ASYMPTOTIC PRESERVING SCHEMES FOR KINETIC EQUATIONS THAT ARE ALSO STATIONARY PRESERVING*

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Abstract. In this work we are interested in the stationary preserving (SP) property of asymptotic preserving (AP) schemes for kinetic models. Our key observation is that as far as some macroscopic quantities can be updated explicitly, a large class of AP schemes have the SP property as well. To illustrate the generality of our observation, three different AP schemes for three different kinetic models are considered. Their SP properties are proved analytically and tested numerically, which confirms our observations.

Key words. Asymptotic preserving, Stationary preserving, Neutron transport equation, Chemotaxis kinetic model, The Boltzmann equation, Parity based schemes, UGKS, Penalty method.

AMS subject classifications. 65M08, 35Q20, 35Q92

1. Introduction.

Kinetic models describe the time evolution of probability density distribution of particles that travel freely for a certain distance and then change their directions due to collision or scattering. They usually include a transport term that takes into account the movement of the particles and integral terms that take into account the scattering, tumbling or colliding. When the average distance between two successive velocity change is small, i.e. the mean free path is small, one has to use resolved space and time steps that are less than the mean free path. Moreover, the probability density function in kinetic models depends not only on space and time but also on velocity. The high dimensionality and the small mean free path lead to an extremely high computational cost and AP schemes that allow mean free path independent meshes become popular in last decades.

AP schemes were first proposed in [15, 14] for the neutron transport equation and have been successfully extended to a lot of applications, we refer to the review paper [21] for more discussions. Different AP schemes have been developed for various kinetic models, including the neutron transport equation [1, 13, 15, 16], the velocity jump model for E.coli chemotaxis [3, 6] and the Boltzmann equation [8, 23, 4, 12].

The Knudsen number is the ratio of the mean free path and the domain typical length scale [14]. To prove that a scheme is AP, one has to show that when the Knudsen number goes to zero in the discretized scheme, it converges to a good discretization of the corresponding limit model. The main advantage of AP schemes is that their stability and convergence are independent of the Knudsen number. On the other hand, there are situations when in applications the solution after some time reaches a quasi-stationary state, meaning that numerically the difference between the global equilibrium and the solution after finite time is smaller than machine precision. In semiconductor models such a state is the mode of operation of the electronic device, namely the state where the applied voltage is in equilibrium so that no current flows. Thus it is of interest to have a numerical scheme maintains stationary solutions up to machine precision. We call such schemes stationary preserving (SP).

*Submitted to the editors 13.01.2020.

Funding: 2nd author: National Council for Scientific Research of Lebanon (CNRS-L). 2nd, 3rd author: Bayerisches Hochschulzentrum for China.

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Our key observation is that as far as some macroscopic quantities can be updated explicitly, a large class of AP schemes have the SP property as well. To illustrate the idea, we present here a glimpse of the proof of the SP property for an AP scheme for the BGK model [4]. The BGK model writes:

$$\partial_t f + v.\nabla x f = \frac{1}{\tau}[M_f - f]. \tag{1.1}$$

where $f(x, v, t)$ is the probability density function at time t, position x and moving with velocity v. M_f is the Maxwellian distribution and τ is the relaxation time. At the macroscopic level, mass, momentum and energy are moments of the distribution function f in velocity space that are given by:

$$\rho(x,t) = \int_V f(x,v,t)dv,$$

$$\rho u(x,t) = \int_V v f(x,v,t)dv,$$

$$E(x,t) = \int_V \frac{1}{2}|v|^2 f(x,v,t)dv.$$

As $\tau \to 0$, these moments solves the Euler equations,

$$\begin{aligned} \rho_t + \nabla.(\rho u) &= 0, \\ (\rho u)_t + \nabla.(\rho u \otimes u + p) &= 0, \\ E_t + \nabla.((E+p)u) &= 0. \end{aligned} \tag{1.2}$$

As in [8], we consider the following AP IMEX scheme

$$\frac{f^{n+1} - f^n}{\Delta t} + v.\nabla_x f^n = \frac{1}{\tau}[M_f^{n+1} - f^{n+1}]. \tag{1.3}$$

Suppose that the solution reaches the stationary state at time t^n, i.e. f^n satisfies the following equation:

$$v.\nabla_x f^n = \frac{1}{\tau}[M_f^n - f^n]. \tag{1.4}$$

Multiplying (1.4) by $\alpha(v)$ with $\alpha(v) = (1, v, \frac{1}{2}|v|^2)$ and integrating over the velocity space leads to

$$\int_V \alpha(v) v.\nabla_x f^n = 0. \tag{1.5}$$

Now multiplying (1.3) by $\alpha(v)$ and integrating over V, one gets

$$\frac{\int_v \alpha(v) f^{n+1} - \int_v \alpha(v) f^n}{\Delta t} + \int_V \alpha(v) v.\nabla_x f^n = 0. \tag{1.6}$$

From (1.5), the moments are preserved. The Maxwellian can be updated explicitly and it is exactly equal to the Maxwellian at the previous time step, i.e. $\mathcal{M}^{n+1} = \mathcal{M}^n$. Hence, the discritized equation can be now written as,

$$\frac{f^{n+1} - f^n}{\Delta t} + v.\nabla_x f^n = \frac{1}{\tau}[M_f^n - f^n + f^n - f^{n+1}].$$

Noting that f^n satisfies (1.4), thus

$$\frac{f^{n+1} - f^n}{\Delta t} + \frac{[f^{n+1} - f^n]}{\tau} = 0,$$

which yields $f^{n+1} = f^n$ and the stationary solution is preserved. Our proof of the SP property is independent of ε. In other words no matter how small ε is, the SP property holds. In the subsequent part, we will consider three different classes of AP schemes for which one can prove their SP properties. To get the SP property, it is crucial to show that the macroscopic quantities in these AP schemes are being updated explicitly, even though the schemes are implicit or IMEX. As one can see from the SP proof, once we are able to show that the macroscopic quantities are preserved, the SP property follows immediately. To show the universality of our observation, we test different kinetic models for different AP schemes, as listed in Table 1.
The paper is structured as follows: In section 2, the parity equations-based AP scheme developed in [21] for the neutron transport equation is considered and then the SP property of the scheme is proved. In section 3, the unified gas kinetic scheme [16, 23, 24](UGKS) is extended to the velocity jump chemotaxis model and then we prove that this extension has the SP property. In section 4, we consider the penalization method proposed in [8] for the Boltzmann equation and prove its SP property. Finally, we present some numerical results to show the AP and SP properties of each numerical scheme in section 5. All three different strategies of developing AP schemes (Parity-equations based scheme, UGKS, penalization method) have been extended to various kinetic models and thus the extension of our observation is natural.

Section	Kinetic Model	Scheme
2	Neutron transport equation	Parity-equations based scheme
3	Chemotaxis kinetic model	UGKS
4	Boltzmann equation	IMEX scheme with the Penalization method

TABLE 1
A list of kinetic models together with their corresponding schemes.

