

Enhanced approximate cloaking by optimal change of variables

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January 13, 2014

Abstract

The aim of (passive) cloaking with respect to electromagnetic (or acoustic) sensing is to surround a region of space with a material layer – the cloak – that renders its contents and even the existence of the layer undetectable by such measurements. At least theoretically this can be achieved using the coordinate invariance of the underlying wave equation, through so-called cloaking by mapping. However, a practical realization of the cloaking by mapping schemes discussed in the literature frequently requires the design of highly anisotropic materials with extreme dielectric properties. In this work we consider, in the electrostatic case, a regularized, approximate cloaking by mapping scheme and discuss the problem of optimal choice of radial maps, that determine the conductivity distribution of the cloak. We consider two different optimality criteria: minimal maximal anisotropy and minimal mean anisotropy of this conductivity distribution. Using both criteria we show that it is possible to achieve significantly lower anisotropy (for a prescribed level of invisibility) or significantly lower visibility (for a prescribed level of anisotropy). For example, in two dimensions one may achieve exponentially small visibility with a cloak, that in terms of anisotropy (and lowest and highest conductivity) is no worse than the traditional affine map cloak, which only yields quadratically small visibility.

Mathematics subject classifications (MSC2000): 78A10, (35Q60).

Keywords: Cloaking by mapping, approximate cloaking, cloak enhancement, optimal cloaking

Short title: Approximate cloaking by optimal change of variables

1 Introduction

Electromagnetic (or acoustic) cloaking, and in particular the passive approach referred to as cloaking by mapping [14, 15, 22, 33], has recently received a lot of attention, see for instance [5, 11, 12, 19] and the references therein. Cloaking by mapping comes in two varieties, in terms of the invisibility achieved: (i) perfect cloaking, and (ii) approximate cloaking. Very broadly speaking: in the perfect case a point is “mapped” to a set of finite size, for example the unit ball, whereas in the approximate case a “small” set is mapped to a larger set of finite size, for example the ball of radius ε is mapped to the unit ball. Perfect cloaking by mapping will introduce media with degenerate and highly anisotropic properties, whereas the approximate analogues restrict the degeneracy as well as the anisotropy. In the perfect case the objects that are being cloaked are entirely invisible to an observer outside the cloak, whereas in the approximate case this is only asymptotically true.

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A most important problem for any approximate cloaking scheme is to estimate the degree of invisibility that is achieved for a given level of anisotropy and degeneracy. Some estimates of this kind are found in [18, 19, 23, 26, 27, 28, 30, 31] for approximate cloaking schemes in the context of the electrical conductivity problem, the Helmholtz problem, and the time-dependent scalar wave problem. Enhanced versions of these schemes have been analyzed in [1, 2, 3, 4, 16, 24, 25]. Most of the cloaking by mapping schemes that have been discussed so far are based on fairly simple affine, radial mappings, though other radial mappings have been briefly mentioned in several places, see e.g. [6, 7, 8, 32]. There has been some preliminary work on the optimal design of cloaks, see for instance [34, 36], where the authors optimize the cloaking of a fixed, perfectly conducting or constant inclusion with respect to scattering measurements, obtaining a better result with less complexity than for the traditional affine, radial cloaking by mapping scheme.

However, to the best of our knowledge no one has systematically addressed any of the “optimality” questions, for instance: given an acceptable amount of anisotropy, what is the maximal invisibility achievable for an arbitrary object; or its dual formulation: given a desired level of invisibility, what is the minimal amount of anisotropy required. Similar, natural questions pose themselves concerning degeneracy. In this paper we shall provide an analysis of such questions of optimality in a restricted setting. The physical problem we consider is that of electrical prospecting at zero frequency, i.e., the electrical conductivity problem, and we restrict the class of cloaks to those of a radially symmetric nature. Given those limitations we construct approximate cloaks that are based on suitably chosen radial maps, and minimize two different measures of anisotropy, subject to a given degree of invisibility: the maximal anisotropy and the mean anisotropy of the conductivity distribution of the approximate cloak. We furthermore show that these optimal approximate cloaks achieve exponential invisibility, with a level of anisotropy (and degeneracy) that is comparable to those of [19] and [2, 16].

The paper is organized as follows. In Section 2 we briefly introduce the mathematical setting and our notion of a cloak. Then in Section 3 we construct the two optimal radially symmetric cloaks, discussed above. Finally in Section 4 we draw some analogies between our optimal cloaks and those achieved by the cloak enhancement strategy introduced by Ammari et al. [2, 16].

2 Mathematical setting

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded smooth domain and suppose

$$\sigma \in \Sigma(\Omega) := \{ \gamma \in L^\infty(\Omega; \mathbb{R}^{n \times n}) \mid \gamma \text{ symm. and pos. definite a.e. in } \Omega, \text{ess inf } \gamma > 0 \},$$

where

$$\text{ess inf } \gamma := \sup \{ m \in \mathbb{R} \mid m |\xi|^2 \leq \xi^T \sigma(x) \xi \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega \}.$$

Throughout σ represents a *conductivity distribution* in Ω . We denote by ν the unit outward normal to $\partial\Omega$, and we write

$$H_\diamond^{\pm 1/2}(\partial\Omega) := \{ \varphi \in H^{\pm 1/2}(\partial\Omega) \mid \int_{\partial\Omega} \varphi \, ds = 0 \}.$$

The *electrostatic potential* u in Ω generated by some *boundary current* $f \in H_\diamond^{-1/2}(\partial\Omega)$ satisfies

$$\text{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad (\sigma \nabla u) \cdot \nu = f \quad \text{on } \partial\Omega.$$

This boundary value problem has a unique solution

$$u \in H_\diamond^1(\Omega) := \left\{ v \in H^1(\Omega) \mid \int_{\partial\Omega} v \, ds = 0 \right\},$$

and the associated *Neumann-to-Dirichlet operator*

$$\Lambda_\sigma : H_\diamond^{-1/2}(\partial\Omega) \rightarrow H_\diamond^{1/2}(\partial\Omega), \quad f \mapsto u|_{\partial\Omega},$$

mapping boundary currents to the corresponding boundary voltages, is bounded. This operator describes all possible outcomes of an idealized electromagnetic sensing experiment in the zero frequency limit, with observations restricted to $\partial\Omega$.

Following [19] we say that a non-negative matrix-valued conductivity distribution σ_c in $\Omega \setminus \overline{D}$ cloaks a subdomain $D \subset \Omega$ if for any $A \in \Sigma(D)$ the Neumann-to-Dirichlet operator Λ_{σ_A} corresponding to the conductivity distribution

$$\sigma_A(x) := \begin{cases} A(x), & x \in D, \\ \sigma_c(x), & x \in \Omega \setminus \overline{D}, \end{cases} \quad (2.1)$$

is well-defined and coincides with Λ_{I_n} , where I_n is the constant isotropic conductivity distribution given by the $n \times n$ -identity matrix in Ω . In particular this implies that σ_A and I_n cannot be distinguished by observations of electrostatic currents and voltages on $\partial\Omega$, and we call the conductivity distribution σ_c in $\Omega \setminus \overline{D}$ a *cloak* for D .

Denoting by $B_r := \{x \in \mathbb{R}^n \mid |x| < r\}$, $r > 0$, the ball of radius r around the origin, we assume for simplicity throughout this work that $B_2 \subset \Omega$ and $D \subset B_1$. We observe that if $\sigma_c \in \Sigma(B_2 \setminus \overline{B_1})$ cloaks B_1 , then its extension to $\Omega \setminus \overline{D}$ by I_n cloaks D , too, and so we shall in the following restrict the discussion to the special case $\Omega = B_2$ and $D = B_1$.

The cloak construction approach we shall take here, frequently referred to as *cloaking by mapping*, relies on the fact that the Neumann-to-Dirichlet map is invariant under certain changes of variables in Ω , preserving points at the boundary $\partial\Omega$. More precisely, let $F : \overline{\Omega} \rightarrow \overline{\Omega}$ be any Lipschitz homeomorphism¹ satisfying $F(x) = x$ for all $x \in \partial\Omega$, and define the *push-forward* of $\sigma \in \Sigma(\Omega)$ by F

$$F_*\sigma(x) := \frac{1}{|\det DF(y)|} DF(y)\sigma(y)DF(y)^T \Big|_{y=F^{-1}(x)}, \quad x \in \Omega, \quad (2.2)$$

where DF denotes the Jacobian matrix of F , then

$$\Lambda_\sigma = \Lambda_{F_*\sigma}.$$

This construction was originally introduced in [20] with an attribution to L. Tartar.

