Summer School

Normal Families in Complex Analysis

Julius-Maximilians-Universität Würzburg

May 22–29, 2015

3.–4. Uniformly normal families and generalisations

Aimo Hinkkanen

University of Illinois at Urbana–Champaign

aimo@math.uiuc.edu

Uniformly normal families

Definition 1

(Hayman 1955)
Suppose that $B \geq 0$. We define $\mathcal{N}(B)$ to be the family of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that whenever $z_1, z_2 \in \mathbb{D}$,
$|f(z_1)| \leq 1$ and $|f(z_2)| \geq e^B$,
we have

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \geq \frac{1}{2}.$$

A subfamily of some $\mathcal{N}(B)$ is said to be **uniformly normal**.

Uniformly normal families

Definition 1

(Hayman 1955)
Suppose that $B \geq 0$. We define $\mathcal{N}(B)$ to be the family of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that whenever $z_1, z_2 \in \mathbb{D}$,
$|f(z_1)| \leq 1$ and $|f(z_2)| \geq e^B$,
we have
$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \geq \frac{1}{2}.$$
A subfamily of some $\mathcal{N}(B)$ is said to be **uniformly normal**.

Remark. If $B = 0$, this means that each f satisfies either $|f| < 1$ throughout $\mathbb{D}$, or $|f| > 1$ throughout $\mathbb{D}$.

Observations.

The family $\mathcal{F} = \mathcal{N}(B)$ has the following properties:

(i) if $f \in \mathcal{F}$ and $|f(0)| \leq 1$, then for all z with $|z| < 1/2$, we have

$$|f(z)| < e^B.$$

Uniformly normal families

Observations.

The family $\mathcal{F} = \mathcal{N}(B)$ has the following properties:
(i) if $f \in \mathcal{F}$ and $|f(0)| \leq 1$, then for all z with $|z| < 1/2$, we have

$$|f(z)| < e^B.$$

(ii) if $f \in \mathcal{F}$ and $|a| < 1 = |c|$, then the function

$$f\left(c\frac{z-a}{1-\overline{a}z}\right)$$

is also in $\mathcal{F}$. That is, $\mathcal{F}$ is **linearly invariant**.

Observations.

The family $\mathcal{F} = \mathcal{N}(B)$ has the following properties:
(i) if $f \in \mathcal{F}$ and $|f(0)| \leq 1$, then for all z with $|z| < 1/2$, we have

$$|f(z)| < e^B.$$

(ii) if $f \in \mathcal{F}$ and $|a| < 1 = |c|$, then the function

$$f\left(c\frac{z-a}{1-\bar{a}z}\right)$$

is also in $\mathcal{F}$. That is, $\mathcal{F}$ is **linearly invariant**.
Every normal family $\mathcal{F}$ satisfies (i) for some $B \geq 0$.

A property of normal families.

Theorem 2

Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \overline{\mathbb{C}}$. Suppose that $0 < a < b$. Then $\mathcal{F}$ is a normal family if, and only if, each $z_0 \in D$ has a neighbourhood U such that each $f \in \mathcal{F}$ satisfies $|f(z)| > a$ for all $z \in U$, or $|f(z)| < b$ for all $z \in U$.

A property of normal families.

Theorem 3

Let $\mathcal{F}$ be a normal family of analytic functions in $\mathbb{D}$. Then there exists a number $B \geq 0$ such that whenever $f \in \mathcal{F}$ and $|f(0)| \leq 1$, we have

$$|f(z)| < e^B$$

for all z with $|z| < 1/2$.

A property of normal families. Proof.

To get a contradiction, suppose that $\mathcal{F}$ is a normal family of analytic functions in $\mathbb{D}$ and that there is a sequence $f_n \in \mathcal{F}$ and a sequence of points z_n such that for all n, we have $|f_n(0)| \leq 1$, $|z_n| < 1/2$, and $|f_n(z_n)| > n$.

A property of normal families. Proof.

To get a contradiction, suppose that $\mathcal{F}$ is a normal family of analytic functions in $\mathbb{D}$ and that there is a sequence $f_n \in \mathcal{F}$ and a sequence of points z_n such that for all n, we have $|f_n(0)| \leq 1$, $|z_n| < 1/2$, and $|f_n(z_n)| > n$.

We may assume, without changing notation, that $f_n \to f$ locally uniformly in $\mathbb{D}$, where f is analytic in $\mathbb{D}$, or $f \equiv \infty$.

A property of normal families. Proof.

To get a contradiction, suppose that $\mathcal{F}$ is a normal family of analytic functions in $\mathbb{D}$ and that there is a sequence $f_n \in \mathcal{F}$ and a sequence of points z_n such that for all n, we have $|f_n(0)| \leq 1$, $|z_n| < 1/2$, and $|f_n(z_n)| > n$.

We may assume, without changing notation, that $f_n \to f$ locally uniformly in $\mathbb{D}$, where f is analytic in $\mathbb{D}$, or $f \equiv \infty$.

We cannot have $f \equiv \infty$ since $|f_n(0)| \leq 1$ for all n. Hence $|f(0)| \leq 1$.

A property of normal families. Proof.

To get a contradiction, suppose that $\mathcal{F}$ is a normal family of analytic functions in $\mathbb{D}$ and that there is a sequence $f_n \in \mathcal{F}$ and a sequence of points z_n such that for all n, we have $|f_n(0)| \leq 1$, $|z_n| < 1/2$, and $|f_n(z_n)| > n$.

We may assume, without changing notation, that $f_n \to f$ locally uniformly in $\mathbb{D}$, where f is analytic in $\mathbb{D}$, or $f \equiv \infty$.

We cannot have $f \equiv \infty$ since $|f_n(0)| \leq 1$ for all n. Hence $|f(0)| \leq 1$.

We may assume that $z_n \to w$, where $|w| \leq 1/2$.
pause

By uniform convergence in $\overline{B}(0, 1/2)$, we have $f_n(z_n) \to f(w)$. Now f is analytic, so $f(w) \neq \infty$. But since $|f_n(z_n)| > n$, we have $f(w) = \infty$. This contradiction proves the theorem.

Uniformly normal families

Every normal linearly invariant family is uniformly normal.

Uniformly normal families

Theorem 4

The family $\mathcal{N}(B)$ is normal and hence every uniformly normal family is normal.

Due to linear invariance, it suffices to show that $\mathcal{N}(B)$ is normal at the origin.
Let F_1 be the set of those functions f in $\mathcal{N}(B)$ that satisfy $|f| \geq 1$ in a given small disk $B(0, r)$.
If f is in $\mathcal{N}(B)$ but not in F_1 then there is a point $z \in B(0, r)$ with $|f(z)| < 1$. Now (moving z first to 0) we see that there is a radius t close to $1/2$ such that in $B(0, t)$, we have $|f| < e^B$.
By Theorem 2, $\mathcal{N}(B)$ is normal.

Examples of uniformly normal families.
Let D_1, D_2, D_3 be (Jordan) domains in $\overline{\mathbb{C}}$ with disjoint closures.

Examples of uniformly normal families.
Let D_1, D_2, D_3 be (Jordan) domains in $\overline{\mathbb{C}}$ with disjoint closures.
1. Let $\mathcal{F}_1(D_1, D_2, D_3)$ be the family of all analytic functions f in $\mathbb{D}$ such that f maps no subdomain of $\mathbb{D}$ conformally onto any D_j.

Examples of uniformly normal families.

Let D_1, D_2, D_3 be (Jordan) domains in $\overline{\mathbb{C}}$ with disjoint closures.

1. Let $\mathcal{F}_1(D_1, D_2, D_3)$ be the family of all analytic functions f in $\mathbb{D}$ such that f maps no subdomain of $\mathbb{D}$ conformally onto any D_j.

2. Let $\mathcal{F}_2(D_1, D_2)$ be the family of all analytic functions f in $\mathbb{D}$ such that f maps no subdomain of $\mathbb{D}$ conformally onto D_1 or D_2, or two-to-one onto D_1.