2. Parity equations-based scheme for the Neutron transport equation. In this section we check the Parity equations-based AP scheme for the neutron transport equation in [21, 22]. This scheme is then proved to be SP as well.

2.1. The neutron transport equation. Consider the one-dimensional neutron transport equation:

$$\partial_t f + \frac{1}{\varepsilon} v.\nabla x f = \frac{\sigma_T}{\varepsilon^2}(\frac{1}{2}\int_{-1}^{1} f dv' - f) - \sigma_a(\frac{1}{2}\int_{-1}^{1} f dv') + q \tag{2.1}$$

with $x \in [x_L, x_R]$ and $v \in [-1, 1]$. We present the scheme for a simplified neutron transport equation with $\sigma_T = 1, \sigma_a = 0$, $q = 0$. The extension to more general cases does not add any difficulties.

2.2. Discretization of the model. When $\sigma_T = 1, \sigma_a = 0$, $q = 0$ in (2.1), the Parity equations-based scheme in [22] can be summarized by the following steps:

- Rewrite (2.1) into two equations. For $v \geq 0$,

$$\begin{aligned} \varepsilon \partial_t f(v) + v \partial_x f(v) &= \frac{1}{\varepsilon}(\frac{1}{2}\int_{-1}^{1} f dv - f(v)), \\ \varepsilon \partial_t f(-v) - v \partial_x f(-v) &= \frac{1}{\varepsilon}(\frac{1}{2}\int_{-1}^{1} f dv - f(-v)). \end{aligned} \tag{2.2}$$

- Introduce the even and odd parities that are

$$r(t,x,v) = \frac{1}{2}[f(t,x,v) + f(t,x,-v)], \qquad j(t,x,v) = \frac{1}{2\varepsilon}[f(t,x,v) - f(t,x,-v)].$$

- Add and subtract the equations in (2.2) and rewrite them into the following diffusive relaxation system,

$$\begin{aligned} \partial_t r + v \partial_x j &= -\frac{1}{\varepsilon^2}(r - \rho_r), \\ \partial_t j + \eta v \partial_x r &= -\frac{1}{\varepsilon^2}[j + (1-\epsilon^2\eta) v \partial_x r], \end{aligned} \tag{2.3}$$

  where $\rho_r = \int_0^1 r dv'$ and $\eta(\varepsilon) = \min(1, \frac{1}{\varepsilon})$.
- Split the equations (2.3) into two steps:
  - Relaxation step:
$$\begin{cases} \partial_t r = -\frac{1}{\epsilon^2}(r - \rho_r), \\ \partial_t j = -\frac{1}{\epsilon^2}[j + (1-\epsilon^2\eta) v \partial_x r]. \end{cases}$$
  - Transport step:
$$\begin{cases} \partial_t r + v \partial_x j = 0, \\ \partial_t j + \eta v \partial_x r = 0. \end{cases}$$
- Discretize the two steps as follows:
  - For the transport step, we use an explicit first order upwind scheme on its diagonal from such that

$$\begin{cases} r_i^{n+\frac{1}{2}} = r_i^n - v\frac{\Delta t}{\Delta x} D^u j_i^n, \\ j_i^{n+\frac{1}{2}} = j_i^n - \eta v \frac{\Delta t}{\Delta x} D^u r_i^n. \end{cases} \tag{2.4}$$

    where $D^u f_i^n = f_{i+1}^n - f_i^n$ and $D^c f_i^n = \frac{f_{i+1}^n - f_{i-1}^n}{2}$ are respectively the upwind and the central spatial differences.
  - For the relaxation step, we use an implicit backward Euler method that writes

$$\begin{cases} \frac{r_i^{n+1} - r_i^{n+\frac{1}{2}}}{\Delta t} = -\frac{1}{\varepsilon^2}(r_i^{n+1} - \rho_{r_i}^{n+1}), \\ \frac{j_i^{n+1} - j_i^{n+\frac{1}{2}}}{\Delta t} = -\frac{1}{\varepsilon^2}(j_i^{n+1} + (1-\epsilon^2\eta) v \frac{D^c}{\Delta x} r_i^{n+1}). \end{cases}$$

By integrating the above first equation over V we find, $\rho_{r_i}^{n+1} = \rho_{r_i}^{n+\frac{1}{2}}$. Then,

$$\begin{cases} r_i^{n+1} = A r_i^{n+\frac{1}{2}} + B\rho_{r_i}^{n+\frac{1}{2}}, \\ j_i^{n+1} = A j_i^{n+\frac{1}{2}} - B(1-\varepsilon^2\eta) v \frac{D^c}{\Delta x} r_i^{n+1}, \end{cases} \tag{2.5}$$

with A and B being defined as:

$$A = \frac{\varepsilon^2}{\varepsilon^2+\Delta t} \quad \text{and} \quad B = \frac{\Delta t}{\varepsilon^2+\Delta t}.$$

The fully space-time discretized parity equations-based AP scheme is given by the transport step (2.4) and the relaxation step (2.5). The boundary conditions for r and j are the same as in [22] and are obtained using the following relations:

$$r + \varepsilon j|_{x=x_L} = F_L(v) \;\; and \;\; r - \varepsilon j|_{x=x_R} = F_R(v) \tag{2.6}$$

when $\varepsilon << 1$, j can be approximated by,

$$j = -v\partial_x r \tag{2.7}$$

from the second equation in (2.3). Hence, the boundary conditions for r and j are (2.8) and (2.9),

$$r - \varepsilon v \partial_x r|_{x=x_L} = F_L(v) \;\; and \;\; r + \varepsilon v \partial_x r|_{x=x_R} = F_R(v) \tag{2.8}$$

$$j = -v\partial_x r \tag{2.9}$$

where $F_L(v)$ and $F_R(v)$ are the inflow boundary conditions of f. The AP proof of the scheme has been done in [22], [21], [3].

2.3. SP Property. We will prove that the above AP scheme is SP as well. Plugging (2.4) in (2.5) and using the fact that $\rho_r^{n+\frac{1}{2}} = \rho_r^{n+1}$, the equations for updating r^{n+1} and j_i^{n+1} can be written as:

$$\frac{r_i^{n+1} - r_i^n}{\Delta t} + v\frac{D^u}{\Delta x} j_i^n = -\frac{1}{\varepsilon^2}(r_i^{n+1} - \rho_{r_i}^{n+1}), \tag{2.10a}$$

$$\frac{j_i^{n+1} - j_i^n}{\Delta t} + \eta v \frac{D^u}{\Delta x} r_i^n = -\frac{1}{\varepsilon^2}(j_i^{n+1} + (1-\varepsilon^2\eta) v \frac{D^c}{\Delta x} r_i^{n+1}). \tag{2.10b}$$

Definition 2.1. *A steady state solution of (2.3) is a function pair (r^n, j^n) that satisfies:*

$$v\partial_x j^n = -\frac{1}{\varepsilon^2}(r^n - \rho_r^n),$$

$$\eta v \partial_x r^n = -\frac{1}{\varepsilon^2}[j^n + (1-\epsilon^2\eta) v \partial_x r^n].$$

with the same boundary conditions as in (2.8) and (2.9).