The change of variables formula (2.2) was used independently in [14, 15] and [33] (see also [22]) to construct a cloak of B_1 ,

$$\sigma_c(x) := (F_0)_*I_n(x), \quad x \in B_2 \setminus \overline{B_1}, \quad (2.3)$$

by means of the singular mapping

$$F_0 : \overline{B_2} \setminus \{0\} \rightarrow \overline{B_2}, \quad F_0(x) := \left(1 + \frac{1}{2}|x|\right) \frac{x}{|x|}, \quad (2.4)$$

¹This means F is bijective and F as well as F^{-1} are Lipschitz continuous.

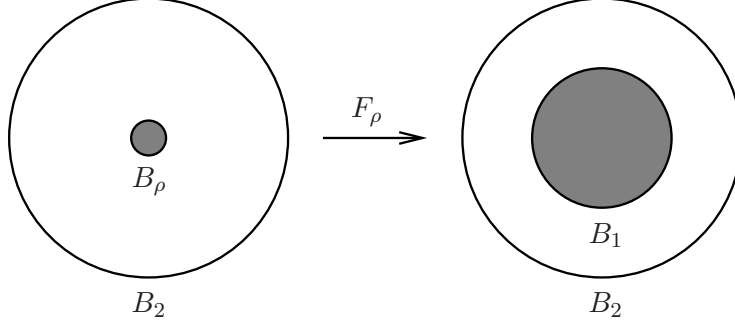


Figure 1: Change of variables F_ρ expanding a small ball B_ρ to the unit ball B_1 .

which “blows up” $\{0\}$ to $\overline{B_1}$, maps $\overline{B_2} \setminus \{0\}$ onto $\overline{B_2} \setminus \overline{B_1}$, and satisfies $F_0(x) = x$ for all $x \in \partial B_2$. Given an arbitrary $A \in \Sigma(B_1)$, the push-forward $(F_0^{-1})_* \sigma_A$ of σ_A from (2.1) coincides with I_n away from the origin. Since, roughly speaking, the difference in the conductivity at a single point should not affect the electrostatic potential, one expects that $\Lambda_{\sigma_A} = \Lambda_{(F_0^{-1})_* \sigma_A} = \Lambda_{I_n}$. Indeed it has been shown in [14, 19] that Λ_{σ_A} , with σ_c from (2.3), is well-defined and cloaks B_1 . However due to the singular behavior of F_0 at the origin this cloak is highly anisotropic and degenerate near ∂B_1 , which in practice would require the design of materials with singular dielectric properties.

To avoid singular material behavior, several regular approximations of the perfect cloak from (2.3) – so-called *approximate* or *near cloaks* – have been discussed in recent years. For instance, such approximations have been obtained in [9, 10, 13, 21, 35] by restricting the perfect cloak from (2.3) to an annulus $B_2 \setminus \overline{B_{1+\rho}}$, $0 < \rho < 1$, thereby truncating the singularity at ∂B_1 . In the limit as $\rho \rightarrow 0$ these approximate cloaks converge to the perfect cloak from (2.3).

On the other hand, the piecewise smooth mapping

$$F_\rho : \overline{B_2} \rightarrow \overline{B_2}, \quad F_\rho(x) := \begin{cases} \frac{x}{\rho}, & x \in B_\rho, \\ \left(\frac{2(1-\rho)}{2-\rho} + \frac{|x|}{2-\rho} \right) \frac{x}{|x|}, & x \in \overline{B_2} \setminus B_\rho, \end{cases} \quad (2.5)$$

$0 < \rho < 1$, which constitutes a regularized version of the mapping in (2.4), and that expands B_ρ to B_1 , maps $\overline{B_2} \setminus B_\rho$ onto $\overline{B_2} \setminus B_1$, and satisfies $F_\rho(x) = x$ for all $x \in \partial B_2$ (see Figure 1 for a sketch), has been used in [19] to construct a regular approximate cloak

$$\sigma_{c,\rho}(x) := (F_\rho)_* I_n(x), \quad x \in B_2 \setminus \overline{B_1}. \quad (2.6)$$

Given $A \in \Sigma(B_1)$ and defining

$$\sigma_{A,\rho}(x) := \begin{cases} A(x), & x \in B_1, \\ \sigma_{c,\rho}(x), & x \in B_2 \setminus \overline{B_1}, \end{cases} \quad (2.7)$$

similar to (2.1), it follows immediately that $(F_\rho^{-1})_* \sigma_{A,\rho}$ coincides with I_n in $B_2 \setminus \overline{B_\rho}$. To estimate the degree of invisibility of this approximate cloak it therefore suffices to estimate the effect on the Neumann-to-Dirichlet map (on ∂B_2) of a small inhomogeneity B_ρ , $0 < \rho < 1$, filled with an arbitrary conductivity distribution $a \in \Sigma(B_\rho)$. Writing $\sigma_{a,\rho}(x) := I_n + \chi_{B_\rho}(x)(a(x) - I_n)$, $x \in B_2$, where χ_{B_ρ} denotes the characteristic function for B_ρ , it has been shown in [29, Cor. 1] that there

exists $\rho_0 > 0$ and a constant $C > 0$ independent of a such that $\|\Lambda_{\sigma_{a,\rho}} - \Lambda_{I_n}\| \leq C\rho^n$ for all $0 < \rho < \rho_0$, where $\|\cdot\|$ denotes the operator norm on the space $\mathcal{L}(H_\diamond^{-1/2}(\partial B_2), H_\diamond^{1/2}(\partial B_2))$ of bounded linear operators from $H_\diamond^{-1/2}(\partial B_2)$ to $H_\diamond^{1/2}(\partial B_2)$. This immediately yields the following *invisibility estimate* originally established in [19, Thm. 1].

Proposition 2.1. *Given $A \in \Sigma(B_1)$ let $\Lambda_{\sigma_{A,\rho}}$ be the Neumann-to-Dirichlet map corresponding to the conductivity distribution $\sigma_{A,\rho}$ from (2.7) with the regular near cloak $\sigma_{c,\rho}$ from (2.6). Then there exists $\rho_0 > 0$ and a constant $C > 0$ independent of A such that*

$$\|\Lambda_{\sigma_{A,\rho}} - \Lambda_{I_n}\| \leq C\rho^n \quad \text{for all } 0 < \rho < \rho_0, \quad (2.8)$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(H_\diamond^{-1/2}(\partial B_2), H_\diamond^{1/2}(\partial B_2))$, and Λ_{I_n} denotes the Neumann-to-Dirichlet map corresponding to the constant conductivity distribution I_n .

Remark 2.1. As has recently been pointed out in [17], the approximate cloak obtained by truncating the singularity in [9, 10, 13, 21, 35] can be rewritten as a near cloak obtained by expanding a small ball using a Lipschitz change of variables similar to (2.5)–(2.7). This can then be used to establish invisibility estimates similar to (2.8) for this class of cloaks, using the technique described above. \diamond

Observing that $I_n - xx^T/|x|^2$ and $xx^T/|x|^2$ are the orthogonal projections onto the tangent space of the sphere S^{n-1} at $x/|x| \in S^{n-1}$ and its orthogonal complement, respectively, a short calculation shows that the approximate cloak from (2.6) is given by

$$\sigma_{c,\rho}(x) = \left(2 - \rho - \frac{2 - 2\rho}{|x|}\right)^{n-2} \left(\alpha_{F_\rho}(|x|) \frac{xx^T}{|x|^2} + \frac{1}{\alpha_{F_\rho}(|x|)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right), \quad x \in B_2 \setminus \overline{B_1}, \quad (2.9)$$

where

$$\alpha_{F_\rho}(r) := 1 - \frac{2 - 2\rho}{(2 - \rho)r}, \quad 1 < r < 2. \quad (2.10)$$

Thus, $\sigma_{c,\rho}(x)$ has eigenvalues

$$\lambda_r^{F_\rho}(x) = \left(2 - \rho - \frac{2 - 2\rho}{|x|}\right)^{n-2} \alpha_{F_\rho}(|x|) \quad \text{and} \quad \lambda_t^{F_\rho}(x) = \left(2 - \rho - \frac{2 - 2\rho}{|x|}\right)^{n-2} \frac{1}{\alpha_{F_\rho}(|x|)}, \quad (2.11)$$

where $\lambda_r^{F_\rho}(x)$ is of multiplicity 1 and corresponds to the eigenvector $x/|x|$ in the radial direction, while $\lambda_t^{F_\rho}(x)$ is of multiplicity $n - 1$ and corresponds to the eigenspace spanned by the columns of $I_n - xx^T/|x|^2$, tangential to S^{n-1} . These eigenvalues carry all information about the material properties that have to be implemented in practice, when actually building the near cloak device characterized by $\sigma_{c,\rho}$. Following [16], we next introduce some degeneracy measures for these material properties.