Examples of uniformly normal families.
Let D_1, D_2, D_3 be (Jordan) domains in $\overline{\mathbb{C}}$ with disjoint closures.
1. Let $\mathcal{F}_1(D_1, D_2, D_3)$ be the family of all analytic functions f in $\mathbb{D}$ such that f maps no subdomain of $\mathbb{D}$ conformally onto any D_j.
2. Let $\mathcal{F}_2(D_1, D_2)$ be the family of all analytic functions f in $\mathbb{D}$ such that f maps no subdomain of $\mathbb{D}$ conformally onto D_1 or D_2, or two-to-one onto D_1.

Clearly each of $\mathcal{F}_1(D_1, D_2, D_3)$ and $\mathcal{F}_2(D_1, D_2)$ is linearly invariant. Each is normal, as was proved by Ahlfors (1933) (also a consequence of the Ahlfors theory of covering surfaces (1935)). Thus they are uniformly normal.

Examples of uniformly normal families

3. Let $\mathcal{F}$ be a family of analytic functions in $\mathbb{D}$ such that for certain distinct finite values a, b, and for certain positive integers h, k with

$$\frac{1}{h} + \frac{1}{k} < 1,$$

each $f \in \mathcal{F}$ takes the value a only with multiplicity at least h, and each $f \in \mathcal{F}$ takes the value b only with multiplicity at least k. Then $\mathcal{F}$ is a uniformly normal family.

Uniformly normal families. Example.

Example of a family that is normal but not uniformly normal.
Let $\mathcal{F}$ consist of all f analytic in $\mathbb{D}$ such that $f \neq 0$ and $f' \neq 1$ in $\mathbb{D}$.
Then $\mathcal{F}$ is normal by Milloux (1940).
Clearly $\mathcal{F}$ is not linearly invariant, due to the condition $f' \neq 1$.

Uniformly normal families

One motivation for the concept of a uniformly normal family.
To develop a framework for obtaining growth estimates for functions in many different families that happen to be uniformly normal.
For example, for the families $\mathcal{F}_1(D_1, D_2, D_3)$ and $\mathcal{F}_2(D_1, D_2)$ this leads to upper bounds for the maximum modulus (Hayman 1955) that improve earlier results (Ahlfors 1933).

Marty's criterion.

Recall Marty's criterion for general normal families of meromorphic functions.

Theorem 5

Let $\mathcal{F}$ be a family of meromorphic functions in $\mathbb{D}$. Then $\mathcal{F}$ is a normal family if, and only if, for each compact subset K of $\mathbb{D}$, there is a constant $M > 0$ such that for all $f \in \mathcal{F}$ and all $z \in K$, we have

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \leq M.$$

Marty's criterion for uniformly normal families.

We obtain the following result for families of meromorphic functions. Applying the result to families of analytic functions, we get a criterion for a family to be uniformly normal.

Marty's criterion for uniformly normal families.

We obtain the following result for families of meromorphic functions. Applying the result to families of analytic functions, we get a criterion for a family to be uniformly normal.

Theorem 6

Let $\mathcal{F}$ be a linearly invariant family of meromorphic functions in $\mathbb{D}$. Then $\mathcal{F}$ is a normal family if, and only if, there is a constant $M > 0$ such that for all $f \in \mathcal{F}$ we have

$$f^{\#}(0) = \frac{|f'(0)|}{1 + |f(0)|^2} \leq M. \tag{1}$$

Marty's criterion for uniformly normal families. Proof.

If $\mathcal{F}$ is normal, then Marty's criterion implies (1).

Marty's criterion for uniformly normal families. Proof.

If $\mathcal{F}$ is normal, then Marty's criterion implies (1).

Suppose then that $\mathcal{F}$ is linearly invariant and that (5) holds. Let $K \subset \mathbb{D}$ be compact with $|z| \leq r < 1$ for all $z \in K$. Pick $a \in K$. Then $g \in \mathcal{F}$, where

$$g(z) = f\left(\frac{z-a}{1-\bar{a}z}\right).$$

We have

$$\frac{(1-|a|^2)|f'(a)|}{1+|f(a)|^2} = g^{\#}(0) \leq M$$

so that

$$f^{\#}(a) \leq \frac{M}{1-|a|^2} \leq \frac{M}{1-r^2}.$$

Hence $\mathcal{F}$ is normal.

The following example is somewhat special, but was mentioned by Montel (1934) as an example of a normal family. We write

$$A(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f'(te^{i\theta})|^2}{(1 + |f(te^{i\theta})|^2)^2} \, t \, dt \, d\theta$$

for the average covering number in the disk $B(0, r)$ under the meromorphic function f. Note that $A(r, f)$ is an increasing function of r.

Examples of uniformly normal families

The following example is somewhat special, but was mentioned by Montel (1934) as an example of a normal family. We write

$$A(r,f) = \frac{1}{\pi}\int_0^r \int_0^{2\pi} \frac{|f'(te^{i\theta})|^2}{(1+|f(te^{i\theta})|^2)^2}\, t\, dt\, d\theta$$

for the average covering number in the disk $B(0,r)$ under the meromorphic function f. Note that $A(r,f)$ is an increasing function of r.

4. For $0 < K < 1$, let $\mathcal{F}_K$ be the family of meromorphic functions in $\mathbb{D}$ such that $A(r,f) \leq K$ for all $r \in (0,1)$. Then $\mathcal{F}_K$ is a linearly invariant normal family. Invariance is clear.

Thus the corresponding family $\mathcal{A}_K$ of functions f analytic in $\mathbb{D}$ such that $A(r, f) \leq K < 1$ for all $r \in (0, 1)$, is a uniformly normal family.

Examples of uniformly normal families

Thus the corresponding family $\mathcal{A}_K$ of functions f analytic in $\mathbb{D}$ such that $A(r, f) \leq K < 1$ for all $r \in (0, 1)$, is a uniformly normal family.

This follows from a theorem of Dufresnoy (1941), which states that

$$f^{\#}(0)^2 \leq \frac{1}{r^2} \frac{A(r, f)}{1 - A(r, f)}$$

for all $r \in (0, 1)$ (if all these $A(r, f)$ are < 1). So if $f \in \mathcal{A}_K$, we obtain, as $r \to 1$, that

$$f^{\#}(0) \leq \sqrt{\frac{K}{1 - K}}.$$

Examples of uniformly normal families

Thus the corresponding family $\mathcal{A}_K$ of functions f analytic in $\mathbb{D}$ such that $A(r,f) \leq K < 1$ for all $r \in (0,1)$, is a uniformly normal family.

This follows from a theorem of Dufresnoy (1941), which states that

$$f^{\#}(0)^2 \leq \frac{1}{r^2}\frac{A(r,f)}{1-A(r,f)}$$

for all $r \in (0,1)$ (if all these $A(r,f)$ are < 1). So if $f \in \mathcal{A}_K$, we obtain, as $r \to 1$, that

$$f^{\#}(0) \leq \sqrt{\frac{K}{1-K}}.$$

The family $f \in \mathcal{A}_1$ is not normal since it contains the non-normal sequence

$$\{f_n(z) = nz : n \geq 1\}.$$

Examples of uniformly normal families

The following example is connected to the Ahlfors theory of covering surfaces (1935).
We write

$$L(r, f) = \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2}\, r\, d\theta$$

for the spherical length of the image under f of the circle $S(0, r) = \{z : |z| = r\}$.

Examples of uniformly normal families

The following example is connected to the Ahlfors theory of covering surfaces (1935).

We write

$$L(r, f) = \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2}\, r\, d\theta$$

for the spherical length of the image under f of the circle $S(0, r) = \{z : |z| = r\}$.

Let B be any disk, not necessarily centred at the origin. We define the quantities $A(B, f)$ and $L(B, f)$ analogously to $A(r, f)$ and $L(r, f)$, using the disk B instead of the disk $B(0, r)$.