Definition 2.2. *A discrete stationary solution to (2.10) are r_i^n and j_i^n that satisfies:*

$$v.\frac{D^u}{\Delta x} j_i^n = -\frac{1}{\varepsilon^2}(r_i^n - \rho_{r_i}^n), \tag{2.11a}$$

$$\eta v. \frac{D^u}{\Delta x} r_i^n = -\frac{1}{\varepsilon^2}[j_i^n + (1-\varepsilon^2\eta) v \frac{D^c}{\Delta x} r_i^n]. \tag{2.11b}$$

LEMMA 2.3. *When r_i^n and j_i^n are discrete stationary solution that satisfies (2.11), the scheme in (2.10) will lead to $r_i^{n+1} = r_i^n$ and $j_i^{n+1} = j_i^n$. Hence the parity equations-based scheme is SP.*

Proof.

- For r: Since $\rho_{r_i}^n = \int_0^1 r_i^n$, integrating (2.11a) over $[0,1]$ yields

$$\int_0^1 v.\frac{D^u}{\Delta x} j_i^n dv = 0. \tag{2.12}$$

Integrating (2.10a) with respect to v over $[0,1]$ and using (2.12), one finds,

$$\frac{\rho_{r_i}^{n+1} - \rho_{r_i}^n}{\Delta t} + \int_0^1 v.\frac{D^u}{\Delta x} j_i^n dv = 0,$$

and thus $\rho_r^{n+1} = \rho_r^n$. Using (2.11a) and $\rho_r^{n+1} = \rho_r^n$, (2.10a) gives

$$\frac{r_i^{n+1} - r_i^n}{\Delta t} - \frac{1}{\varepsilon^2}(r_i^n - \rho_{r_i}^n) = -\frac{1}{\varepsilon^2}(r_i^{n+1} - \rho_{r_i}^n).$$

Hence,

$$(\frac{1}{\Delta t} + \frac{1}{\epsilon^2})(r_i^{n+1} - r_i^n) = 0.$$

and then $r_i^{n+1} = r_i^n$.

- For j: Using $r^{n+1} = r^n$, (2.10b) becomes

$$\frac{j_i^{n+1} - j_i^n}{\Delta t} + \eta v \frac{D^u}{\Delta x} r_i^n = -\frac{1}{\epsilon^2}[j_i^{n+1} + (1-\epsilon^2\eta) v \frac{D^c}{\Delta x} r_i^n]. \tag{2.13}$$

From (2.11b), (2.13) writes,

$$\frac{j_i^{n+1} - j_i^n}{\Delta t} - \frac{1}{\epsilon^2}[j_i^n + (1-\epsilon^2\eta) v \frac{D^c}{\Delta x} r_i^n] = -\frac{1}{\epsilon^2}[j_i^{n+1} + (1-\epsilon^2\eta) v \frac{D^c}{\Delta x} r_i^n].$$

Then,

$$(\frac{1}{\Delta t} + \frac{1}{\epsilon^2})(j_i^{n+1} - j_i^n) = 0$$

and thus $j_i^{n+1} = j_i^n$.

The SP property of the parity equations-based AP scheme is concluded. ∎

3. UGKS scheme for the chemotaxis kinetic model. In this section we first extend the UGKS in [16, 23, 24] to the time evolutionary chemotaxis model, then show its AP and SP properties.

3.1. The chemotaxis kinetic model. The chemotaxis kinetic model models bacteria that undergo run and tumble process as mentioned in [10, 19, 20]. During the run phase, bacteria move along a straight line and change their directions during the tumble phase. This is called the velocity jump process and can be modeled by the Othmer-Dunbar-Alt model that writes [2, 17]:

$$\begin{cases} \partial_t f + \frac{1}{\varepsilon} v.\nabla x f = \frac{1}{\varepsilon^2}[\frac{1}{|V|}\int_V (1+\varepsilon\phi(v'.\partial_x S)) f(v') dv' - (1+\varepsilon\phi(v.\partial_x S)) f(v)], \\ \partial_t S - D\Delta S + \alpha S = \beta\rho, \qquad \rho(x,t) := \frac{1}{|V|}\int_V f(v) dv. \end{cases} \tag{3.1}$$

Here $f(x,v,t)$ is the probability density function at time t, position x and moving with velocity v; ϕ is an odd decreasing function such that $\phi(-u) = -\phi(u)$; $S(x,t)$ is the concentration of a chemical substance where the parameters D, α, β are positive constants; ε is the Knudsen number. When $\phi = 0$, the chemotaxis kinetic model reduces to the neutron transport equation.

As $\varepsilon \to 0$, $f(x,v,t)$ converges to $\rho_0(x,t)$ where $\rho_0(x,t)$ solves the following Keller-Segel equation [5, 11, 18]:

$$\begin{cases} \partial_t \rho_0 = \frac{1}{3}\Delta\rho_0 + \nabla((\frac{1}{|V|}\int_V v\phi(v\partial_x S)dv)\rho_0), \\ \partial_t S - D\Delta S + \alpha S = \beta\rho_0. \end{cases} \tag{3.2}$$

3.2. Discretization of the model. Before discussing about the more complex equation for f, we first discretize the equation for the chemical concentration S. Let $S_i^n \approx S(x_i, t^n)$, the following centered finite difference method is used to update S:

$$\frac{S_i^{n+1} - S_i^n}{\Delta t} = D\frac{S_{i+1}^{n+1} - 2S_i^{n+1} + S_{i-1}^{n+1}}{\Delta x^2} - \alpha S_i^{n+1} + \beta\rho_i^n. \tag{3.3}$$

. After S_i^{n+1} are obtained, we approximate $\partial_x S^{n+1}$ by a piecewise constant function such that

$$\partial_x S(x, t^{n+1}) \approx \partial_x S(x_{i+\frac{1}{2}}, t^{n+1}) \approx \frac{S_{i+1}^{n+1} - S_i^{n+1}}{\Delta x} := \sigma_{i+\frac{1}{2}}, \qquad \text{for } \forall x \in [x_i, x_{i+1}). \tag{3.4}$$

The UGKS is a finite volume approach for discretizing the kinetic equation f. By integrating the chemotaxis kinetic model (3.1) over $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}] \times V$ and letting $f_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,v,t^n)\,dx$, $\rho_i^n = \frac{1}{|V|}\int_V f_i^n\,dv$, the total density ρ_i^{n+1} and density fluxes f_i^{n+1} are updated as follows

