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AN ASYMPTOTIC FACTORIZATION METHOD FOR INVERSE ELECTROMAGNETIC SCATTERING IN LAYERED MEDIA*

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Abstract. We consider the inverse problem to reconstruct the number and the positions of a collection of finitely many small perfectly conducting scatterers buried within the lower halfspace of an unbounded two-layered background medium from near field measurements of time harmonic electromagnetic fields. For this purpose we first study the direct scattering problem and derive an asymptotic expansion of the scattered field, as the size of the scatterers tends to zero. Integral equation methods and a factorization of the corresponding near field measurement operator are applied to prove this expansion. In the second part of this work we use the asymptotic expansion to justify a noniterative reconstruction algorithm, which is a combination of factorization methods and MUSIC-type methods. We illustrate the feasibility of this method by a numerical example.

Key words. inverse scattering, Maxwell's equations, small scatterers, layered media, asymptotic expansions

AMS subject classifications. 35C20, 78A46, 35Q60

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1. Introduction.

We consider a simple but fully three dimensional model for the electromagnetic exploration of perfectly conducting objects buried within the lower half-space of an unbounded two-layered background medium. In possible applications, such as, e.g., humanitarian demining or, more generally, the exploration of the ground's subsurface to detect and identify buried objects, the two layers would correspond to air and soil. Moving a set of electric devices parallel to the surface of ground to generate a time harmonic field, the induced field is measured within the same devices. The goal is to retrieve information about buried scatterers from these data.

This work originated in the project [23] on humanitarian demining. In the course of this project mathematical methods for analyzing data obtained from standard off-the-shelf metal detectors have been developed. The aim of the project has been to reduce the number of false alarms produced by such devices used for humanitarian demining.

In mathematical terms, we consider an inverse scattering problem for Maxwell's equations in a two-layered background medium. An iterative method for such a problem was recently proposed by Delbary et al. [17]. Among the so-called qualitative methods (see Cakoni and Colton [10]), the linear sampling method was studied by Gebauer et al. [20] and by Cakoni, Fares, and Haddar [11]. Moreover, the factorization method was applied by Kirsch [29] and by Gebauer, Hanke, and Schneider [21]. In numerical experiments these methods turned out to be quite sensitive to noise. This is of course due to the ill-posedness of the inverse problem.

In order to handle this ill-posedness it is generally advisable to incorporate all available a priori knowledge about the measurement device and the scatterers and

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to determine very specific features. Standard off-the-shelf metal detectors used for humanitarian demining work at relatively low frequencies around 20 kHz; cf., e.g., [22]. In vacuum this corresponds to wavelengths of approximately 15 km. Thus the typical size of the objects of interest, which is only a few centimeters, is very small with respect to the wavelength of the incident field. We use this a priori knowledge to justify a noniterative reconstruction method that determines only the number and the position of the unknown scatterers but is more robust against noise in the data.

This method is a generalization of a method which was originally developed for electrical impedance tomography by Brühl, Hanke, and Vogelius [8]. It is based on an asymptotic expansion of the scattered field on the measurement device as the size of the scatterers tends to zero. A similar reconstruction method was recently investigated by Ammari et al. [2] for homogeneous background media and by Iakovleva et al. [24] for two-layered background media. In contrast to the present investigation, these works study a discrete measurement array, which can be considered as a special case of the measurement device studied here. We expect that the theoretical results obtained for nondiscrete measurement devices can be applied to even more realistic models for the measurement process; cf., e.g., [17]. Moreover, the asymptotic expansions of the scattered field were obtained only formally in [2] and [24]. Here we give a rigorous justification of these formulas for two-layered background media. For bounded background domains related formulas were rigorously proven by Ammari et al. [6, 4, 5], and these results were extended to unbounded homogeneous media and plane wave incident fields by Ammari and Volkov [7]. But this analysis applies neither to layered media nor to near field measurements such as considered here.

Our proof of the asymptotic formula employs a factorization of the near field measurement operator that maps magnetic dipole distributions on the measurement device to the corresponding scattered field on the same device. We apply layer potential techniques to describe the three operators occurring in this factorization and expand them separately as the size of the scatterers tends to zero. Then these expansions are combined to calculate the leading order term in the asymptotic expansion of the scattered field. This generalizes the approach we used in [1] for a boundary value problem in electrostatics. By contrast, in [6, 4, 5, 7] variational methods were applied.

Then, we derive a characterization of the location of the scatterers in terms of the range of the leading order term of the asymptotic expansion of the near field measurement operator, similar to range criteria known from factorization methods, introduced first by Kirsch [28], and MUSIC-type methods, applied first to inverse scattering problems by Devaney [18]. We use a MUSIC-type strategy to implement this range criterion numerically; basically, MUltiple SIgnal Classification is a method of characterizing the range of finite rank operators on Hilbert spaces; see Cheney [12].