Examples of uniformly normal families

The following example is connected to the Ahlfors theory of covering surfaces (1935).
We write

$$L(r, f) = \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2}\, r\, d\theta$$

for the spherical length of the image under f of the circle $S(0, r) = \{z : |z| = r\}$.

Let B be any disk, not necessarily centred at the origin. We define the quantities $A(B, f)$ and $L(B, f)$ analogously to $A(r, f)$ and $L(r, f)$, using the disk B instead of the disk $B(0, r)$.

5. For real $h > 0$, let $\mathcal{G}_h$ be the family of functions f analytic in $\mathbb{D}$ such that for all disks B whose closure is a subset of $\mathbb{D}$, we have

$$A(B, f) \leq hL(B, f).$$

Then $\mathcal{G}_h$ is a uniformly normal family.
We omit the proof of this claim.

A property of uniformly normal families

Definition 7

The Ahlfors–Shimizu characteristic of a function f meromorphic in $\mathbb{D}$ is given by

$$T_0(r,f) = \int_0^r \frac{A(t)}{t}\,dt$$

for $0 < r < 1$.

A property of uniformly normal families

Definition 7

The Ahlfors–Shimizu characteristic of a function f meromorphic in $\mathbb{D}$ is given by

$$T_0(r, f) = \int_0^r \frac{A(t)}{t}\, dt$$

for $0 < r < 1$.

Theorem 8

If $\mathcal{G}$ is a uniformly normal family in $\mathbb{D}$ then there is a constant $C > 0$ such that whenever $f \in \mathcal{G}$ and $0 < r < 1$, we have

$$T_0(r, f) \leq C \log \frac{1}{1 - r^2}.$$

We omit the proof. The same result is valid for linearly invariant normal families of meromorphic functions in $\mathbb{D}$.

Uniformly normal families. Estimates on the derivative.

Due to linear invariance, estimates at the origin can be used to obtain estimates at any point of $\mathbb{D}$.
Basic result:

Theorem 9

Suppose that $f \in \mathcal{N}(B)$*,*

$$f(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n$$

for $|z| < 1$*. Then, with*

$$\mu = \max\{1, |a_0|\} \geq 1,$$

we have

$$|a_1| \leq 2\mu(\log \mu + B_1),$$

where B_1 *depends on* B *only (we may take* $B_1 = 4e^{2B}$*).*

Uniformly normal families. Estimates on the derivative.

Application, along the lines of Hayman's (1947) results on Schottky's theorem:

Theorem 10

Suppose that $f \in \mathcal{N}(B)$ *and* $|z_0| = r < 1$*. Then, with* $\mu_0 = \max\{1, |f(z_0)|\}$*, we have*

$$|f'(z_0)| \leq \frac{2}{1-r^2}\mu_0(\log \mu_0 + B_1)$$

and with $\mu = \max\{1, |a_0|\}$*,*

$$M(r,f) = \max\{|f(z)| : |z| = r\} \leq \mu^{\frac{1+r}{1-r}} e^{\frac{2B_1 r}{1-r}}. \tag{2}$$

Here $B_1 = 4e^{2B}$ as in Theorem 9.

Proof of Theorem 10, assuming Theorem 9.
Define

$$g(z) = f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)$$

so that $g \in \mathcal{N}(B)$, and

$$a_0(g) = f(z_0),\ \ \mu(g) = \mu_0(f),$$

$$a_1(g) = g'(0) = f'(z_0)(1-r^2).$$

Apply Theorem 9 to g to get

$$|f'(z_0)|(1-r^2) \leq 2\mu_0(f)(\log \mu_0(f) + B_1).$$

Uniformly normal families. Estimates on the derivative.

Pick $r \in (0, 1)$.
If $M(r, f) \le 1$, then (2) is trivial.

Uniformly normal families. Estimates on the derivative.

Pick $r \in (0,1)$.
If $M(r,f) \leq 1$, then (2) is trivial.

If $M(r,f) > 1$, choose z_0 with $|z_0| = r$ and $|f(z_0)| = M(r,f)$.
We have

$$\begin{aligned} & \frac{d}{dt} \log\log(e^{B_1}|f(tz_0/r)|) \\ & = \frac{1}{\log|f(tz_0/r)| + B_1} \mathrm{Re}\left(\frac{z_0}{r}\frac{f'}{f}(tz_0/r)\right) \end{aligned}$$

so that when $|f(tz_0/r)| \geq 1$, we have

$$\begin{aligned} \left| \frac{d}{dt} \log\log(e^{B_1}|f(tz_0/r)|) \right| & \leq \frac{|(f'/f)(tz_0/r)|}{\log|f(tz_0/r)| + B_1} \\ & \leq \frac{2}{1-t^2}. \end{aligned}$$

Uniformly normal families. Estimates on the derivative.

Choose $t_0 \in [0,1)$ so that $|f(tz_0/r)| > 1$ for $t_0 < t \le r$ and either $t_0 = 0$ or $|f(t_0 z_0/r)| = 1$. In either case, $|f(t_0 z_0/r)| \le \mu = \max\{1, |f(0)|\}$. Then

$$\begin{aligned} &|\log\log(e^{B_1}|f(z_0)|) - \log\log(e^{B_1}|f(t_0 z_0/r)|)| \\ &\le \int_{t_0}^{r} \frac{2}{1-t^2}\,dt \le \int_0^r \frac{2}{1-t^2}\,dt = \log\left(\frac{1+r}{1-r}\right), \end{aligned}$$

hence

$$\begin{aligned} &\log(e^{B_1}|f(z_0)|) \le \frac{1+r}{1-r}\log(e^{B_1}|f(t_0 z_0/r)|) \\ &\le \frac{1+r}{1-r}\log(e^{B_1}\mu), \end{aligned}$$

so

$$\begin{aligned} M(r,f) &= |f(z_0)| \le e^{-B_1}\mu^{\frac{1+r}{1-r}} e^{B_1\frac{1+r}{1-r}} \\ &= \mu^{\frac{1+r}{1-r}} e^{\frac{2B_1 r}{1-r}}. \end{aligned}$$

This proves Theorem 10.

Uniformly normal families. Growth estimates.

Theorem 10 implies that if $f \in \mathcal{N}(B)$, then, as $r \to 1$, we have

$$\log M(r, f) = O\left(\frac{1}{1-r}\right).$$

Uniformly normal families. Growth estimates.

Theorem 10 implies that if $f \in \mathcal{N}(B)$, then, as $r \to 1$, we have

$$\log M(r, f) = O\left(\frac{1}{1-r}\right).$$

An earlier estimate of Ahlfors (1933) for the families $\mathcal{F}_1(D_1, D_2, D_3)$ and $\mathcal{F}_2(D_1, D_2)$ had been

$$\log M(r, f) = \frac{O(1)}{1-r} \log\left(\frac{1}{1-r}\right).$$

Harnack's inequality

For the proof of Theorem 9, we need a lemma. It is partly essentially a form of Schwarz's lemma (or the Borel–Carathéodory inequality) and partly a consequence of Harnack's inequality for positive harmonic functions u in the unit disk:

$$\frac{1-|z|}{1+|z|}u(0) \leq u(z) \leq \frac{1+|z|}{1-|z|}u(0)$$

whenever $|z| < 1$.