Definition 2.2. Let $\sigma \in \Sigma(B_2 \setminus \overline{B_1})$ represent a conductivity distribution in $B_2 \setminus \overline{B_1}$. Writing

$$\lambda_{\min}(x) := \min\{\lambda(x) \mid \lambda(x) \text{ eigenvalue of } \sigma(x)\}, \quad x \in B_2 \setminus \overline{B_1},$$

and

$$\lambda_{\max}(x) := \max\{\lambda(x) \mid \lambda(x) \text{ eigenvalue of } \sigma(x)\}, \quad x \in B_2 \setminus \overline{B_1},$$

the *maximal anisotropy* of σ is defined by

$$\chi_{\max} := \operatorname{ess\,sup}_{x \in B_2 \setminus \overline{B_1}} \frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}, \quad (2.12)$$

and the *mean anisotropy* of σ is defined by

$$\chi_{\text{mean}} := \frac{1}{|B_2 \setminus \overline{B_1}|} \int_{B_2 \setminus \overline{B_1}} \frac{\lambda_{\max}(x)}{\lambda_{\min}(x)} \, dx, \quad (2.13)$$

where $|B_2 \setminus \overline{B_1}|$ denotes the volume of $B_2 \setminus \overline{B_1}$. Furthermore the *minimal* and *maximal directional conductivity* are defined by

$$\Lambda_{\min} := \operatorname{ess\,inf}_{x \in B_2 \setminus \overline{B_1}} \lambda_{\min}(x) \quad \text{and} \quad \Lambda_{\max} := \operatorname{ess\,sup}_{x \in B_2 \setminus \overline{B_1}} \lambda_{\max}(x).$$

The regular near cloak $\sigma_{c,\rho}$ from (2.6) satisfies

$$\chi_{\max} = \frac{(2-\rho)^2}{\rho^2}, \quad \chi_{\text{mean}} = \frac{n}{2^n - 1} \left(\frac{2}{\rho} - (n+1) \log(\rho) \right) + O(1), \quad (2.14a)$$

and

$$\Lambda_{\min} = \frac{\rho^{n-1}}{2-\rho}, \quad \Lambda_{\max} = \begin{cases} (2-\rho)/\rho, & n=2, \\ 2-\rho, & n \geq 3. \end{cases} \quad (2.14b)$$

In this work we discuss some simple modifications of (2.5)–(2.7) that yield significantly enhanced invisibility estimates when compared to (2.8) without worsening the maximal anisotropy and the minimal and maximal directional conductivity too much, or even improving some of these measures, when compared to (2.14). Of course replacing ρ by ρ^L , $L \geq 1$, or even by $e^{-1/\rho}$ in (2.5)–(2.7) would immediately give a better invisibility estimate of order ρ^{nL} or $e^{-n/\rho}$. However, in this case χ_{\max} , χ_{mean} , Λ_{\min} and Λ_{\max} would worsen tremendously.

In [2] (see also [1, 3, 4]) enhanced approximate cloaks have been constructed by adding a finite number of suitably selected isotropic layers of constant conductivity to the near cloak from (2.5)–(2.7), leading to an invisibility estimate similar to (2.8) but of order ρ^{2L+2} , where $L \geq 1$ denotes the number of additional layers. In order to determine the conductivities of these layers, a system of algebraic equations has to be solved, and although numerical evidence suggests that it should be possible to achieve arbitrary powers of ρ by adding sufficiently many layers (see [2, 16]), the solvability of the corresponding systems of algebraic equations has so far only been established for $1 \leq L \leq 4$. In [2] it has been assumed that either the conductivity distribution inside the cloaked region B_1 is constant and known, or that the boundary ∂B_1 carries a perfectly conducting or perfectly insulating boundary condition. In [16] this enhanced near cloaking scheme has been extended to obtain approximate cloaks for arbitrary, unknown spatially varying conductivity distributions inside B_1 , and a detailed analysis of the enhanced approximate cloaks has been provided. It has also been pointed out that the layer construction neither significantly worsens the maximal anisotropy nor the maximal directional conductivity of the enhanced approximate cloak when compared to the approximate cloak from (2.6). However the minimal directional conductivity of the enhanced approximate cloak is much smaller than the one from (2.14).

3 Approximate cloaking by optimal radial coordinate stretching

In this section we discuss a new family of approximate cloaks that are obtained by a regular change of variables in the radial direction, such that these cloaks satisfy a given invisibility estimate, but at the same time are optimal, in the sense that they have either smallest possible maximal anisotropy or smallest possible mean anisotropy.

Let $0 < \varepsilon < 1$ and consider a C^1 diffeomorphism $\psi_\varepsilon : [1, 2] \rightarrow [\varepsilon, 2]$ satisfying $\psi_\varepsilon(1) = \varepsilon$, $\psi_\varepsilon(2) = 2$, and $\psi'_\varepsilon(r) > 0$ for all $r \in (1, 2)$. We define the mapping

$$H_\varepsilon : \overline{B_2} \rightarrow \overline{B_2}, \quad H_\varepsilon(x) := \begin{cases} \frac{x}{\varepsilon}, & x \in B_\varepsilon, \\ \psi_\varepsilon^{-1}(|x|) \frac{x}{|x|}, & x \in \overline{B_2} \setminus B_\varepsilon. \end{cases} \quad (3.1)$$

This is a Lipschitz homeomorphism that expands B_ε to B_1 , maps $\overline{B_2} \setminus B_\varepsilon$ onto $\overline{B_2} \setminus B_1$, and satisfies $H_\varepsilon(x) = x$ for all $x \in \partial B_2$. A short calculation shows, that the push-forward by H_ε of the constant conductivity distribution I_n in $B_2 \setminus \overline{B_\varepsilon}$, is given by

$$(H_\varepsilon)_* I_n(x) = \left(\frac{\psi_\varepsilon(|x|)}{|x|} \right)^{n-2} \left(\alpha_{H_\varepsilon}(|x|) \frac{xx^T}{|x|^2} + \frac{1}{\alpha_{H_\varepsilon}(|x|)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right), \quad x \in B_2 \setminus \overline{B_1}, \quad (3.2)$$

where

$$\alpha_{H_\varepsilon}(r) := \frac{\psi_\varepsilon(r)}{r\psi'_\varepsilon(r)}, \quad 1 < r < 2. \quad (3.3)$$

Accordingly, $(H_\varepsilon)_* I_n(x)$, $x \in B_2 \setminus \overline{B_1}$, has eigenvalues

$$\lambda_r(x) = \left(\frac{\psi_\varepsilon(|x|)}{|x|} \right)^{n-2} \alpha_{H_\varepsilon}(|x|) \quad \text{and} \quad \lambda_t(x) = \left(\frac{\psi_\varepsilon(|x|)}{|x|} \right)^{n-2} \frac{1}{\alpha_{H_\varepsilon}(|x|)}, \quad (3.4)$$

the latter with multiplicity $n - 1$. We notice that

$$\frac{\partial}{\partial r} \left(\frac{\psi_\varepsilon(r)}{r} \right) = \frac{r\psi'_\varepsilon(r) - \psi_\varepsilon(r)}{r^2} = \left(\frac{1}{\alpha_{H_\varepsilon}(r)} - 1 \right) \frac{\psi_\varepsilon(r)}{r^2}, \quad 1 < r < 2,$$

and so, $\psi_\varepsilon(r)/r$ is monotonically increasing in r if $0 < \alpha_{H_\varepsilon}(r) < 1$.

Remark 3.1. The transformation F_ρ from (2.5) can also be written in the form (3.1) choosing $\varepsilon = \rho$ and

$$\psi_\varepsilon(r) = \psi_\rho(r) := (2 - \rho)r - 2(1 - \rho), \quad 1 \leq r \leq 2. \quad (3.5)$$

In this sense (2.9)–(2.11) is a special case of (3.2)–(3.4). \diamond

If we define

$$\sigma_{c,\varepsilon}(x) := (H_\varepsilon)_* I_n(x), \quad x \in B_2 \setminus \overline{B_1}, \quad (3.6)$$

and

$$\sigma_{A,\varepsilon}(x) := \begin{cases} A(x), & x \in B_1, \\ \sigma_{c,\varepsilon}(x), & x \in B_2 \setminus \overline{B_1}, \end{cases} \quad (3.7)$$

for an arbitrary $A \in \Sigma(B_1)$, then we find that $(H_\varepsilon^{-1})_* \sigma_{A,\varepsilon}$ coincides with I_n in $B_2 \setminus \overline{B_\varepsilon}$. Thus, using [29, Cor. 1] as before, we obtain the following analogue of Proposition 2.1.