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x} = 0, \tag{3.5}$$

$$\begin{aligned} \frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{\Phi_{i+\frac{1}{2}}^n - \Phi_{i-\frac{1}{2}}^n}{\Delta x} &= \frac{1}{\varepsilon^2}\left(\rho_i^{n+1} - f_i^{n+1}\right) \\ &+ \frac{1}{\varepsilon}\left(\frac{1}{|\,V\,|}\int_V \phi(v'\sigma_{i+\frac{1}{2}})f_i^n(v')\,dv' - \phi(v\sigma_{i+\frac{1}{2}})f_i^n\right). \end{aligned} \tag{3.6}$$

Here the numerical fluxes are given by

$$\begin{aligned} \Phi_{i+\frac{1}{2}}^n &= \frac{1}{\varepsilon\Delta t}\int_{t^n}^{t^{n+1}} vf(x_{i+\frac{1}{2}}, v, t)\,dt, \\ F_{i+\frac{1}{2}}^n &= \frac{1}{|V|}\int_V \Big(\frac{1}{\varepsilon\Delta t}\int_{t^n}^{t^{n+1}} vf(x_{i+\frac{1}{2}}, v, t)\,dt\Big)\,dv. \end{aligned} \tag{3.7}$$

It is important to note that $\sigma_{i+\frac{1}{2}}$ approximates $\partial_x S$ in the interval $[x_i, x_{i+1})$ while f_i^n is the average density over the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. This choice is important to get the correct advection term in the limit Keller-Segel model when ε becomes small.

We use discrete ordinate method for the velocity discretization, but for convenience of explanation, we write the scheme in continuous velocity. The most crucial step for UGKS is to determine $\Phi_{i+\frac{1}{2}}^n$ and $F_{i+\frac{1}{2}}^n$. the details are listed below:

- **Find the approximation of $f(x_{i+\frac{1}{2}}, v, t)$.** The 1d chemotaxis model (3.1) can be rewritten as:

$$\partial_t f + \frac{1+\varepsilon\phi(v\partial_x S^\varepsilon)}{\varepsilon^2} f + \frac{v}{\varepsilon}\partial_x f = \frac{1}{\varepsilon^2}\mathcal{T}^1 f, \tag{3.8}$$

where $(\mathcal{T}^1 f)(x,t) := \dfrac{1}{|V|}\displaystyle\int_V \big(1+\varepsilon\phi(v'\partial_x S)\big) f(x,v',t)dv'$.

Consider the interval $[x_i, x_{i+1})$, multiplying both sides of (3.8) by $\exp\left(\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}t\right)$ yields

$$\frac{d}{dt}\left[f(x+\frac{v}{\varepsilon}t, v, t)\exp\left(\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}t\right)\right] = \frac{\mathcal{T}^1 f(x,t)}{\varepsilon^2}\exp\left(\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}t\right).$$

Integrating the above equation over (t^n, t) yields to,

$$\begin{aligned} f(x_{i+\frac{1}{2}}, v, t) = {} & f(x_{i+\frac{1}{2}} - \frac{v}{\varepsilon}(t-t^n), v, t^n)\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right) \\ & + \frac{1}{\varepsilon^2}\int_{t^n}^t \mathcal{T}^1 f(x_{i+\frac{1}{2}} - \frac{v}{\varepsilon}(t-s), s)\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-s)\right)ds. \end{aligned} \tag{3.9}$$

This is an exact expression for $f(x_{i+\frac{1}{2}}, v, t)$ that will be used to determine $\Phi^n_{i+\frac{1}{2}}$, $F^n_{i+\frac{1}{2}}$ in (3.7). At this stage, we need to approximate $f(x, v, t^n)$ and $(\mathcal{T}^1 f)(x,t)$ on the right hand side of (3.9). f is approximated by a piecewise constant function and $\mathcal{T}^1 f$ by a piecewise linear function as follows:

$$f(x,v,t^n) = \begin{cases} f^n_i, & x < x_{i+\frac{1}{2}}, \\ f^n_{i+1}, & x > x_{i+\frac{1}{2}}, \end{cases}$$

$$\mathcal{T}^1 f(x,t) = \begin{cases} \mathcal{T}^1 f^n_{i+\frac{1}{2}} + \delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}}(x - x_{i+\frac{1}{2}}), & x < x_{i+\frac{1}{2}}, \\ \mathcal{T}^1 f^n_{i+\frac{1}{2}} + \delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}}(x - x_{i+\frac{1}{2}}), & x > x_{i+\frac{1}{2}}. \end{cases}$$

Here $\mathcal{T}^1 f^n_{i+\frac{1}{2}}$, $\delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}}$, $\delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}}$ are defined by:

$$\begin{cases} \mathcal{T}^1 f^n_{i+\frac{1}{2}} := \dfrac{1}{|V|}\displaystyle\int_{V^-}(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))f^n_{i+1} + \frac{1}{|V|}\int_{V^+}(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))f^n_i, \\ \delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}} := \dfrac{\mathcal{T}^1 f^n_{i+\frac{1}{2}} - \mathcal{T}^1 f^n_i}{\Delta x/2}, \\ \delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}} := \dfrac{\mathcal{T}^1 f^n_{i+1} - \mathcal{T}^1 f^n_{i+\frac{1}{2}}}{\Delta x/2}, \end{cases}$$

with $V^+ = V \cap \mathbb{R}^+$ and $V^- = V \cap \mathbb{R}^-$.
Substituting the above approximations into (3.9) yields an expression for $f(x_{i+\frac{1}{2}}, v, t)$ such

that:
For $v>0$,

$$
\begin{aligned}
f(x_{i+\frac{1}{2}},v,t) &= f_i^n \exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)+\frac{\mathcal{T}^1 f^n_{i+\frac{1}{2}}}{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}\\
&\times\left(1-\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)\right)+v\varepsilon\frac{\delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}}}{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))^2}\\
&\times\left[\left(1+\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)-1\right],
\end{aligned}
\tag{3.10}
$$

and for $v<0$,

$$
\begin{aligned}
f(x_{i+\frac{1}{2}},v,t) &= f_{i+1}^n \exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)+\frac{\mathcal{T}^1 f^n_{i+\frac{1}{2}}}{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}\\
&\times\left(1-\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)\right)+v\varepsilon\frac{\delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}}}{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))^2}\\
&\times\left[\left(1+\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)\exp\left(-\frac{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}(t-t^n)\right)-1\right].
\end{aligned}
\tag{3.11}
$$