Lemma 11

Suppose that ϕ is analytic and satisfies $|\phi| > 1$ in $\mathbb{D}$, with

$$\phi(z) = b_0 + b_1 z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$|b_1| \leq 2|b_0| \log |b_0|,$$

and for $z_1, z_2 \in \mathbb{D}$, we have

$$|\phi(z_1)| \geq |\phi(z_2)|^{\frac{1-t}{1+t}},$$

where

$$t = \left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|.$$

Proof of Lemma 11. We may assume that $z_1 = 0$, b_0 real, positive (so > 1). Write

$$g(z) = \log \phi(z) = g_0 + g_1 z + \sum_{n=2}^{\infty} g_n z^n, \; g_0 > 0.$$

Proof of Lemma 11. We may assume that $z_1 = 0$, b_0 real, positive (so > 1). Write

$$g(z) = \log \phi(z) = g_0 + g_1 z + \sum_{n=2}^{\infty} g_n z^n, \; g_0 > 0.$$

Then $\mathrm{Re}\, g > 0$ and

$$\psi(z) = \frac{g(z) - g_0}{g(z) + g_0}$$

satisfies the assumptions of Schwarz's lemma. Hence

$$1 \geq |\psi'(0)| = \frac{|g'(0)|}{2g_0} = \frac{|b_1|}{2|b_0| \log |b_0|}.$$

Also

$$|\psi(z_2)| \le |z_2| = t,$$

Also

$$|\psi(z_2)| \leq |z_2| = t,$$

$$\frac{g(z_2)}{g_0} = \frac{1+\psi(z_2)}{1-\psi(z_2)}, \; |g(z_2)| \leq g_0 \frac{1+t}{1-t},$$

Uniformly normal families

Also

$$|\psi(z_2)| \leq |z_2| = t,$$

$$\frac{g(z_2)}{g_0} = \frac{1+\psi(z_2)}{1-\psi(z_2)}, \ |g(z_2)| \leq g_0 \frac{1+t}{1-t},$$

$$|\phi(z_2)| = e^{\mathrm{Re}\, g(z_2)} \leq |b_0|^{\frac{1+t}{1-t}} = |\phi(z_1)|^{\frac{1+t}{1-t}}. \square$$

Proof of Theorem 9. Recall
$f(z) = a_0 + a_1 z + \ldots$, $\mu = \max\{1, |a_0|\}$, want to prove
$|a_1| \leq 2\mu(\log \mu + B_1)$.

Proof of Theorem 9. Recall
$f(z) = a_0 + a_1 z + \dots$, $\mu = \max\{1, |a_0|\}$, want to prove
$|a_1| \le 2\mu(\log \mu + B_1)$.

Case 1. $|a_0| \le e^B$.
Then $|f| > 1$ in $B(0, 1/8)$, or $|f| < e^B$ in $B(0, 1/8)$. Otherwise there are
$z_1, z_2 \in B(0, 1/8)$
with $|f(z_1)| \le 1$, $|f(z_2)| \ge e^B$, and

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \le \frac{1/8 + 1/8}{1 - 1/64} < \frac{1}{2},$$

a contradiction.

Uniformly normal families

If $|f| < e^B$ in $B(0, 1/8)$, then $|a_1| = |f'(0)| \leq 8e^B$ by Cauchy's estimates.

If $|f| < e^B$ in $B(0, 1/8)$, then $|a_1| = |f'(0)| \leq 8e^B$ by Cauchy's estimates.

If $|f| > 1$, so $|1/f| < 1$ in $B(0, 1/8)$, then by Cauchy's estimates $|f'(0)|/|f(0)|^2 \leq 8$, so $|a_1| \leq 8|a_0|^2 \leq 8e^{2B}$.

$f(z) = a_0 + a_1 z + \dots$, $\mu = \max\{1, |a_0|\}$, want to prove $|a_1| \le 2\mu(\log\mu + B_1)$.

Case 2. $|a_0| > e^B$.

Let $\rho \in (0,1]$ be maximal such that $|f| > 1$ in $B(0,\rho)$. If $\rho = 1$, the Lemma gives $|a_1| \le 2\mu\log\mu$ $(B_1 = 0)$. Suppose $\rho < 1$. Let $r \in (0,\rho)$ be maximal such that $|f| > e^B$ in $B(0,r)$.

Choose θ with $|f(re^{i\theta})| = e^B$, set $\phi(z) = f(\rho z)$. Apply Lemma11 to ϕ, $z_2 = 0$, $z_1 = (r/\rho)e^{i\theta}$, get

$$e^B = |\phi(z_1)| \ge |a_0|^{\frac{\rho - r}{\rho + r}},$$

$$\frac{\rho - r}{\rho + r}\log|a_0| \le B.$$

With β s.t. $|f(\rho e^{i\beta})| = 1$, $|f(re^{i\beta})| \ge e^B$.

Since $f \in \mathcal{N}(B)$, $\frac{\rho - r}{1 - \rho r} \ge \frac{1}{2}$.

Also $\rho \ge 1/2$ since $|a_0| = |f(0)| > e^B$.

Case 2. $|a_0| > e^B$, continued.
We know that

$$\frac{\rho - r}{\rho + r} \log |a_0| \leq B, \ \frac{\rho - r}{1 - \rho r} \geq \frac{1}{2},$$

so

$$B \geq \frac{\rho - r}{\rho + r} \log |a_0| \geq \frac{1}{4}(1 - \rho r) \log |a_0|,$$

$$1 - \rho \leq 1 - \rho r \leq \frac{4B}{\log |a_0|}.$$

Then Lemma 11 applied to $\phi(z) = f(\rho z)$ gives

$$\rho |a_1| = \rho |f'(0)| = |\phi'(0)| \leq 2|a_0| \log |a_0|.$$

Recall

$$\rho|a_1| = \rho|f'(0)| = |\phi'(0)| \leq 2|a_0| \log|a_0|.$$

Hence (recall $\rho \geq 1/2$)

$$\begin{aligned} |a_1| &\leq \frac{2}{\rho}|a_0| \log|a_0| \\ &= 2|a_0| \log|a_0| + \frac{2(1-\rho)}{\rho}|a_0| \log|a_0| \\ &\leq 2|a_0| \log|a_0| + 16B|a_0|. \end{aligned}$$

Also $B_1 = 4e^{2B} \geq 8B$ works in all cases.

Uniformly normal families. Other estimates.

Some other results of Hayman (1955) without proofs, for $f \in \mathcal{N}(B)$:

Theorem 12

For $\lambda \geq 1$, $0 \leq r < 1$,

$$T_\lambda(r) := \frac{1}{2\pi} \int_0^{2\pi} \left\{ \log \sqrt{1 + |f(re^{i\theta})|^2} \right\}^\lambda d\theta$$
$$\leq T_\lambda(0) + B_2 \log \frac{1}{1-r}, \ \lambda = 1,$$

$$T_\lambda(r) \leq \frac{B(a_0, \lambda)}{(1-r)^{\lambda - 1}}, \ \lambda > 1.$$

Here B_2 depends on B only, and $B(a_0, \lambda)$ depends on B, a_0, and λ.

Uniformly normal families. Other estimates.

Theorem 13

Suppose that $e^g \in \mathcal{N}(B)$,

$$g(z) = u + iv = \alpha + i\beta + \sum_{n=1}^{\infty} g_n z^n, \; |z| < 1.$$

Then $(0 \le r < 1)$

$$|g_1| \le 2(|\alpha| + B_3),$$

$$|g_n| \le B_4(|\alpha| + \log n), \; n \ge 2,$$

$$M(r,g) \le |g(0)| \frac{1+r}{1-r} + \frac{2B_3 r}{1-r},$$

Uniformly normal families. Other estimates.

$$I_\lambda(r,g) := \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^\lambda \, d\theta$$
$$\leq B(g(0),\lambda)(1-r)^{1-\lambda}, \ \ \lambda > 1,$$

$$I_1(r,g) \leq |g(0)| + |\alpha| \log \frac{1+r}{1-r}$$
$$+ rB_5 \left[1 + \left(\log \frac{1}{1-r}\right)^2\right],$$

$$I_1(r,u) \leq |\alpha| + B_6 \left(1 + \log \frac{1}{1-r}\right).$$

The condition that $e^g \in \mathcal{N}(B)$, makes an appearance in the following geometric characterisation of $\mathcal{N}(B)$:

Theorem 14

Let $\mathcal{R}$ be a family of Riemann surfaces spread over the $w-$plane, and let $\mathcal{F}$ be the family of functions e^g, where $w = g(z)$ maps $\mathbb{D}$ conformally onto a surface in $\mathcal{R}$. Then $\mathcal{F}$ is uniformly normal if, and only if, the radii of schlicht disks in the surfaces in $\mathcal{R}$ with centres on the imaginary $w-$axis are uniformly bounded.