²Note that if we define $\psi_\varepsilon(r) = \varepsilon r$ for $0 \leq r < 1$, then the formulas (3.2)–(3.3) hold in B_1 as well.

Proposition 3.1. *Given $A \in \Sigma(B_1)$ let $\Lambda_{\sigma_{A,\varepsilon}}$ be the Neumann-to-Dirichlet map corresponding to the conductivity distribution $\sigma_{A,\varepsilon}$ from (3.7) with the regular near cloak $\sigma_{c,\varepsilon}$ from (3.6). There exists $\varepsilon_0 > 0$ and a constant $C > 0$ independent of A such that*

$$\|\Lambda_{\sigma_{A,\varepsilon}} - \Lambda_{I_n}\| \leq C\varepsilon^n \quad \text{for all } 0 < \varepsilon < \varepsilon_0,$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(H_\diamond^{-1/2}(\partial B_2), H_\diamond^{1/2}(\partial B_2))$, and Λ_{I_n} denotes the Neumann-to-Dirichlet map corresponding to the constant conductivity distribution I_n .

3.1 Minimizing the maximal anisotropy χ_{\max} of the approximate cloak

A change of variables as in (3.1) with smallest possible maximal anisotropy χ_{\max} from (2.12) is characterized by a mapping $\psi_\varepsilon : [1, 2] \rightarrow [\varepsilon, 2]$ solving the constrained optimization problem

$$\text{minimize} \quad \sup_{r \in (1,2)} \left\{ (\alpha_{H_\varepsilon}(r))^2, \frac{1}{(\alpha_{H_\varepsilon}(r))^2} \right\} \quad \text{subject to} \quad \psi_\varepsilon(1) = \varepsilon \quad \text{and} \quad \psi_\varepsilon(2) = 2 \quad (3.8)$$

with α_{H_ε} from (3.3). On the other hand a diffeomorphism $\psi_\varepsilon \in C^1([1, 2])$ solves (3.8) if and only if

$$\psi_\varepsilon(r) = \varepsilon e^{\int_1^r (\alpha_{H_\varepsilon}(t))^{-1} dt}, \quad 1 \leq r \leq 2,$$

where α_{H_ε} solves

$$\begin{aligned} \text{minimize} \quad & \sup_{r \in (1,2)} \left\{ \alpha_{H_\varepsilon}(r), \frac{1}{\alpha_{H_\varepsilon}(r)} \right\} \\ \text{subject to} \quad & \alpha_{H_\varepsilon}(r) > 0 \quad \text{for all } r \in (1,2), \quad \text{and} \quad \int_1^2 \frac{1}{r\alpha_{H_\varepsilon}(r)} dr = \log\left(\frac{2}{\varepsilon}\right). \end{aligned} \quad (3.9)$$

Given any admissible α_{H_ε} satisfying the constraint from (3.9) and introducing

$$a_{\min} := \inf_{r \in (1,2)} \alpha_{H_\varepsilon}(r) \leq \alpha_{H_\varepsilon} \leq \sup_{r \in (1,2)} \alpha_{H_\varepsilon}(r) =: a_{\max} \quad \text{in } (1,2),$$

we find that

$$\int_1^2 \frac{1}{ra_{\max}} dr \leq \int_1^2 \frac{1}{r\alpha_{H_\varepsilon}(r)} dr \leq \int_1^2 \frac{1}{ra_{\min}} dr.$$

Thus there exists $a_* \in [a_{\min}, a_{\max}]$ such that

$$\int_1^2 \frac{1}{a_* r} dr = \log\left(\frac{2}{\varepsilon}\right),$$

namely

$$a_* = \frac{\log(2)}{\log(2) - \log(\varepsilon)} < 1. \quad (3.10)$$

Since

$$\sup_{r \in (1,2)} \left\{ \alpha_{H_\varepsilon}(r), \frac{1}{\alpha_{H_\varepsilon}(r)} \right\} = \max \left\{ a_{\max}, \frac{1}{a_{\min}} \right\} \geq \max \left\{ a_*, \frac{1}{a_*} \right\},$$

where the last inequality is strict unless α_{H_ε} is constant, we get that the constant function a_* in (1, 2) is the unique solution of the constrained optimization problem (3.9).

The corresponding optimal radial coordinate transform ψ_ε^* , $0 < \varepsilon < 1$, satisfies

$$\frac{(\psi_\varepsilon^*)'(r)}{\psi_\varepsilon^*(r)} = \frac{1}{ra_*}, \quad r \in (1, 2), \quad \psi_\varepsilon^*(1) = \varepsilon, \quad \psi_\varepsilon^*(2) = 2,$$

and so

$$\psi_\varepsilon^*(r) = \varepsilon e^{\frac{1}{a_*} \int_1^r \frac{1}{t} dt} = \varepsilon r^{1/a_*} = \varepsilon r^{\frac{\log(2) - \log(\varepsilon)}{\log(2)}}. \quad (3.11)$$

We therefore also obtain

$$\chi_{\max} = \chi_{\text{mean}} = \frac{1}{a_*^2} = \left(\frac{\log(2) - \log(\varepsilon)}{\log(2)} \right)^2,$$

and

$$\Lambda_{\min} = \varepsilon^{n-2} \frac{\log(2)}{\log(2) - \log(\varepsilon)}, \quad \Lambda_{\max} = \frac{\log(2) - \log(\varepsilon)}{\log(2)}.$$

Example 3.2. Let $\varepsilon = \rho^L$, $L > 0$. Then $\log(\varepsilon) = L \log(\rho)$ and accordingly

$$a_* = \frac{\log(2)}{\log(2) - L \log(\rho)}, \quad \psi_\varepsilon^*(|x|) = \rho^L |x|^{\frac{\log(2) - L \log(\rho)}{\log(2)}},$$

and

$$(H_\varepsilon)_* I_n(x) = \left(\rho^L |x|^{\frac{L \log(\rho)}{\log(2)}} \right)^{n-2} \left(\frac{\log(2)}{\log(2) - L \log(\rho)} \frac{xx^T}{|x|^2} + \frac{\log(2) - L \log(\rho)}{\log(2)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right)$$

for all $x \in B_2 \setminus \overline{B_1}$. Therefore, for $0 < \rho < 1$:

$$\chi_{\max} = \chi_{\text{mean}} = \left(\frac{\log(2) - L \log(\rho)}{\log(2)} \right)^2$$

and

$$\Lambda_{\min} = \rho^{(n-2)L} \frac{\log(2)}{\log(2) - L \log(\rho)}, \quad \Lambda_{\max} = \frac{\log(2) - L \log(\rho)}{\log(2)}.$$

In particular, the maximal anisotropy, the mean anisotropy, and the maximal directional conductivity are less extreme, or very comparable to those in (2.14), but the invisibility estimate from Proposition 3.1 is of order ρ^{nL} . \diamond

Example 3.3. Let $\varepsilon = e^{-1/\rho}$, which corresponds to setting $L = -1/(\rho \log(\rho))$ in Example 3.2. Then $\log(\varepsilon) = -1/\rho$ and accordingly

$$a_* = \frac{\log(2)}{\log(2) + 1/\rho}, \quad \psi_\varepsilon^*(|x|) = e^{-1/\rho} |x|^{\frac{\log(2) + 1/\rho}{\log(2)}},$$

and

$$(H_\varepsilon)_* I_n(x) = \left(e^{-1/\rho} |x|^{\frac{1}{\rho \log(2)}} \right)^{n-2} \left(\frac{\log(2)}{\log(2) + 1/\rho} \frac{xx^T}{|x|^2} + \frac{\log(2) + 1/\rho}{\log(2)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right)$$

for all $x \in B_2 \setminus \overline{B_1}$. Therefore, for $0 < \rho < 1$:

$$\chi_{\max} = \chi_{\text{mean}} = \frac{1}{a_*^2} = \left(\frac{\log(2) + 1/\rho}{\log(2)} \right)^2,$$

and

$$\Lambda_{\min} = e^{-\frac{n-2}{\rho}} \frac{\log(2)}{\log(2) + 1/\rho}, \quad \Lambda_{\max} = \frac{\log(2) + 1/\rho}{\log(2)}.$$

In particular, the maximal anisotropy is comparable to that of (2.14). The maximal directional conductivity is comparable to that of (2.14), for $n = 2$, and only slightly worse for $n \geq 3$, but the invisibility estimate from Proposition 3.1 is order $e^{-\frac{n}{\rho}}$! \diamond

3.2 Minimizing the mean anisotropy χ_{mean} of the approximate cloak

Instead of minimizing the maximal anisotropy χ_{\max} of the approximate cloak we may also minimize its mean anisotropy χ_{mean} from (2.13). This means we are looking for a C^1 diffeomorphism $\psi_\varepsilon : [1, 2] \rightarrow [\varepsilon, 2]$ that solves the constrained optimization problem

$$\text{minimize } \int_1^2 \max \left\{ (\alpha_{H_\varepsilon}(r))^2, \frac{1}{(\alpha_{H_\varepsilon}(r))^2} \right\} r^{n-1} dr \quad \text{subject to } \psi_\varepsilon(1) = \varepsilon \text{ and } \psi_\varepsilon(2) = 2 \quad (3.12)$$

with α_{H_ε} from (3.3). In this connection the following simple lemma will prove helpful.

