\mathbf{u}\cdot\nabla\phi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t&\\ +\int_\Omega\Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+P(\varrho_0)+\frac{1}{2}b_0^2\Big)\phi(0,\cdot)\,\mathrm{d}\mathbf{x}=0&\end{aligned} \quad (19)$$

holds.

The Cauchy problem for the isentropic MHD equations is ill–posed, too:

Corollary 3.2. *Let $\varrho_0\in L^\infty(\Omega;(0,\infty))$ and $b_0\in L^\infty(\Omega)$ be arbitrary piecewise constant functions. Then there exists $\mathbf{u}_0\in L^\infty(\Omega;\mathbb{R}^2)$ such that there are infinitely many energy–conserving weak solutions to* (14) *with initial data $\varrho_0,\mathbf{u}_0,b_0$ and impermeability boundary condition. These solutions have the property that ϱ and b do not depend on time; in other words $\varrho\equiv\varrho_0$ and $b\equiv b_0$.*

Proof. Let $\varrho_0 \in L^\infty(\Omega;(0,\infty))$ and $b_0 \in L^\infty(\Omega)$ given piecewise constant functions. Set furthermore $p_0 := p(\varrho_0)$. Then $p_0 \in L^\infty(\Omega;(0,\infty))$ is a piecewise constant function. Additionally we can choose the function $e(\varrho,p)$ in such a way that $\varrho_0 e(\varrho_0,p_0) = P(\varrho_0)$. We know from Theorem 1.3 that there exists an initial velocity $\mathbf{u}_0 \in L^\infty(\Omega;\mathbb{R}^2)$ such that there are infinitely many entropy–conserving weak solutions $(\varrho \equiv \varrho_0, p \equiv p_0, \mathbf{u}, b \equiv b_0)$ to (2) with initial data $\varrho_0, p_0, \mathbf{u}_0, b_0$. It is easy to check that for each of these solutions, the triple $(\varrho \equiv \varrho_0, \mathbf{u}, b \equiv b_0)$ is an energy–conserving weak solution to the isentropic MHD equations (14) with initial data $\varrho_0, \mathbf{u}_0, b_0$ in the sense of Definition 3.1. □

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