We note that clearly $\mathcal{F}$ is linearly invariant.

Proof of Theorem 14.

Suppose that all radii in question are $< A$ and let $\mathcal{H}$ consist of all analytic functions in $\mathbb{D}$ of the form

$$h(z) = \frac{g(z) - iv}{A}, \ v \in \mathbb{R},$$

arising in this way from elements of $\mathcal{R}$. Set $D_1 = \mathbb{D}$, $D_2 = B(3i, 1)$, $D_3 = B(-3i, 1)$. Then the closures of the D_j are disjoint, and no $h \in \mathcal{H}$ maps any subdomain of $\mathbb{D}$ conformally onto any D_j. By Ahlfors, $\mathcal{H}$ is a normal family. Hence there is $A_1 > 0$ such that $h(0) = 0$ implies $\operatorname{Re} h(z) < A_1$ whenever $|z| \le 1/2$. Since v is arbitrary, we have

$$\operatorname{Re} g(0) = 0 \Rightarrow \operatorname{Re} g(z) < AA_1, \ |z| < 1/2.$$

For $f = e^g$, this means that

$$|f(0)| = 1 \Rightarrow |f(z)| < \exp(AA_1), \ |z| < 1/2.$$

Hence $\mathcal{F} \subset \mathcal{N}(AA_1)$, so $\mathcal{F}$ is uniformly normal.

Proof of Theorem 14, continued.
Suppose that $\mathcal{F}$ is uniformly normal. To get a contradiction, suppose that the radii are not bounded above and for some surface in $\mathcal{R}$ there is a schlicht disk $B(iv, d)$. Let g be a conformal mapping of $\mathbb{D}$ onto this surface with $g(0) = iv$. Let h be a branch of g^{-1} mapping $B(iv, d)$ onto a subdomain of $\mathbb{D}$, $h(iv) = 0$, $z_1 = h(iv + d/2)$. By Schwarz's lemma, $|z_1| \le 1/2$. If $f = e^g$, this means that

$$f(0) = e^{iv},\ |f(0)| = 1,\ |f(z_1)| = e^{d/2}.$$

If d can be arbitrarily large, then $\mathcal{F}$ is not uniformly normal. Hence the radii d are bounded.
This proves Theorem 14.

Uniformly normal families

References

L.V. Ahlfors, Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée, Acta Soc. Sci. Fennicae, Nova Ser. A, 2, No. 2 (1933), 1–17.

W.K. Hayman, Uniformly normal families,
pp. 199–212 in Lectures on Functions of a Complex Variable, University of Michigan Press, edited by Wilfred Kaplan, 1955.

Generalisations of uniformly normal families

We discuss generalisations of uniformly normal families, as given in the paper
W.K. Hayman and A. Hinkkanen, Generalisations of uniformly normal families, Transcendental dynamics and complex analysis, pp. 179–186, London Math. Soc. Lecture Note Ser., 348, Cambridge Univ. Press, Cambridge, 2008.

Recall that for a number $B \geq 0$, the family $\mathcal{N}(B)$ consists of all analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that
whenever $z_1, z_2 \in \mathbb{D}$,
$|f(z_1)| \leq 1$ and $|f(z_2)| \geq e^B$,
we have

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \geq \frac{1}{2}.$$

Generalisations of uniformly normal families, definition.

We now replace the parameter B by two positive functions, $\delta(r)$ (replacing the constant $1/2$) and $\lambda(r)$ (replacing, effectively, the constant $B/2$), defined for $0 < r < 1$, such that

$$0 < \delta(r) < 1,$$

$\delta(r)$ is decreasing (that is, non-increasing), and $\lambda(r)$ is increasing (that is, non-decreasing). This leads us to consider f analytic in $\mathbb{D}$ such that
whenever $|z_1| \le r < 1$ and $|z_2| \le r$,
the conditions
$|f(z_1)| \le e^{-\lambda(r)}$ and $|f(z_2)| \ge e^{\lambda(r)}$
imply that

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \ge \delta(r).$$

Let us denote the family of all such f by

$$\mathcal{N}(\delta(r), \lambda(r)).$$

Generalisations of uniformly normal families, definition.

Definition 15

Let $\delta(r)$ and $\lambda(r)$ be positive functions defined for $0 < r < 1$, such that $0 < \delta(r) < 1$, $\delta(r)$ is decreasing, and $\lambda(r)$ is increasing. We define the family $\mathcal{N}(\delta(r), \lambda(r))$ to consist of all functions f analytic in $\mathbb{D}$ such that whenever $|z_1| \leq r < 1$ and $|z_2| \leq r$, the conditions
$|f(z_1)| \leq e^{-\lambda(r)}$ and $|f(z_2)| \geq e^{\lambda(r)}$
imply that

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \geq \delta(r).$$

Generalisations of uniformly normal families, definition.

Definition 15

Let $\delta(r)$ and $\lambda(r)$ be positive functions defined for $0 < r < 1$, such that $0 < \delta(r) < 1$, $\delta(r)$ is decreasing, and $\lambda(r)$ is increasing. We define the family $\mathcal{N}(\delta(r), \lambda(r))$ to consist of all functions f analytic in $\mathbb{D}$ such that whenever $|z_1| \leq r < 1$ and $|z_2| \leq r$, the conditions
$|f(z_1)| \leq e^{-\lambda(r)}$ and $|f(z_2)| \geq e^{\lambda(r)}$
imply that

$$\left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right| \geq \delta(r).$$

The family $\mathcal{N}(\delta(r), \lambda(r))$ is not linearly invariant except in the special case when the functions $\delta(r)$ and $\lambda(r)$ are constant functions.

Generalisations of uniformly normal families, definition.

The family $\mathcal{N}(\delta(r), \lambda(r))$ is not linearly invariant except in the special case when the functions $\delta(r)$ and $\lambda(r)$ are constant functions.

Generalisations of uniformly normal families, definition.

The family $\mathcal{N}(\delta(r), \lambda(r))$ is not linearly invariant except in the special case when the functions $\delta(r)$ and $\lambda(r)$ are constant functions.

If $\delta(r)$ is bounded below by a positive number and $\lambda(r)$ is bounded above, then $\mathcal{N}(\delta(r), \lambda(r))$ is contained in a linearly invariant normal family.

Theorem 16

The family $\mathcal{N}(\delta(r), \lambda(r))$ is a normal family.

Proof. Pick $z_0 \in \mathbb{D}$. It suffices to show that $\mathcal{N}(\delta(r), \lambda(r))$ is normal at z_0. Choose $r \in (|z_0|, 1)$. The numbers $\delta(r)$ and $\lambda(r)$ play the role of fixed positive numbers in the argument that follows.

Let U be a small disk neighbourhood of z_0 with the following properties:

(i) $U \subset B(0, r)$;

(ii) whenever $z_1, z_2 \in U$, we have

$$\left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right| < \frac{\delta(r)}{2}.$$

Proof, continued. Let F_1 consist of those elements f of $\mathcal{N}(\delta(r), \lambda(r))$ such that $|f(z)| \geq \exp(-\lambda(r))$ for all $z \in U$.
If f lies in $\mathcal{N}(\delta(r), \lambda(r))$ but not in F_1, then there is a point $z_1 \in U$ such that $|f(z_1)| < \exp(-\lambda(r))$. Therefore, if $|z_2| \leq r$ and $|f(z_2)| \geq \exp(\lambda(r))$, we have

$$\left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right| \geq \delta(r)$$

so that $z_2 \notin U$. Hence, for this f, we have $|f(z)| < \exp(\lambda(r))$ for all $z \in U$. Now it follows from Theorem 2 that $\mathcal{N}(\delta(r), \lambda(r))$ is normal at z_0. This completes the proof of Theorem 16.