Lemma 3.4. *The optimization problem*

$$\text{minimize } \int_1^2 \frac{1}{(\alpha_{H_\varepsilon}(r))^2} r^{n-1} dr \quad \text{subject to } \psi_\varepsilon(1) = \varepsilon, \psi_\varepsilon(2) = 2, \text{ and } \varepsilon \leq \psi_\varepsilon \leq 2, \quad (3.13)$$

posed in $H^1((1, 2))$ has the unique minimizer

$$\psi_\varepsilon^*(r) = e^{-\frac{2^n \log(2/\varepsilon)}{r^n(2^n-1)} + \log(2) + \frac{\log(2/\varepsilon)}{2^n-1}},$$

a smooth diffeomorphism from $[1, 2]$ to $[\varepsilon, 2]$, with the associated $\alpha_{H_\varepsilon}^*(r) = (2^n - 1)r^n / (n2^n \log(2/\varepsilon))$.

Proof. Introducing $\beta_\varepsilon = \log(\psi_\varepsilon)$ the minimization problem (3.13) is equivalent to

$$\text{minimize } \int_1^2 (\beta_\varepsilon'(r))^2 r^{n+1} dr \quad \text{s.t. } \beta_\varepsilon(1) = \log(\varepsilon), \beta_\varepsilon(2) = \log(2), \text{ and } \log(\varepsilon) \leq \beta_\varepsilon \leq \log(2), \quad (3.14)$$

also posed in $H^1((1, 2))$. If we drop the constraint $\log(\varepsilon) \leq \beta_\varepsilon \leq \log(2)$ from the latter optimization problem then we get a simple variational problem whose unique minimizer $\beta_\varepsilon^* \in H^1((1, 2))$ is a weak solution of

$$(r^{n+1}(\beta_\varepsilon^*)'(r))' = 0, \quad r \in (1, 2), \quad \beta_\varepsilon^*(1) = \log(\varepsilon), \beta_\varepsilon^*(2) = \log(2).$$

A simple calculation gives

$$\beta_\varepsilon^*(r) = -\frac{2^n \log(2/\varepsilon)}{2^n - 1} r^{-n} + \log(2) + \frac{\log(2/\varepsilon)}{2^n - 1}, \quad 1 < r < 2.$$

Since the coefficient in front of r^{-n} is negative β_ε^* automatically satisfies $\log(\varepsilon) \leq \beta_\varepsilon^* \leq \log(2)$ and so β_ε^* is indeed the unique solution to the optimization problem (3.14). One easily calculates

$$\psi_\varepsilon^*(r) = e^{-\frac{2^n \log(2/\varepsilon)}{r^n(2^n-1)} + \log(2) + \frac{\log(2/\varepsilon)}{2^n-1}}, \quad 1 \leq r \leq 2, \quad (3.15)$$

and

$$\alpha_{H_\varepsilon}^*(r) = \frac{1}{r(\beta_\varepsilon^*)'(r)} = \frac{(2^n-1)r^n}{n2^n \log(2/\varepsilon)}, \quad 1 < r < 2, \quad (3.16)$$

as desired. \square

We note that for $\varepsilon < 2e^{(1-2^n)/n} < 1/2$ we have that $0 < \alpha_{H_\varepsilon}^*(r) < 1$, and so

$$\int_1^2 \max \left\{ (\alpha_{H_\varepsilon}^*(r))^2, \frac{1}{(\alpha_{H_\varepsilon}^*(r))^2} \right\} r^{n-1} dr = \int_1^2 \frac{1}{(\alpha_{H_\varepsilon}^*(r))^2} r^{n-1} dr.$$

Since

$$\int_1^2 \frac{1}{(\alpha_{H_\varepsilon}^*(r))^2} r^{n-1} dr \leq \int_1^2 \frac{1}{(\alpha_{H_\varepsilon}(r))^2} r^{n-1} dr \leq \int_1^2 \max \left\{ (\alpha_{H_\varepsilon}(r))^2, \frac{1}{(\alpha_{H_\varepsilon}(r))^2} \right\} r^{n-1} dr,$$

for any other C^1 diffeomorphism ψ_ε , it follows immediately that ψ_ε^* is also the unique minimizer for (3.12), provided that $\varepsilon < 2e^{(1-2^n)/n}$. We now calculate that for this transformation which minimizes the mean anisotropy

$$\chi_{\max} = \frac{1}{(\alpha_{H_\varepsilon}^*(1))^2} = \left(\frac{n2^n \log(2/\varepsilon)}{2^n - 1} \right)^2, \quad \chi_{\text{mean}} = \frac{n^2 2^n (\log(2/\varepsilon))^2}{(2^n - 1)^2},$$

and

$$\Lambda_{\min} = \varepsilon^{n-2} \frac{2^n - 1}{n2^n \log(2/\varepsilon)}, \quad \Lambda_{\max} \leq \frac{n2^n \log(2/\varepsilon)}{2^n - 1}.$$

Example 3.5. Let $\varepsilon = \rho^L$, $L > 0$. Then $\log(\varepsilon) = L \log(\rho)$ and accordingly

$$\alpha_{H_\varepsilon}^*(|x|) = \frac{(2^n - 1)|x|^n}{n2^n (\log(2) - L \log(\rho))}, \quad \psi_\varepsilon^*(|x|) = e^{-\frac{2^n (\log(2) - L \log(\rho))}{|x|^n (2^n - 1)} + \log(2) + \frac{\log(2) - L \log(\rho)}{2^n - 1}},$$

and

$$(H_\varepsilon)_* I_n(x) = \left(\frac{\psi_\varepsilon^*(|x|)}{|x|} \right)^{n-2} \left(\frac{(2^n - 1)|x|^n}{n2^n (\log(2) - L \log(\rho))} \frac{xx^T}{|x|^2} + \frac{n2^n (\log(2) - L \log(\rho))}{(2^n - 1)|x|^n} \left(I_n - \frac{xx^T}{|x|^2} \right) \right)$$

for all $x \in B_2 \setminus \overline{B_1}$. Furthermore,

$$\chi_{\max} = \left(\frac{n2^n (L \log(\rho) - \log(2))}{2^n - 1} \right)^2, \quad \chi_{\text{mean}} = \frac{n^2 2^n (\log(2) - L \log(\rho))^2}{(2^n - 1)^2},$$

which is better than (2.14), and

$$\Lambda_{\min} = \rho^{(n-2)L} \frac{2^n - 1}{n2^n (\log(2) - L \log(\rho))}, \quad \Lambda_{\max} \leq \frac{n2^n (\log(2) - L \log(\rho))}{2^n - 1},$$

which is also better than (2.14) for $n = 2$, and not much worse (at least for Λ_{\max}) for $n \geq 3$. But the invisibility estimate from Proposition 3.1 is of the order ρ^{nL} . \diamond

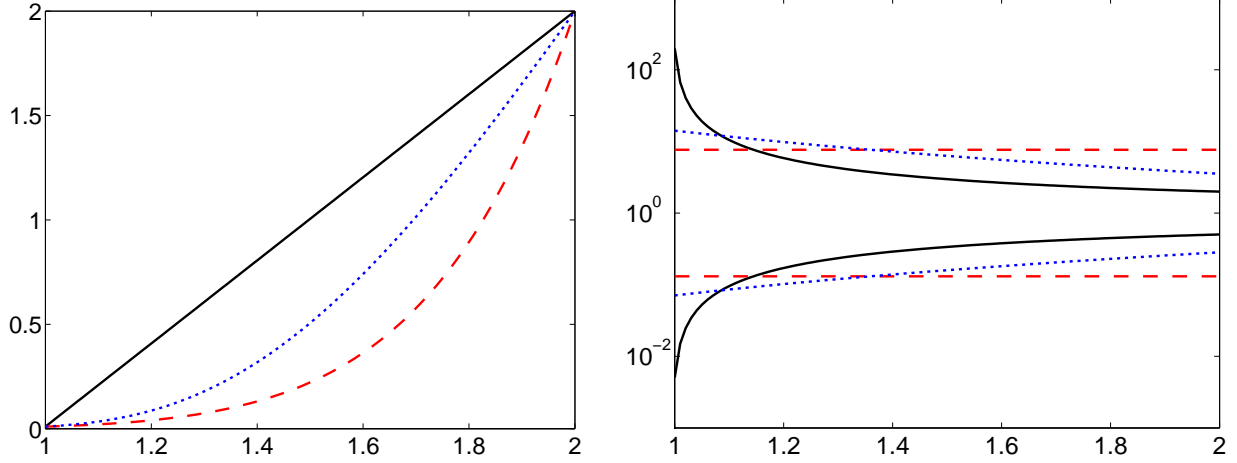


Figure 2: Left: Traditional affine radial stretching function from (3.5) (solid line) and optimal radial stretching functions from (3.11) (dashed line) and (3.15) (dotted line) for $\rho = \varepsilon = 0.01$ and $n = 2$. Right: Eigenvalues of the corresponding near cloaks.