- **Determine $\Phi^n_{i+\frac{1}{2}}$, $F^n_{i+\frac{1}{2}}$.** The flux $\Phi^n_{i+\frac{1}{2}}(v)$ in (3.7) can be approximated by

$$
\begin{aligned}
\Phi_{i+\frac{1}{2}}(v) &= Avf^n_{i+1}+Bv\mathcal{T}^1 f^n_{i+\frac{1}{2}}+Cv^2\delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}}, \quad \text{for } v<0,\\
\Phi_{i+\frac{1}{2}}(v) &= Avf^n_{i}+Bv\mathcal{T}^1 f^n_{i+\frac{1}{2}}+Cv^2\delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}}, \quad \text{for } v>0,
\end{aligned}
\tag{3.12}
$$

where the coefficients $A(v,\varepsilon,\Delta t), B(v,\varepsilon,\Delta t), C(v,\varepsilon,\Delta t)$ can be determined explicitly such that

$$
\begin{aligned}
A(v,\varepsilon,\Delta t) :&= \frac{\varepsilon}{\Delta t\big(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})\big)}\left(1-\exp\big(-\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}\Delta t\big)\right),\\
B(v,\varepsilon,\Delta t) :&= \frac{1}{\varepsilon(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))}\\
&\quad-\frac{\varepsilon}{\Delta t(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))^2}\left(1-\exp\big(-\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}\Delta t\big)\right),\\
C(v,\varepsilon,\Delta t) :&= \frac{2\varepsilon^2}{\Delta t(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))^3}\left(1-\exp\big(-\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}\Delta t\big)\right)\\
&\quad-\frac{1}{(1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}}))^2}\left(1+\exp\big(-\frac{1+\varepsilon\phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2}\Delta t\big)\right).
\end{aligned}
\tag{3.13}
$$

Furthermore, $F^n_{i+\frac{1}{2}}$ in (3.7) is given by

(3.14)
$$F^n_{i+\frac{1}{2}} = \frac{1}{|V|}\int_{V^-} Avf^n_{i+1}dv + \frac{1}{|V|}\int_{V^+} Avf^n_i dv + \frac{1}{|V|}\mathcal{T}^1 f^n_{i+\frac{1}{2}}\int_V vBdv + \frac{1}{|V|}\delta^R\mathcal{T}^1 f^n_{i+\frac{1}{2}}\int_{V^-} Cv^2dv + \frac{1}{|V|}\delta^L\mathcal{T}^1 f^n_{i+\frac{1}{2}}\int_{V^+} Cv^2dv.$$

This concludes the construction of the scheme. For the proof of its AP property, one can refer to Appendix A.

3.3. SP Property. Assume that we start from a steady state solution that at the discrete level satisfies,

(3.15)
$$\frac{\Phi^n_{i+\frac{1}{2}} - \Phi^n_{i-\frac{1}{2}}}{\Delta x} = \frac{1}{\varepsilon^2}\left(\rho^n_i - f^n_i\right) + \frac{1}{\varepsilon}\left(\frac{1}{|V|}\int_V \phi(v'\sigma_{i+\frac{1}{2}})f^n_i(v')\,dv' - \phi(v\sigma_{i+\frac{1}{2}})f^n_i\right).$$

Integrating (3.15) over v yields

$$\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = 0,$$

From (3.5) one can deduce that,

(3.16)
$$\rho^{n+1}_i = \rho^n_i,$$

which indicates that the macroscopic density is preserved. Using (3.15), the equation of updating f^{n+1} in (3.6) can be written as,

$$\frac{f^{n+1}_i - f^n_i}{\Delta t} = \frac{1}{\varepsilon^2}\Big((\rho^{n+1}_i - \rho^n_i) - (f^{n+1}_i - f^n_i)\Big).$$

Then from (3.16),

$$\big(1 + \frac{\Delta t}{\epsilon^2}\big)(f^{n+1}_i - f^n_i) = 0,$$

which gives $f^{n+1}_i = f^n_i$. This concludes the SP property of the UGKS.

4. IMEX scheme with the Penalization method for the Boltzmann equation. In this section, we consider the penalization method developed in [8] for the Boltzmann equation. This method together with an IMEX discretization of the equation give an AP scheme for the Boltzmann equation. One can find the AP proof in [8]. Here we show that the penalization method is not only AP but also SP as well.

4.1. The Boltzmann equation. The Boltzmann equation describes the time evolution of the density distribution of gas particles. It is given by

$$\partial_t f + v\nabla_x f = \frac{\mathcal{Q}(f)}{\varepsilon}.$$

Here $f(x, v, t)$ is the probability density distribution of particles at time t, position x and with velocity v. $\mathcal{Q}$ is the Boltzmann collision operator where only binary interactions are considered.

Let (v, v_*) and (v', v'_*) be respectively the velocities of the two colliding particles before and after the collision related by

$$\begin{cases} v' = \frac{1}{2}((v - v_*) - |v - v_*|\sigma), \\ v'_* = \frac{1}{2}((v - v_*) + |v - v_*|\sigma). \end{cases}$$

with $\sigma \in \mathcal{S}^{d_v - 1}$. $\mathcal{Q}$ is given by

$$\mathcal{Q}(f)(v) = \int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v - 1}} B(|v - v_*|, \cos\theta)(f(v'_*)f(v') - f(v_*)f(v))d\sigma dv_*.$$

The collision kernel B is a non-negative function given by $B(|u|, \cos\theta) = C_\alpha |u|^\alpha$ where $u = \frac{(v - v_*)}{|v - v_*|}$ and $\cos\theta = u \cdot \sigma$. For more details, one can look at the Boltzmann equation description in [8]. ε is the dimensionless Knudsen number and $\int_v \alpha(v)\mathcal{Q}(f)dv = 0$ for $\alpha(v) = (1, v, |v|^2)$. The equilibrium distribution of $\mathcal{Q}$ is the Maxwellian distribution $\mathcal{M}_{\rho,u,T}$, i.e. $\mathcal{Q}(\mathcal{M}_{\rho,u,T}) = 0$. As $\varepsilon \to 0$, the moments of the distribution function solve the Euler equations (1.2).

4.2. IMEX scheme with the Penalization method. The penalization method was originally developed in [8, 21]. The idea is to split the collision term of the Boltzmann equation into a stiff part and less stiff part. More precisely, the Boltzmann equation is written in the following form:

$$\partial_t f + v\nabla_x f = \frac{\mathcal{Q}(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon},$$

where $\mathcal{Q}(f)$ is the Boltzmann collision operator and $P(f)$ is a relaxation operator, namely $P(f) = \beta[\mathcal{M}_{\rho,u,T}(v) - f(v)]$ where β is a strictly positive parameter. $P(f)$ has the same equilibrium as $\mathcal{Q}(f)$. It satisfies $\int_v P(f)\alpha(v)dv = 0$ for $\alpha(v) = (1, v, |v|^2)$ and $P(\mathcal{M}_{\rho,u,T}) = 0$. As in [8], β^n is chosen to be $2\pi\rho^n$ such that both operators $P(f)$ and the full Boltzmann operator $\mathcal{Q}(f)$ have the same loss term corresponding to the dissipative part.
The following IMEX discretization of the Boltzmann equation is proposed in [8]:

$$\frac{f^{n+1} - f^n}{\Delta t} + v.\nabla_x f^n = \frac{\mathcal{Q}(f^n) - P(f^n)}{\varepsilon} + \frac{P(f^{n+1})}{\varepsilon}. \tag{4.1}$$

For the discretization of the Boltzmann operator one can use a fast spectral Fourier-Galerkin method [7], and for the transport part, a first or second order finite volume scheme can be employed. This gives an AP discretization for the Boltzmann equation as proved in [8].