The main result.

Theorem 17

Suppose that $f \in \mathcal{N}(\delta(r), \lambda(r))$. *Suppose that* $|z_0| < r < 1$ *and*

$$\log |f(z_0)| > \frac{8\lambda(r)}{\delta(r)}.$$

Then

$$|f'(z_0)| < \frac{2r|f(z_0)|}{r^2 - |z_0|^2} \left\{ \log |f(z_0)| + \frac{8\lambda(r)}{\delta(r)} \right\}.$$

Generalisations of uniformly normal families, main result.

In most applications,

$$\frac{8\lambda(r)}{\delta(r)} = a + b \log \frac{1}{1-r}$$

for some constants $a \geq 0$, $b \geq 0$. In this case, define t by

$$r = 1 - \frac{1}{2}(1-t)^2$$

and define

$$\lambda^*(t) = a + b \log 2 + 2b \log \frac{1}{1-t} = a + b \log \frac{1}{1-r}.$$

Generalisations of uniformly normal families, main result.

$$\lambda^*(t) = a + b \log 2 + 2b \log \frac{1}{1-t} = a + b \log \frac{1}{1-r}$$

Corollary 18

Let f, $\delta(r)$, $\lambda(r)$, t, and $\lambda^*(t)$ be as above. If $|z| = t$ then

$$\log |f(z)| \leq \lambda^*(t)$$

or

$$\frac{|f'(z)|}{|f(z)|} < \left\{ \frac{2}{1-t^2} + 8 \right\} (\log |f(z)| + \lambda^*(t)).$$

Further, in all cases,

$$\log |f(z)| \leq e^8 \left\{ 3a + 8b + \log^+ |f(0)| \right\} \frac{1+|z|}{1-|z|}.$$

Generalisations of uniformly normal families, remarks on proof.

The proof of Theorem 17 is structured so that one first proves a lemma in which $\delta(r)$ and $\lambda(r)$ are constants. Then Theorem 17 is proved using the lemma and a suitable change of variables. All of this is very technical, and before getting to it, we consider some applications of these concepts.

Examples and applications.

For a positive integer k, let M_k and A_k be the families of meromorphic functions f in $\mathbb{D}$ and analytic functions f in $\mathbb{D}$, such that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $z \in \mathbb{D}$.

Then M_k and A_k are normal families in $\mathbb{D}$.

Hence there exist positive constants λ_k and δ_k depending only on k such that:

if $f \in M_k$ and $|z_j| \leq \delta_k$, for $j = 1, 2$, then we cannot have $|f(z_1)| \leq e^{-\lambda_k}$ and $|f(z_2)| \geq e^{\lambda_k}$.

The conclusion must hold for every sufficiently small δ_k, with λ_k depending on δ_k and k only.

Proof of the claim that $\delta_k > 0$ and $\lambda_k > 0$ exist as stated above.

To get a contradiction, suppose that there is no pair (δ_k, λ_k) with the required properties. Then there are sequences of functions $f_n \in M_k$ and points $z_n \to 0$ and $w_n \to 0$ such that $f_n(z_n) \to 0$ and $f_n(w_n) \to \infty$. Since M_k is normal, we may assume that $f_n \to f$ uniformly in a neighbourhood of 0, where f is meromorphic or $f \equiv \infty$. By uniform convergence, we have $f(0) = \lim_{n\to\infty} f_n(z_n) = 0$ and $f(0) = \lim_{n\to\infty} f_n(w_n) = \infty$, which is a contradiction. This proves the claim.

Generalisations of uniformly normal families, a property of normal families.

We apply the above conclusion with $|z_1| < 1$ and

$$F(z) = (1 - |z_1|)^{-k} f(z_1 + (1 - |z_1|)z)$$

instead of $f(z)$. Then F belongs to M_k or A_k if f does.

Generalisations of uniformly normal families, $f \neq 0$, $f^{(k)} \neq 1$

Corollary 19

If k is a positive integer and $f \in M_k$, then $1/f$ satisfies the hypotheses and hence the conclusion of Theorem 17 and Corollary 18 with $\delta(r) = \frac{1}{3}\delta_k$ for $0 < r < 1$, and

$$\frac{8\lambda(r)}{\delta(r)} = \frac{24}{\delta_k}\left\{\lambda_k + k\log\frac{1}{1-r}\right\},$$

$$\lambda^*(t) = \frac{24}{\delta_k}\left\{\lambda_k + k\log 2 + 2k\log\frac{1}{1-t}\right\},$$

where δ_k and λ_k are positive constants depending only on k. If f is also analytic then f satisfies the same conclusions.

Generalisations of uniformly normal families, $f \neq 0$, $f^{(k)} \neq 1$

Significance of Corollary 19.

Recall that a family $\mathcal{F}$ of analytic functions in $\mathbb{D}$ is called linearly invariant if , whenever $f \in \mathcal{F}$ and $|a| < 1 = |c|$, the function

$$f\left(c\frac{z-a}{1-\bar{a}z}\right)$$

is also in $\mathcal{F}$.

Every linearly invariant normal family is uniformly normal.

The family A_k is not linearly invariant, and hence not uniformly normal.

Therefore growth results for members of uniformly normal families do not provide such results for functions in A_k.

Corollary 19 provides growth results for members of A_k, and also results for $1/f$, even for non-analytic meromorphic members f of M_k.

Generalisations of uniformly normal families, $f \neq 0$, $f^{(k)} \neq 1$

On the function $1/f$.

The hypotheses of Theorem 17 (apart from the requirement of analyticity) are the same for f and $1/f$. Hence if f satisfies them and $1/f$ is analytic, then $1/f$ satisfies the conclusions of Theorem 17. In particular if $f \in M_k$, so that $f \neq 0$ in $\mathbb{D}$, we can apply the conclusion of Theorem 17 to $1/f$ instead of f, with $\lambda(r)$ and $\lambda^*(r)$ as in Corollary 19.

Generalisations of uniformly normal families, another example.

Another example.
We denote by $\mathcal{M}_k$ and $\mathcal{A}_k$ respectively the family of meromorphic functions f and analytic functions f in $\mathbb{D}$ such that $f'(z)f(z)^k \neq 1$ in $\mathbb{D}$. The families $\mathcal{A}_k$ and $\mathcal{M}_k$ are normal. (These problems have a long and illustrious history which we do not get into here.)

Corollary 20

If k is a positive integer and $f \in \mathcal{A}_k$, then f satisfies the hypotheses and hence the conclusion of Theorem 17 and Corollary 18 with $\delta(r) = \eta_k$, and

$$\lambda(r) = \mu_k + \frac{1}{k+1} \log \frac{1}{1-r},$$

where η_k and μ_k are positive constants depending only on k.

Generalisations of uniformly normal families, locally bounded characteristic.

The functions f in (M_k and) $\mathcal{M}_k$ are of locally bounded characteristic, which can be used to deduce upper bounds for their spherical derivatives

$$\mu(r) = \sup_{|z|\leq r} f^\sharp(z) = \sup_{|z|\leq r} \frac{|f'(z)|}{1+|f(z)|^2},$$

of the form

$$\mu(r) = O\left(\frac{1}{1-r}\right)\log\frac{1}{1-r},$$

so that

$$\int_0^1 (1-r)^{1/2}\mu(r)\,dr < \infty,$$

which condition leads to further conclusions. We do not give further details here.

Generalisations of uniformly normal families, a lemma.

We now move to proofs. For the proof of Theorem 17, we need a lemma.