Example 3.6. Let $\varepsilon = e^{-1/\rho}$, which corresponds to setting $L = -1/(\rho \log(\rho))$ in Example 3.5. Then $\log(\varepsilon) = -1/\rho$ and accordingly

$$\alpha_{H_\varepsilon}^*(|x|) = \frac{(2^n - 1)|x|^n}{n2^n(\log(2) + 1/\rho)}, \quad \psi_\varepsilon^*(|x|) = e^{-\frac{2^n(\log(2)+1/\rho)}{|x|^n(2^n-1)} + \log(2) + \frac{\log(2)+1/\rho}{2^n-1}},$$

and

$$(H_\varepsilon)_* I_n(x) = \left(\frac{\psi_\varepsilon^*(|x|)}{|x|} \right)^{n-2} \left(\frac{(2^n - 1)|x|^n}{n2^n(\log(2) + 1/\rho)} \frac{xx^T}{|x|^2} + \frac{n2^n(\log(2) + 1/\rho)}{(2^n - 1)|x|^n} \left(I_n - \frac{xx^T}{|x|^2} \right) \right)$$

for all $x \in B_2 \setminus \overline{B_1}$. Therefore,

$$\chi_{\max} = \left(\frac{n2^n(\log(2) + 1/\rho)}{2^n - 1} \right)^2, \quad \chi_{\text{mean}} = \frac{n^2 2^n (\log(2) + 1/\rho)^2}{(2^n - 1)^2},$$

which is comparable to (2.14) (at least for χ_{\max}), and

$$\Lambda_{\min} = e^{-(n-2)/\rho} \frac{2^n - 1}{n2^n(\log(2) + 1/\rho)}, \quad \Lambda_{\max} \leq \frac{n2^n(\log(2) + 1/\rho)}{2^n - 1},$$

which is comparable to (2.14) for $n = 2$, and only slightly worse (at least for Λ_{\max}) for $n \geq 3$. But the invisibility estimate from Proposition 3.1 is of the order $e^{-\frac{n}{\rho}}$! \diamond

To further illustrate the results of this section we include in Figure 2 (left) a plot of the traditional affine radial stretching function ψ_ε from (3.5) (solid line) together with plots of the optimal transformations ψ_ε^* from (3.11) (dashed line) and (3.15) (dotted line) for $\rho = \varepsilon = 0.01$ and $n = 2$. The eigenvalues α_{F_ρ} and $1/\alpha_{F_\rho}$ of the traditional near cloak from (2.10) (solid lines), the eigenvalues a_* and $1/a_*$ of the near cloak with minimal maximal anisotropy from (3.10) (dashed lines), as well as the eigenvalues $\alpha_{H_\varepsilon}^*$ and $1/\alpha_{H_\varepsilon}^*$ of the near cloak with minimal mean anisotropy from (3.16) (dotted lines) are shown in Figure 2 (right).

Note that the χ_{\max} , χ_{mean} , Λ_{\min} and Λ_{\max} obtained in Sections 3.1 and 3.2 are equivalent (up to constants) as functions of ε (or ρ), in spite of the fact that the optimality criteria are different. Note that unlike the affine ψ_ε defined in (3.5), the optimal ψ_ε^* in Sections 3.1 and 3.2 do not have regular function limits as $\varepsilon \rightarrow 0$; in other words there is no corresponding “limiting perfect cloak”. Indeed the optimal ψ_ε^* from Sections 3.1 and 3.2 converge to 0 for any $1 \leq r < 2$. The eigenvalues of the cloak converge to 0 and ∞ (for $n = 2$) and 0 (for $n \geq 3$) everywhere in $1 \leq r < 2$, whereas the local anisotropy measure $1/(\alpha_{H_\varepsilon}^*)^2$ converges to ∞ everywhere in $1 \leq r \leq 2$.

4 Near cloak enhancing layers

Inspired by the layering approach from [2, 16], we discuss in this section a simple strategy to improve the near cloak from (2.6) by adding a cloak enhancing layer. To this end we compose the regular change of variables F_ρ on B_2 , which is used in the push-forward construction to obtain the near cloak in (2.6), with another suitably defined regular change of variables $G_{\rho,\varepsilon}$ on B_2 , that significantly enhances the cloaking effect of this near cloak.

Let $0 < \rho < 1$ and $0 < \varepsilon < \rho/2$, and consider a Lipschitz homeomorphism $\phi_{\rho,\varepsilon} : [\rho/2, \rho] \rightarrow [\varepsilon, \rho]$ satisfying $\phi_{\rho,\varepsilon}(\rho/2) = \varepsilon$ and $\phi_{\rho,\varepsilon}(\rho) = \rho$. Similar to (3.1) we define the mapping

$$G_{\rho,\varepsilon} : \overline{B_2} \rightarrow \overline{B_2}, \quad G_{\rho,\varepsilon}(x) := \begin{cases} \frac{\rho x}{2\varepsilon}, & x \in B_\varepsilon, \\ \phi_{\rho,\varepsilon}^{-1}(|x|) \frac{x}{|x|}, & x \in B_\rho \setminus B_\varepsilon, \\ x, & x \in \overline{B_2} \setminus B_\rho, \end{cases}$$

that expands B_ε to $B_{\rho/2}$, maps $\overline{B_2} \setminus B_\varepsilon$ onto $\overline{B_2} \setminus B_{\rho/2}$, and satisfies $G_{\rho,\varepsilon}(x) = x$ for all $x \in \overline{B_2} \setminus B_\rho$. If we extend $\phi_{\rho,\varepsilon}$ by $\phi_{\rho,\varepsilon}(r) = r$, for $\rho < r \leq 2$, then $G_{\rho,\varepsilon}$ may be written as $G_{\rho,\varepsilon}(x) = \phi_{\rho,\varepsilon}^{-1}(|x|)x/|x|$ for all $x \in \overline{B_2} \setminus B_\varepsilon$. Accordingly, the push-forward by $G_{\rho,\varepsilon}$ of the constant conductivity distribution I_n in $B_2 \setminus \overline{B_\varepsilon}$ is given by

$$(G_{\rho,\varepsilon})_* I_n(x) = \left(\frac{\phi_{\rho,\varepsilon}(|x|)}{|x|} \right)^{n-2} \left(\alpha_{G_{\rho,\varepsilon}}(|x|) \frac{xx^T}{|x|^2} + \frac{1}{\alpha_{G_{\rho,\varepsilon}}(|x|)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right), \quad x \in B_2 \setminus \overline{B_{\rho/2}},$$

where, analogously to (3.3),

$$\alpha_{G_{\rho,\varepsilon}}(r) := \frac{\phi_{\rho,\varepsilon}(r)}{r\phi'_{\rho,\varepsilon}(r)}, \quad \rho/2 < r < 2. \quad (4.1)$$

Composing $G_{\rho,\varepsilon}$ with F_ρ from (2.5) yields another regular Lipschitz change of variables

$$H_{\rho,\varepsilon} := F_\rho \circ G_{\rho,\varepsilon}$$

on $\overline{B_2}$ that expands B_ε to $B_{1/2}$, maps $\overline{B_2} \setminus B_\varepsilon$ onto $\overline{B_2} \setminus B_{1/2}$, and satisfies $H_{\rho,\varepsilon}(x) = x$ for all $x \in \partial B_2$ (see Figure 3 for a sketch). Defining

$$\sigma_{c,\rho,\varepsilon}(x) := (H_{\rho,\varepsilon})_* I_n(x), \quad x \in B_2 \setminus \overline{B_{1/2}}, \quad (4.2)$$

and correspondingly

$$\sigma_{A,\rho,\varepsilon}(x) = \begin{cases} A(x), & x \in B_{1/2}, \\ \sigma_{c,\rho,\varepsilon}(x), & x \in B_2 \setminus \overline{B_{1/2}}, \end{cases} \quad (4.3)$$