4.3. SP property. Suppose that the solution satisfies the stationary equation at time t^n, i.e.

$$v.\nabla_x f^n = \frac{\mathcal{Q}(f^n) - P(f^n)}{\varepsilon} + \frac{P(f^n)}{\varepsilon}. \tag{4.2}$$

It follows from the properties of the collision operator $\mathcal{Q}$ and the relaxation operator P that:

$$\int_v \alpha(v)v.\nabla_x f^n = 0, \tag{4.3}$$

with $\alpha(v) = (1, v, |v|^2)$.
Now multiply (4.1) by $\alpha(v)$ and integrate over the velocity space. Using the conservation properties

of $\mathcal{Q}$, P and (4.3), one gets that the macroscopic quantities are preserved. Then $\mathcal{M}^{n+1} = \mathcal{M}^n$ and $\beta^{n+1} = \beta^n$. By adding and subtracting $\beta^n f^n/\varepsilon$, Eq.(4.1) can be written as:

$$\frac{f^{n+1} - f^n}{\Delta t} + v.\nabla_x f^n = \frac{\mathcal{Q}(f^n) - P(f^n)}{\varepsilon} + \frac{\beta^n[\mathcal{M}^n - f^n]}{\varepsilon} - \frac{\beta^n[f^{n+1} - f^n]}{\varepsilon}.$$

Noting that f^n satisfies the steady state equation (4.2) , one gets

$$\frac{f^{n+1} - f^n}{\Delta t} + \frac{\beta^n[f^{n+1} - f^n]}{\varepsilon} = 0,$$

Thus $f^{n+1} = f^n$. The steady state is preserved.

5. Experimental results. Three test cases are considered in this section, each validates the AP and SP properties of one scheme presented in section 2, 3 or 4.

5.1. Neutron transport equation-Parity equations-based scheme. To validate the AP and SP properties of the parity equations-based scheme, we use the same initial and boundary conditions as problem 1 in section 6 in [22]. The initial distribution is $f(x, v, t = 0) = 0$ and the computational domain is $x \in [0, 1]$. The boundary conditions are as in (2.8) and (2.9) with

$$F_L(v) = 1 \text{ and } F_R(v) = 0.$$

This data is consistent as can be seen by (2.8) and (2.9). The mesh and time step sizes are respectively $\Delta x = 0.025$ and $\Delta t = 0.0002$. In Figure 1, we plot the density at time $t = 0.05$ for $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-6}$ and compare it to its diffusion limit. The curves get close to each other when ε gets very small. The curve corresponding to $\varepsilon = 10^{-6}$ is exactly on top of the curve of the diffusion limit equation. This verifies the AP property. Furthermore, we plot in Figure 2 the time evolution of the distance between the numerical stationary solution ρ_r^s and the numerical solution ρ_r of the time evolutionary equation given by the L^∞ norm

$$||\rho_r - \rho_r^s||_\infty = \max_j\{\rho_{rj} - \rho_{rj}^s\}.$$

One can see that this distance does not change after we reach the steady state at $t = 10$. After that we give the norm at discrete times in Table 2 where we also show that the SP property is valid for all $\varepsilon << 1$. Figure 2 and Table 2 indicate that the SP property is well satisfied.

$\varepsilon = 10^{-3}$	Time	0	2	4	10	20
	L_∞-norm	1	9.730×10^{-4}	5.042×10^{-4}	5.042×10^{-4}	5.042×10^{-4}
$\varepsilon = 10^{-8}$	Time	0	4	8	10	20
	L_∞-norm	1	1.263×10^{-6}	1.053×10^{-12}	1.269×10^{-12}	1.269×10^{-12}

TABLE 2

Neutron Transport: L_∞-norm of the difference between the solution and the stationary solution in the time interval [0,20] for different ε.

5.2. Chemotaxis kinetic model-UGKS scheme. Parameters in (3.1) are chosen as in [9] such that,

$$\chi_S = 1, D = 15, \beta = 60, \alpha = 3.$$

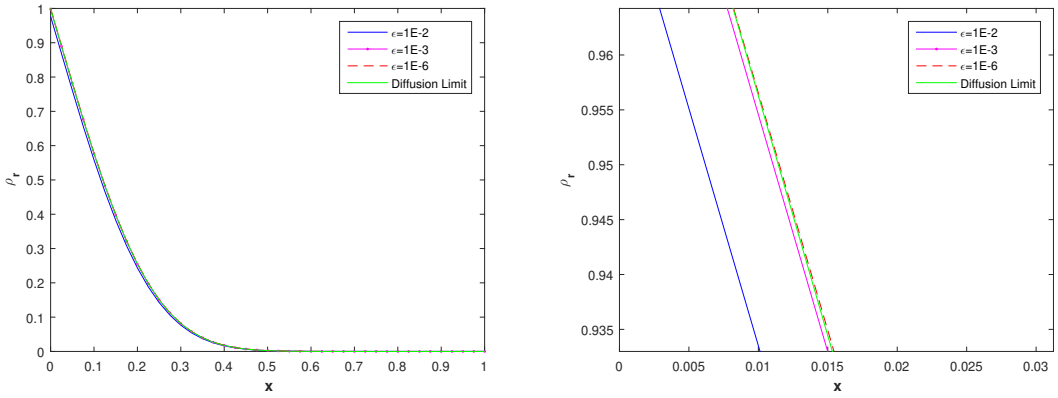

FIG. 1. *Verification of the AP property of the parity equations-based scheme for the neutron Transport equation. Left: The density ρ_r at time $t = 0.05$ for $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-6}$ and the solution of the diffusion limit equation; right: a zoomed part of the left plot.*

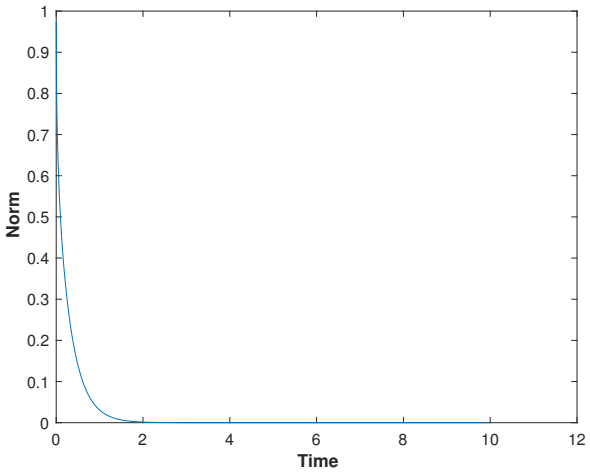

FIG. 2. *Neutron Transport: Time evolution of the L_∞-norm of the difference between the solution(starting from initial data $f = 0$) and the stationary solution(starting from the steady f at time $t = 10$) in the time interval [0,10] for $\varepsilon = 10^{-8}$.*

and ϕ is of the form

$$\phi(u) = -\chi_S \tanh u.$$

The computational domain is set to be $x \in [-1, 1]$. We impose specular boundary conditions for f and Dirichlet conditions for S. The initial density distribution is composed of two bumps located