Lemma 21

Suppose that $\lambda(r) = \lambda$ *and* $\delta(r) = \delta$, *where* λ *and* δ *are positive constants, that* $f \in \mathcal{N}(\delta, \lambda)$, *and that*

$$\alpha = \log|f(z_0)| > \frac{8\lambda}{\delta}.$$

Then

$$|f(z)| > e^{\lambda} \quad \textit{when} \ \left|\frac{z - z_0}{1 - \overline{z_0}z}\right| < r_1, \quad \textit{where} \ 1 - r_1 = \frac{4\lambda}{\alpha\delta} < \frac{1}{2}.$$

Also,

$$|f'(z_0)| < \frac{2|f(z_0)|}{1 - |z_0|^2}\left\{\log|f(z_0)| + \frac{8\lambda}{\delta}\right\}.$$

Proof of Lemma 21. Since the hypotheses are conformally invariant, we assume $z_0 = 0$. Let r_2 be maximal such that $|f(z)| > e^{-\lambda}$ if $|z| < r_2$. Suppose first $r_2 < 1$. Let r_1 be the largest number such that

$$|f(z)| > e^{\lambda} \quad \text{for } |z| < r_1.$$

Then $0 < r_1 < r_2 < 1$ and there exists $z_2 = r_2 e^{i\theta}$ such that

$$|f(z_2)| = e^{-\lambda}.$$

We set $z_1 = r_1 e^{i\theta}$. Then

$$\left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| = \frac{r_2 - r_1}{1 - r_1 r_2} \geq \delta,$$

i.e.,

$$r_2 \geq \frac{r_1 + \delta}{1 + \delta r_1}.$$

Proof of Lemma, continued.

Apply Harnack's inequality to $\log|\phi|$ where

$$\phi(z) = e^{\lambda} f(r_2 z),$$

choose z so that $|z| = r_1/r_2$ and $|f(r_2 z)| = e^{\lambda}$. This is possible since r_1 is maximal subject to $|f(z)| > e^{\lambda}$ for $|z| < r_1$. Then $|\phi(z)| > 1$ for $|z| < 1$, and so

$$(*) \qquad \log|\phi(z)| \geq \frac{1-|z|}{1+|z|} \log|\phi(0)|,$$

i.e.,

$$(**) \qquad 2\lambda \geq \frac{r_2 - r_1}{r_2 + r_1}(\lambda + \alpha).$$

If $r_2 = 1$, then either $r_1 = 1$, in which case the conclusions are trivial; or we can choose z such that $|z| = r_1$ and $|f(z_1)| = e^{\lambda}$. Now (*) still holds and we obtain (**) as before with $r_2 = 1$. Thus (**) is always true. Also $r_2 \geq \frac{r_1+\delta}{1+\delta r_1}$ is true if $r_2 = 1$ and $r_1 < 1$, since $\delta < 1$. Thus this inequality and (**) always hold.

Proof of Lemma, continued.

We substitute r_2 from $r_2 \geq \frac{r_1+\delta}{1+\delta r_1}$ to (**) and obtain

$$\begin{aligned} \frac{2\lambda}{\lambda+\alpha} &\geq \left(\frac{r_1+\delta}{1+\delta r_1} - r_1\right) \Big/ \left(\frac{r_1+\delta}{1+\delta r_1} + r_1\right) \\ &= \frac{\delta(1-r_1^2)}{\delta(1+r_1^2)+2r_1} \\ &\geq \frac{\delta(1-r_1^2)}{(1+r_1)^2} = \delta\frac{1-r_1}{1+r_1} \geq \frac{\delta}{2}(1-r_1). \end{aligned}$$

Thus

$$1 - r_1 \leq \frac{4\lambda}{\delta(\lambda+\alpha)} \leq \frac{4\lambda}{\delta\alpha} < \frac{1}{2}.$$

This proves the first claims of Lemma 21.

Proof of Lemma, continued.

The function

$$\Psi(z) = e^{-\lambda} f(r_1 z)$$

satisfies $|\Psi(z)| > 1$ if $|z| < 1$. Thus, as we saw in the proof of Lemma 11,

$$|\Psi'(0)| \le 2|\Psi(0)| \log |\Psi(0)|,$$

i.e.,

$$\begin{aligned}
\frac{|f'(0)|}{|f(0)|} &\le \frac{2}{r_1}(\log|f(0)| - \lambda) = \frac{2}{r_1}(\alpha - \lambda) \\
&\le 2(\alpha - \lambda)\frac{1}{1 - \frac{4\lambda}{\delta\alpha}} \\
&= 2(\alpha - \lambda) + 2(\alpha - \lambda)\frac{4\lambda}{\delta\alpha - 4\lambda} \\
&< 2\alpha + \frac{16\alpha\lambda}{\delta\alpha} = 2\left(\alpha + \frac{8\lambda}{\delta}\right)
\end{aligned}$$

since $\delta\alpha - 4\lambda > \delta\alpha/2$.

Proof of Lemma, continued.

This proves the last claim if $z_0 = 0$.

If $z_0 \neq 0$, we apply the above result to F instead of f at the origin, where $F(z) = f((z + z_0)/(1 + \overline{z_0}z))$.

This is legitimate now since $\delta(r)$ and $\lambda(r)$ are constants.

This yields the last claim in general, hence completes the proof of Lemma 21.

Proof of the main result.

Proof of Theorem 17.
Fix $r \in (0,1)$, apply Lemma 21 with $F(z) = f(rz)$ instead of f and with $\lambda = \lambda(r)$, $\delta = \delta(r)$.
We first need to check exactly which conditions are satisfied by F.
Write $z_1 = rZ_1$, $z_2 = rZ_2$. By hypothesis on f, if $|Z_1| < 1$, $|Z_2| < 1$ and

$$|F(Z_1)| \le e^{-\lambda}, \quad |F(Z_2)| \ge e^{\lambda},$$

we deduce that

$$\left| \frac{r(Z_2 - Z_1)}{1 - r^2 \overline{Z_1} Z_2} \right| \ge \delta,$$

with $\delta = \delta(r)$ and $\lambda = \lambda(r)$.

Proof of the main result, continued.

We next prove that

$$(*) \qquad \frac{|Z_2 - Z_1|}{|1 - \overline{Z_1}Z_2|} > \frac{r|Z_2 - Z_1|}{|1 - r^2\overline{Z_1}Z_2|}.$$

To see this, note that $Z_1 \neq Z_2$ and

$$|1 - r^2\overline{Z_1}Z_2|^2 - r^2|1 - \overline{Z_1}Z_2|^2$$

$$= (1 - r^2)(1 - r^2|Z_1|^2|Z_2|^2) > 0,$$

which yields (*). Thus $\left|\frac{r(Z_2 - Z_1)}{1 - r^2\overline{Z_1}Z_2}\right| \geq \delta$ implies

$$\left|\frac{Z_2 - Z_1}{1 - \overline{Z_1}Z_2}\right| > \delta.$$

So we apply Lemma 21 to $F(z) = f(rz)$ instead of $f(z)$, and with $Z_0 = z_0/r$ instead of z_0. This yields the claim of Theorem 17.

Proof of the main corollary.

Proof of Corollary 18.

We have

$$\frac{1}{r^2-t^2}-\frac{1}{1-t^2}=\frac{(1-r)(1+r)}{(r-t)(r+t)(1-t)(1+t)}$$

$$\leq \frac{(1-t)^2}{\frac{1}{2}(1-t)^2\cdot\frac{1}{2}}=4,$$

since $r=1-\frac{1}{2}(1-t)^2\geq\frac{1}{2}$ and $r-t\geq\frac{1}{2}(1-t)$.

Thus Theorem 17, with $\alpha=\log|f(z_0)|$, gives

$$|f'(z_0)|<\frac{2|f(z_0)|}{r^2-t^2}\{\alpha+\lambda^*(t)\}$$

$$<2|f(z_0)|\left\{\frac{1}{1-t^2}+4\right\}\{\alpha+\lambda^*(t)\}.$$

This proves the first claim of Corollary 18.

Proof of the main corollary, continued.