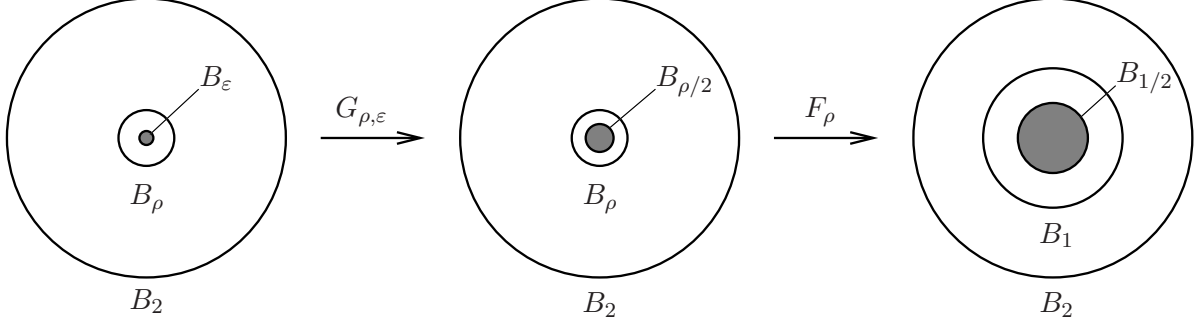


Figure 3: Change of variables $H_{\rho,\varepsilon} = F_\rho \circ G_{\rho,\varepsilon}$ expanding a very small ball B_ε to the ball $B_{1/2}$.

for an arbitrary $A \in \Sigma(B_{1/2})$, we obtain that $(H_{\rho,\varepsilon}^{-1})_*\sigma_{A,\rho,\varepsilon}$ coincides with I_n in $B_2 \setminus \overline{B_\varepsilon}$, and therefore this near cloak satisfies the same invisibility estimate as in Proposition 3.1.

Proposition 4.1. *Given $A \in \Sigma(B_{1/2})$ let $\Lambda_{\sigma_{A,\rho,\varepsilon}}$ be the Neumann-to-Dirichlet map corresponding to the conductivity distribution $\sigma_{A,\rho,\varepsilon}$ from (4.3) with the enhanced regular near cloak $\sigma_{c,\rho,\varepsilon}$ from (4.2). Then there exists $\varepsilon_0 > 0$ and a constant $C > 0$ independent of A such that*

$$\|\Lambda_{\sigma_{A,\rho,\varepsilon}} - \Lambda_{I_n}\| \leq C\varepsilon^n \quad \text{for all } 0 < \varepsilon < \varepsilon_0,$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(H_\diamond^{-1/2}(\partial B_2), H_\diamond^{1/2}(\partial B_2))$, and Λ_{I_n} denotes the Neumann-to-Dirichlet map corresponding to the constant conductivity distribution I_n .

The push-forward of the constant conductivity distribution I_n in $B_2 \setminus \overline{B_\varepsilon}$ by $H_{\rho,\varepsilon}$ is given by

$$\begin{aligned} (H_{\rho,\varepsilon})_*I_n(x) &= (F_\rho)_*(G_{\rho,\varepsilon})_*I_n(x) \\ &= \left(\frac{\phi_{\rho,\varepsilon}(\psi_\rho(|x|))}{|x|} \right)^{n-2} \left(\alpha_{F_\rho}(|x|)\alpha_{G_{\rho,\varepsilon}}(\psi_\rho(|x|)) \frac{xx^T}{|x|^2} \right. \\ &\quad \left. + \frac{1}{\alpha_{F_\rho}(|x|)\alpha_{G_{\rho,\varepsilon}}(\psi_\rho(|x|))} \left(I_n - \frac{xx^T}{|x|^2} \right) \right) \end{aligned} \quad (4.4)$$

for $x \in B_2 \setminus \overline{B_{1/2}}$, with ψ_ρ from (3.5) and α_{F_ρ} from (2.10) (extended by $\psi_\rho(r) = \rho r$ and $\alpha_{F_\rho}(r) = 1$ for $1/2 < r < 1$).

The fact that $\alpha_{F_\rho}(r) = 1$ for all $1/2 < r < 1$ implies that to control the maximal anisotropy or the mean anisotropy of the enhanced near cloak $\sigma_{c,\rho,\varepsilon}$ in $B_1 \setminus \overline{B_{1/2}}$, it suffices to control the maximal anisotropy or the mean anisotropy of $(G_{\rho,\varepsilon})_*I_n$ in $B_\rho \setminus \overline{B_{\rho/2}}$. Furthermore, the fact that $\alpha_{G_{\rho,\varepsilon}}(r) = 1$ and $\phi_{\rho,\varepsilon}(r) = r$ for all $\rho < r < 2$, implies that $\sigma_{c,\rho,\varepsilon} = (F_\rho)_*I_n$ in $B_2 \setminus \overline{B_1}$ (and thus $\sigma_{c,\rho,\varepsilon}$ is independent of $G_{\rho,\varepsilon}$ in $B_2 \setminus \overline{B_1}$).

Example 4.1. Similar to Section 3 we consider cloak enhancing layers that meet a prescribed cloak enhancement $(\varepsilon/\rho)^n$ and have at the same time minimal maximal anisotropy χ_{\max} or minimal mean anisotropy χ_{mean} .

To obtain a cloak enhancing layer of minimal maximal anisotropy, we seek a C^1 diffeomorphism $\phi_{\rho,\varepsilon} : [\rho/2, \rho] \rightarrow [\varepsilon, \rho]$ such that the corresponding $\alpha_{G_{\rho,\varepsilon}}$ from (4.1) solves the constrained

optimization problem

$$\begin{aligned} \text{minimize} \quad & \sup_{r \in (\rho/2, \rho)} \left\{ \alpha_{G_{\rho, \varepsilon}}(r), \frac{1}{\alpha_{G_{\rho, \varepsilon}}(r)} \right\} \\ \text{subject to} \quad & \alpha_{G_{\rho, \varepsilon}}(r) > 0 \text{ for all } r \in (\rho/2, \rho), \text{ and } \int_{\rho/2}^{\rho} \frac{1}{r \alpha_{G_{\rho, \varepsilon}}(r)} dr = \log\left(\frac{\rho}{\varepsilon}\right) \end{aligned} \quad (4.5)$$

(cf. (3.9)). As in Section 3.1 we find that the solution $\alpha_{G_{\rho, \varepsilon}}^*$ to (4.5) is given by the constant function

$$\alpha_{G_{\rho, \varepsilon}}^*(r) = \frac{\log(2)}{\log(\rho) - \log(\varepsilon)}, \quad \rho/2 < r < \rho,$$

and that the corresponding optimal diffeomorphism $\phi_{\rho, \varepsilon}^*$ satisfies

$$\phi_{\rho, \varepsilon}^*(r) = \left(\frac{2\varepsilon}{\rho}\right)^{\log(\rho)/\log(2)} r^{\frac{\log(\rho) - \log(\varepsilon)}{\log(2)}}, \quad \rho/2 \leq r \leq \rho.$$

Accordingly, recalling (4.4), we obtain that

$$\alpha_{H_{\rho, \varepsilon}}(r) := \alpha_{F_{\rho}}(r) \alpha_{G_{\rho, \varepsilon}}^*(\psi_{\rho}(r)) = \frac{\log(2)}{\log(\rho) - \log(\varepsilon)}, \quad 1/2 < r < 1. \quad (4.6)$$

On the other hand, minimizing the mean anisotropy of the cloak enhancing layer, for $\varepsilon < \rho e^{(1-2^n)/n}$ amounts to constructing a C^1 diffeomorphism $\phi_{\rho, \varepsilon} : [\rho/2, \rho] \rightarrow [\varepsilon, \rho]$, such that the corresponding $\alpha_{G_{\rho, \varepsilon}}^*$ from (4.1) solves the constrained optimization problem

$$\text{minimize} \quad \int_{\rho/2}^{\rho} \frac{1}{(\alpha_{G_{\rho, \varepsilon}}(r))^2} r^{n-1} dr \quad \text{subject to} \quad \phi_{\rho, \varepsilon}(\rho/2) = \varepsilon \text{ and } \phi_{\rho, \varepsilon}(\rho) = \rho \quad (4.7)$$