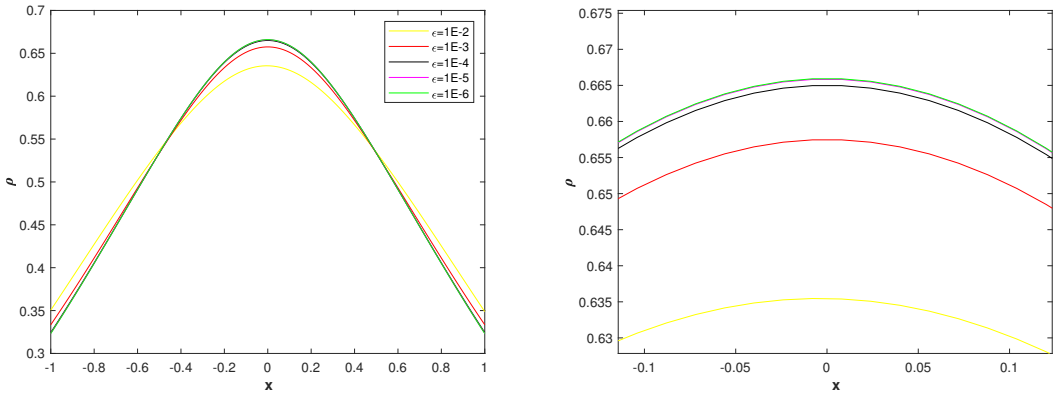

FIG. 3. *Verification of the AP property of the UGKS for the chemotaxis kinetic model. Left: The density ρ at time $t = 1$ for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$; right: a zoomed part of the left plot.*

at $x = \pm 0.65$ given by:

$$f(x, v, 0) = 5(\exp(-10(x - 0.65)^2 - 20(v + 0.45)^2) + \exp(-10(x + 0.65)^2 - 20(v - 0.45)^2)).$$

We use $\Delta x = 2/500$ for the space discretization and $v \in [-1, 1]$ with the S_{32} Gaussian quadrature points for the velocity. The limiting scheme of the UGKS is an explicit solver for the diffusion equation. Therefore, to ensure the stability of the numerical scheme, the time step Δt is chosen as below

$$\Delta t = \begin{cases} 0.5\Delta x^2, & \text{for } \varepsilon < \Delta x, \\ 0.5\varepsilon\Delta x, & \text{else.} \end{cases}$$

In order to verify the AP property of our scheme, the total densities ρ at time $t = 1$ are displayed in Figure 3 for different values of ε ranging from 10^{-2} to 10^{-6}. In order to check the SP property, we ran our simulations till it reaches a steady state then we consider this as our initial data. Time evolution of the L_∞-norm of the difference between the solution (starting from initial data) and the stationary solution in the time interval [0,100] is displayed in Table 3 for $\varepsilon = 1$ and $\varepsilon = 10^{-3}$. These results ensure that the SP property is independent of ε.

$\varepsilon = 1$	Time	0	30	60	65	100
	L_∞-norm	0.9064	8.192×10^{-7}	3.737×10^{-11}	7.385×10^{-12}	1.573×10^{-12}
$\varepsilon = 10^{-3}$	Time	0	5	10	50	100
	L_∞-norm	0.6493	2.968×10^{-7}	9.999×10^{-10}	1.476×10^{-10}	1.476×10^{-10}

TABLE 3

Chemotaxis: L_∞-norm of the difference between the solution and the stationary solution in the time interval [0,100] for different ε.

5.3. The Boltzmann equation-IMEX scheme with the Penalization method . In this section we consider the 2D Bose gas experiment 3.3 in [12] to test the AP and the SP property of

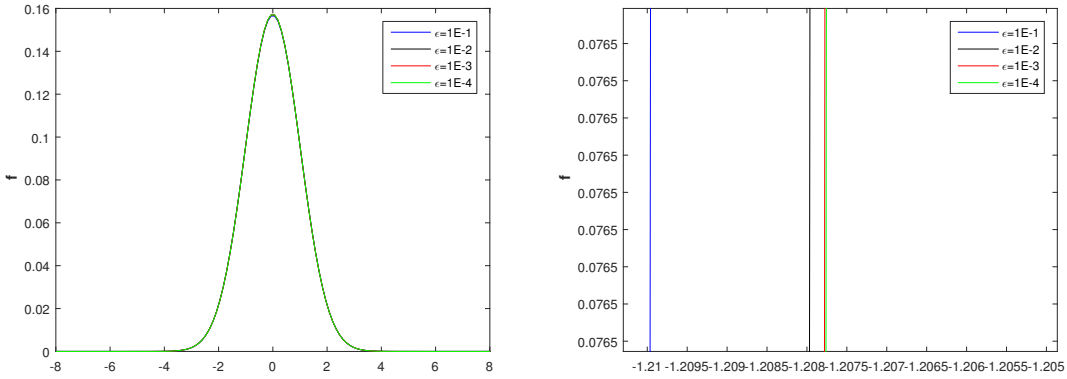

FIG. 4. *Boltzmann: cross section of the distribution function for different values of ε(left) and a zoomed part of the plot(right).*

the penalization method presented in [8]. We solve the space homogeneous quantum Boltzmann equation in 2D velocity space which is a special case of the classical Boltzmann equation for a particular collision operator $\mathcal{Q}_q$(the quantum collision operator) [12].

$$\partial_t f = \frac{\mathcal{Q}_q(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon}.$$

The idea can be extended to more general collision operators. Hence, scheme (4.1) is simplified to

$$f^{n+1} = \frac{\varepsilon}{\varepsilon + \beta^{n+1}\Delta t} f^n + \Delta t \frac{\mathcal{Q}_q(f^n) - P(f^n)}{\varepsilon + \beta^{n+1}\Delta t} + \frac{\beta^{n+1}\Delta t}{\varepsilon + \beta^{n+1}\Delta t}\mathcal{M}^{n+1}.$$

The initial distribution function is given as in [12],

$$f_0(v) = \frac{\rho_0}{4\pi T_0}\left(exp\left(\frac{-|v-u_0|^2}{2T_0}\right) + exp\left(\frac{-|v+u_0|^2}{2T_0}\right)\right),$$

where $\rho_0 = 1$, $T_0 = 3/8$, and $u_0 = (1, 1/2)$. The computational domain is $[-8, 8]^2$ with 64 grid points. The quantum Maxwellian [12] is given as,

$$M_q(v) = \frac{1}{\theta_0}\frac{1}{z^{-1}\exp\frac{(v-u)^2}{2T} - 1}.$$

where $\theta_0 = 0.1^2$, $z = 0.001590$, $T = 1$ is the temperature and $u = 0$ is the macroscopic velocity. In Figure 4 we test the AP property of the penalization method. A cross section of the distribution function for different values of ε is plotted on the left and a zoomed part of the plot on the right. The curves are getting closer to each other as ε converges to 0 which implies that the AP property is satisfied. Moreover, we investigate the SP property. Figure 5 is a comparison between the distribution function at the final time $t = 100$ and the Maxwellian distribution. We run our simulation till $t = 100$ where the equilibrium is reached and then use this equilibrium as new initial