We move to the proof of the second claim of Corollary 18. Recall that that claim was as follows:

$$\log|f(z)| \leq e^8 \left\{3a + 8b + \log^+|f(0)|\right\} \frac{1+|z|}{1-|z|}.$$

Also,

$$\lambda^*(t) = a + b\log 2 + 2b\log\frac{1}{1-t} = a + b\log\frac{1}{1-r}.$$

Next, fix $\theta \in [0, 2\pi)$, write $y(t) = \log|f(te^{i\theta})|$ for $0 \le t < 1$. If $y(t) \le \lambda^*(t)$ or if $t = 0$, the second claim of Corollary 18 clearly holds for $z = te^{i\theta}$. Suppose for some $t \in (0, 1)$,

$$y(t) > \lambda^*(t).$$

Choose $t_0 \in [0, t)$ maximal, s.t. $y(t_0) \le \lambda^*(t_0)$. If $y(\tau) > \lambda^*(\tau)$ for all $\tau \in [0, t)$, set $t_0 = 0$. Then

$$y(\tau) > \lambda^*(\tau) \quad \text{for } t_0 < \tau < t.$$

Thus we can apply the first claim with τ instead of t in this range and obtain

$$y'(\tau) - \left\{\frac{2}{1-\tau^2} + 8\right\} y(\tau) < \left\{\frac{2}{1-\tau^2} + 8\right\} \lambda^*(\tau),$$

for $t_0 < \tau < t$.

Proof of the main corollary, continued.

We have

$$y'(\tau) - \left\{ \frac{2}{1-\tau^2} + 8 \right\} y(\tau) < \left\{ \frac{2}{1-\tau^2} + 8 \right\} \lambda^*(\tau),$$

for $t_0 < \tau < t$. Multiplying by

$$P(\tau) = e^{-8\tau} \frac{1-\tau}{1+\tau}$$

and integrating w.r.t. τ from t_0 to t get

$$y(t) \leq$$

$$\frac{1}{P(t)} \left\{ y(t_0)P(t_0) + \int_{t_0}^{t} \left(\frac{2}{1-\tau^2} + 8 \right) P(\tau)\lambda^*(\tau)\, d\tau \right\}.$$

If $t_0 = 0$, we get $y(t_0)P(t_0) = y(t_0) \leq \log^+ |f(0)|$.
If $t_0 > 0$, we have $y(t_0)P(t_0) = \lambda^*(t_0)P(t_0)$.

Proof of the main corollary, continued.

We write

$$\lambda^*(t) = a^* + b^* \log \frac{1}{1-t}$$

with $a^* = a + b \log 2$ and $b^* = 2b$. Then

$$\begin{aligned}
\lambda^*(t)P(t) &= \left(a^* + b^* \log \frac{1}{1-t}\right) \frac{1-t}{1+t} e^{-8t} \\
&\leq a^* + b^* \sup_{0 \leq t < 1} (1-t) \log \frac{1}{1-t} \\
&= a^* + \frac{b^*}{e}.
\end{aligned}$$

We will apply this to the term $y(t_0)P(t_0) = \lambda^*(t_0)P(t_0)$.

Again

$$\int_0^1 \left(\frac{2}{1-t^2}+8\right)\frac{1-t}{1+t}e^{-8t}\,dt < \int_0^1 10e^{-8t}\,dt < \frac{5}{4}$$

while

$$\begin{aligned}&\int_0^1 \left(\frac{2}{1-t^2}+8\right)\left(\frac{1-t}{1+t}\log\frac{1}{1-t}\right)e^{-8t}\,dt\\ < \quad & 2\int_0^1 \log\frac{1}{1-t}\,dt + \frac{8}{e}\int_0^1 e^{-8t}\,dt < 2+\frac{1}{e}.\end{aligned}$$

Thus

$$\int_0^1 \left(\frac{2}{1-t^2}+8\right)P(t)\lambda^*(t)\,dt \le \frac{5}{4}a^* + \left(2+\frac{1}{e}\right)b^*.$$

Proof of the main corollary, continued.

Hence

$$
\begin{aligned}
& y(t) \\
\leq \; & \frac{1}{P(t)}\left\{a^* + \frac{b^*}{e} + \frac{5}{4}a^* + \left(2 + \frac{1}{e}\right) b^* + \log^+ |f(0)|\right\} \\
\leq \; & \frac{1}{P(t)}\left\{3a + 8b + \log^+ |f(0)|\right\},
\end{aligned}
$$

where $a^* = a + b\log 2$ and $b^* = 2b$.
This implies the second claim of Corollary 18.
This proves Corollary 18.

Proof of Corollary 19.

Suppose $|z_1| \leq |z_2| = r$ and $f \in M_k$. Consider

$$F(Z) = \frac{1}{(1 - |z_1|)^k} f(z_1 + (1 - |z_1|)Z).$$

Then $F(Z) \neq 0$, $F^{(k)}(Z) = f^{(k)}(z_1 + (1 - |z_1|)Z) \neq 1$ when $Z \in \mathbb{D}$, so $F \in M_k$. In particular if $Z_1 = 0$ and $Z_2 = (z_2 - z_1)/(1 - |z_1|)$ and

$$|Z_2| < \delta_k$$

we cannot have

$$|F(Z_j)| \leq e^{-\lambda_k}, \qquad |F(Z_{j'})| \geq e^{\lambda_k},$$

where (j, j') is a permutation of $(1, 2)$.

Returning to f, we see that

$$|f(z_j)| < e^{-\lambda_k}(1 - |z_1|)^k, \quad |f(z_{j'})| > e^{\lambda_k}(1 - |z_1|)^k$$

imply that

$$|z_{j'} - z_j| > \delta_k(1 - |z_1|),$$

so

$$\frac{|z_2 - z_1|}{1 - |z_1|} > \delta_k.$$

Proof of results when $f \neq 0$, $f^{(k)} \neq 1$.

If $\eta = |z_2 - z_1|/(1 - |z_1|) \leq 1$, then

$$\begin{aligned} |1 - \overline{z_1} z_2| &= |1 - \overline{z_1} z_1 + \overline{z_1} z_1 - \overline{z_1} z_2| \\ &\leq 1 - |z_1|^2 + |z_2 - z_1| \\ &= (1 - |z_1|)(1 + |z_1| + \eta) \\ &< 3(1 - |z_1|). \end{aligned}$$

Thus if $|z_2 - z_1| \leq 1 - |z_1|$, we have

$$(*) \qquad \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| > \frac{|z_2 - z_1|}{3(1 - |z_1|)} > \frac{\delta_k}{3}$$

if $\frac{|z_2 - z_1|}{1 - |z_1|} > \delta_k$.

Proof of results when $f \neq 0$, $f^{(k)} \neq 1$.

We have assumed that $|z_2 - z_1| \le 1 - |z_1|$. But if this is false, (*) is still true by the maximum principle. Thus $|z_{j'} - z_j| > \delta_k(1 - |z_1|)$ always implies (*), and so does

$$|f(z_j)| < (1-r)^k e^{-\lambda_k}, \qquad |f(z_{j'})| > (1-r)^{-k} e^{\lambda_k}$$

since this implies $|f(z_j)| < e^{-\lambda_k}(1-|z_1|)^k, |f(z_{j'})| > e^{\lambda_k}(1-|z_1|)^k$ and hence $|z_{j'} - z_j| > \delta_k(1-|z_1|)$, because $r = \max\{|z_j|, |z_{j'}|\}$. This proves Corollary 19.

Proof of Corollary 20.

We write

$$F(z) = (1 - |z_1|)^{-1/(k+1)} f(z_1 + (1 - |z_1|)z)$$

and proceed as in the proof of Corollary 19.

References

L.V. Ahlfors, Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée, Acta Soc. Sci. Fennicae, Nova Ser. A, 2, No. 2 (1933), 1–17.

W.K. Hayman, Uniformly normal families,
pp. 199–212 in Lectures on Functions of a Complex Variable, University of Michigan Press, edited by Wilfred Kaplan, 1955.

W.K. Hayman and A. Hinkkanen, Generalisations of uniformly normal families, Transcendental dynamics and complex analysis, pp. 179–186, London Math. Soc. Lecture Note Ser., 348, Cambridge Univ. Press, Cambridge, 2008.