(cf. Lemma 3.4 and the remarks following). As in Section 3.2 we find that the solution to (4.7) is given by

$$\alpha_{G_{\rho, \varepsilon}}^*(r) = \frac{(2^n - 1)r^n}{n\rho^n \log(\rho/\varepsilon)}, \quad \rho/2 < r < \rho,$$

and that the corresponding optimal diffeomorphism $\phi_{\rho, \varepsilon}$ satisfies

$$\phi_{\rho, \varepsilon}^*(r) = e^{-\frac{\rho^n \log(\rho/\varepsilon)}{r^n(2^n - 1)} + \log(\rho) + \frac{\log(\rho/\varepsilon)}{2^n - 1}}, \quad \rho/2 \leq r \leq \rho.$$

Accordingly, we obtain from (4.4) that

$$\alpha_{H_{\rho, \varepsilon}}(r) := \alpha_{F_{\rho}}(r) \alpha_{G_{\rho, \varepsilon}}^*(\psi_{\rho}(r)) = \frac{(2^n - 1)r^n}{n \log(\rho/\varepsilon)}, \quad 1/2 < r < 1. \quad (4.8)$$

Note that in both cases $\phi_{\rho, \varepsilon}^*$, $\alpha_{G_{\rho, \varepsilon}}^*$ as well as $\alpha_{H_{\rho, \varepsilon}}$ depend non-trivially on the characteristic parameter ρ of the near cloak to be enhanced. \diamond

Example 4.2. Finally we compare qualitative properties of examples of (continuous) anisotropic cloak enhancing layers as discussed above to the corresponding properties of the two dimensional (piecewise constant) isotropic cloak enhancing layers developed in [2, 16].

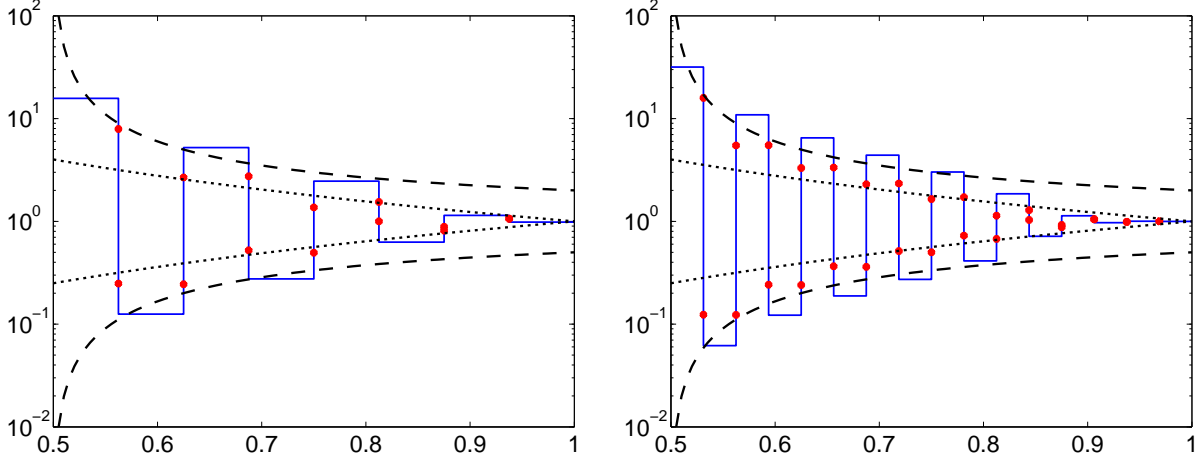


Figure 4: The conductivities of 8 (left) and 16 (right) isotropic cloak enhancing sublayers from [16] in comparison with two anisotropic cloak enhancing layers described in Example 4.2.

The solid lines in Figure 4 are graphs of numerical approximations of the conductivity distributions of two radially symmetric isotropic cloak enhancing layers from [16], consisting of 8 and 16 sublayers of constant conductivity in $B_1 \setminus \overline{B_{1/2}}$. As has been shown in [16] (subject to the solvability of a finite set of algebraic equations), combining these conductivity distributions with the regular near cloak $\sigma_{c,\rho}$ (cf. (2.6)) in $B_2 \setminus \overline{B_1}$, and with an additional layer of very small conductivity in $B_{1/2} \setminus \overline{B_{1/4}}$, we obtain an approximate cloak satisfying a visibility estimate similar to (2.8), with a right hand side of order $\mathcal{O}(\rho^{2L+2})$. Here L denotes the number of cloak enhancing layers – for more details see the remarks following Theorem 1 of [16]. To compare qualitatively the shape of these isotropic conductivity distributions with shapes of anisotropic cloak enhancing layers, we also add to Figure 4 solid dots, indicating the arithmetic and harmonic averages of the constant conductivities of adjacent sublayers of the isotropic conductivity distributions. These averages are well known asymptotically to approximate the eigenvalues of a homogenized anisotropic conductivity distribution with similar dielectric properties as the isotropic conductivity distributions themselves. The fact that the isotropic layers jump between a and $1/a$ is consistent with the fact that the anisotropic cloak enhancing layers discussed above have eigenvalues $\alpha_{G_{\rho,\varepsilon}}^*(r)$ and $1/\alpha_{G_{\rho,\varepsilon}}^*(r)$ (in dimension two).

First we compare these isotropic layers to the optimal cloak enhancing layers from Example 4.1. In contrast to the isotropic layers from [16], the optimal anisotropic layers from Example 4.1 depend non-trivially on the parameter ρ of the near cloak to be enhanced and degenerate as ρ tends to zero. For fixed $\rho > 0$ the eigenvalues of the cloak enhancing layer of minimal maximal anisotropy from (4.6) are constant with respect to the radial variable, i.e., the corresponding conductivity distributions look qualitatively very different from the isotropic cloak enhancing layers from [16], and are thus not shown in Figure 4. On the other hand, the eigenvalues of the cloak enhancing layer of minimal mean anisotropy from (4.8) are monotonically decreasing and increasing with respect to the radial variable, respectively. The dotted lines in Figure 4, corresponding to the functions $r \mapsto r^2$ and $r \mapsto 1/r^2$, visualize the qualitative behavior of these eigenvalues (see (4.8)), omitting the dependency on ρ by neglecting the constant factor in (4.8). These plots indicate that the eigenvalues of this optimal cloak enhancing layer change at a slightly slower rate than the corresponding isotropic conductivities from [16] would warrant, as the radial variable decreases (i.e., as we approach $\partial B_{1/2}$)

One way to obtain anisotropic cloak enhancing layers that do not depend on the parameter ρ of the cloak to be enhanced is to consider perfect cloak enhancing layers. Such a layer can, e.g., be obtained by means of the radial coordinate transform $\psi : [1/2, 1] \rightarrow [0, 1]$, $\psi(r) := 2(r - 1/2)$, which satisfies $\psi(1/2) = 0$ and $\psi(1) = 1$. In this fashion we obtain

$$\sigma_c(x) := \left(\frac{\psi(|x|)}{|x|} \right)^{n-2} \left(\alpha(|x|) \frac{xx^T}{|x|^2} + \frac{1}{\alpha(|x|)} \left(I_n - \frac{xx^T}{|x|^2} \right) \right), \quad x \in B_1 \setminus \overline{B_{1/2}},$$

with $\alpha(r) := 1 - 1/(2r)$, $1/2 < r < 1$, which is just a rescaled version of the perfect cloak from (2.3). The eigenvalues of this anisotropic conductivity distribution, shown in Figure 4 in dashes, are relatively close to the arithmetic and harmonic averages of the isotropic layers from [16], for $1/2 < |x| < 3/4$. \diamond

Conclusions

We have considered a slightly generalized version of the cloaking by mapping scheme from [19], replacing the traditional affine, radial map by more general radial coordinate transforms. We have studied the optimization problem of choosing those transforms which lead to minimal (maximal or mean) anisotropy for the resulting approximate cloak, given a prescribed upper bound for the visibility. We have demonstrated that the corresponding optimal transformations yield significantly enhanced approximate cloaks, when compared to the traditional approximate cloak from [19]. These results are relevant in practice, since they characterize the level of anisotropy that is necessary in order to obtain a certain level of invisibility.

Acknowledgments

This work was initiated during a visit of Roland Griesmaier to Rutgers University. The kind hospitality during this visit is gratefully acknowledged. The research of Michael Vogelius is partially supported by NSF grant DMS-12-11330. The authors would like to thank Holger Heumann for discussions related to the comparisons with the numerical results from [16].

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