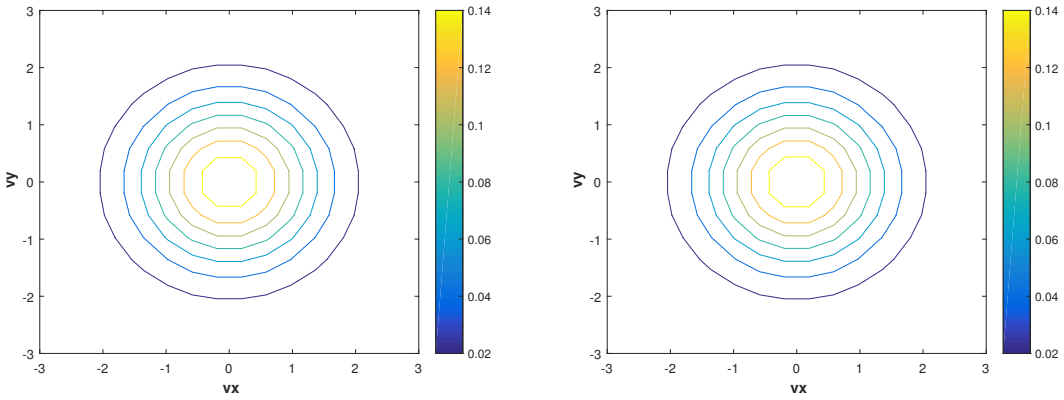

FIG. 5. *Boltzmann: contours of the 2D distribution function at the final time $t = 100$ (left) and the Maxevillian distribution function (right).*

data till $t = 100$. We computed the L_∞-norm of the difference between f and its equilibrium in the time interval $[0, 100]$ in Figure 6 as an evidence that f converges exponentially to the equilibrium. Table 4 presents the discrete times where one can find exactly when the initial distribution function reaches its equilibrium.

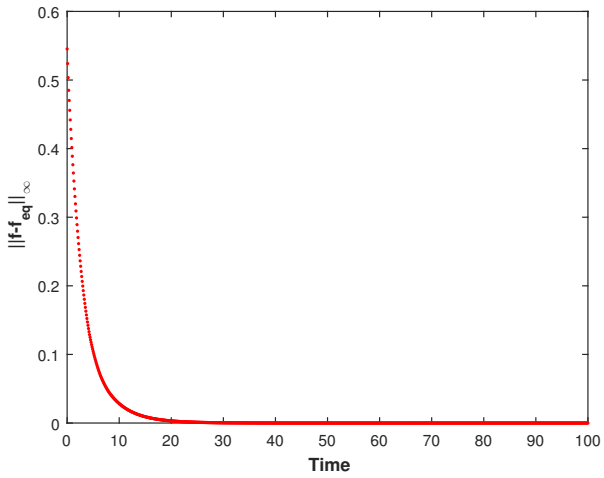

FIG. 6. *Boltzmann: Time evolution of the L_∞-norm of the difference between the distribution function f and its equilibrium in the time interval [0,100].*

6. Conclusion. In this work we find out that these three AP schemes have something in common. In the three schemes, once one is able to show that the macroscopic quantities can be updated explicitly then we are able to prove that scheme is SP. Whether this is true in general or

Time	0	20	60	85	90	100
L_∞-norm	0.5453	3.1×10^{-3}	1.393×10^{-5}	1.464×10^{-5}	1.464×10^{-5}	1.464×10^{-5}

TABLE 4

Boltzmann: L_∞-norm of the difference between f and its equilibrium starting from t=0 till the final time t=100 for $\varepsilon = 1$.

not remains a future work.

Appendix A. AP property of the UGKS. In this part, we give a formal derivation of the AP property for the UGKS proposed in (3.5)–(3.6).

When ε goes to zero, asymptotic expansions of A, B, C given in (3.13) read $A = O(\varepsilon)$, $B = \frac{1}{\varepsilon} - \phi(v\sigma_{i+\frac{1}{2}}) + O(\varepsilon)$, $C = -1 + O(\varepsilon)$. The leading order term of (3.6) yields $f_i^{n+1} = \rho_i^{n+1} + O(\varepsilon)$ and we only need to show that (3.5) satisfies the equation for ρ in (3.2), at the discrete level. Suppose that $f_i^n = \rho_i^n + O(\varepsilon)$, then

$$\begin{cases} \mathcal{T}^1 f^n_{i+\frac{1}{2}} = \dfrac{1}{2}\left(\rho_i^n + \rho_{i+1}^n\right) + O(\varepsilon), \\ \delta^L \mathcal{T}^1 f^n_{i+\frac{1}{2}} = \dfrac{\rho_{i+1}^n - \rho_i^n}{\Delta x} + O(\varepsilon), \\ \delta^R \mathcal{T}^1 f^n_{i+\frac{1}{2}} = \dfrac{\rho_{i+1}^n - \rho_i^n}{\Delta x} + O(\varepsilon). \end{cases}$$

We deduce that the expansion of $F^n_{i+\frac{1}{2}}$ reads:

$$F^n_{i+\frac{1}{2}} = -\frac{\rho_i^n + \rho_{i+1}^n}{2|V|}\left(\int_V v\phi(v\sigma_{i+\frac{1}{2}})dv\right) - \frac{\rho_{i+1}^n - \rho_i^n}{3\Delta x} + O(\varepsilon).$$

Therefore,

$$\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} \tag{A.1}$$

$$= -\frac{\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n}{3(\Delta x)^2} + \Big(-\Big(\frac{1}{|V|}\int_V v\phi(v\sigma_{i+\frac{1}{2}})dv\Big)\frac{\rho_i^n + \rho_{i+1}^n}{2} \tag{A.2}$$

$$+ \Big(\frac{1}{|V|}\int_V v\phi(v\sigma_{i-\frac{1}{2}})dv\Big)\frac{\rho_i^n + \rho_{i-1}^n}{2}\Big) + O(\varepsilon). \tag{A.3}$$

In the limit of $\varepsilon \to 0$, the discretization (3.5) becomes

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n}{3(\Delta x)^2} + \left(\frac{1}{|V|}\left(\int_V v\phi(v\sigma_{i+\frac{1}{2}})dv\right)\frac{\rho_i^n + \rho_{i+1}^n}{2} - \frac{1}{|V|}\left(\int_V v\phi(v\sigma_{i-\frac{1}{2}})dv\right)\frac{\rho_i^n + \rho_{i-1}^n}{2}\right).$$

which is a consistent discretization of the equation for ρ in (3.2). Therefore, the proposed scheme is AP after coupling with the discretization for $S(x,t)$ in (3.3).

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