The article is organized as follows. After introducing some notation in the next section we describe our model and define the measurement operator in section 3. In section 4 we derive a factorization of this operator, and in section 5 we collect some facts concerning boundary integral operators arising in electromagnetic scattering theory for layered background media. Sections 6 and 7 are devoted to the asymptotic expansion of the measurement operator. Then, in section 9 we derive a characterization of the scatterers in terms of a range criterion, and in section 10 we comment on how to implement this criterion numerically. Finally, we present numerical results.

2. Preliminaries. We introduce our notation and recall some facts concerning function spaces used in the context of Maxwell's equations. For further details we refer the reader to [9, 31, 32]. Suppose $D \subset \mathbb{R}^3$ is a bounded domain of class $C^{2,\alpha}$, $0<\alpha<1$. Denote by $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the usual Cartesian basis of $\mathbb{R}^3$, by $\boldsymbol{x} = (x_1, x_2, x_3)^\top$ a generic point in $\mathbb{R}^3$, and by $\boldsymbol{\nu}$ the unit outward normal to ∂D. Throughout let $\boldsymbol{x}\cdot\boldsymbol{y}$ and $\boldsymbol{x}\times\boldsymbol{y}$ be the scalar product and the vector product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, respectively, and let $|\boldsymbol{x}|$ denote the Euclidean norm of $\boldsymbol{x}$. The standard complex valued Sobolev spaces $H^r(D)$, $H^r_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{D})$ for any $r \in \mathbb{R}$ and $H^s(\partial D)$ for $s \in [-2, 2]$ are defined on D, $\mathbb{R}^3\setminus\overline{D}$ and on the boundary ∂D, respectively; see [30]. Let $\gamma_0 : H^r(D) \to H^{r-1/2}(\partial D)$, $1/2 < r \le 2$, be the standard trace operator. We also need the spaces $\boldsymbol{H}(\mathbf{curl}, D)$, $\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3\setminus\overline{D})$, $\boldsymbol{H}(\mathrm{div}, D)$, and $\boldsymbol{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3\setminus\overline{D})$ of (locally) square integrable vector fields with (locally) square integrable curl and divergence, respectively.

The surface gradient $\nabla_{\partial D}$ and the surface vector curl $\mathbf{curl}_{\partial D}$ are defined on ∂D in the usual way by a localization argument. The adjoint operators of $-\nabla_{\partial D}$ and $\mathbf{curl}_{\partial D}$ are the surface divergence $\mathrm{div}_{\partial D}$ and the surface scalar curl $\mathrm{curl}_{\partial D}$, respectively. We introduce the Hilbert space $\boldsymbol{H}_t^{-1/2}(\partial D)$ of tangential vector fields in $H^{-1/2}(\partial D)^3$ and the Hilbert spaces $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$ of vector fields in $\boldsymbol{H}_t^{-1/2}(\partial D)$ with surface divergence and surface scalar curl in $H^{-1/2}(\partial D)$, respectively. The space $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$ is naturally identified with the dual space of $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$. We denote the corresponding duality pairing by $\langle \boldsymbol{b}, \boldsymbol{a}\rangle_{\partial D} = \int_{\partial D} \boldsymbol{b}\cdot\boldsymbol{a}\,\mathrm{d}s$ for any $\boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{b} \in \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$.

For any regular vector field $\boldsymbol{u}$ we define the normal trace $\gamma_n(\boldsymbol{u}) := \boldsymbol{u}|_{\partial D}\cdot\boldsymbol{\nu}$, the tangential trace $\gamma_t(\boldsymbol{u}) := \boldsymbol{\nu}\times\boldsymbol{u}|_{\partial D}$, and the projection on the tangent plane $\pi_t(\boldsymbol{u}) := (\boldsymbol{\nu}\times\boldsymbol{u}|_{\partial D})\times\boldsymbol{\nu}$. Furthermore, let $r(\boldsymbol{a}) := \boldsymbol{\nu}\times\boldsymbol{a}$ for any regular vector field $\boldsymbol{a}$ on ∂D. Then γ_n, γ_t, π_t, and r can be extended to continuous linear, surjective operators

$$\gamma_n : \boldsymbol{H}(\mathrm{div}, D) \to H^{-1/2}(\partial D), \qquad \gamma_t : \boldsymbol{H}(\mathbf{curl}, D) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D),$$
$$\pi_t : \boldsymbol{H}(\mathbf{curl}, D) \to \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D), \qquad r : \boldsymbol{H}_t^{-1/2}(\partial D) \to \boldsymbol{H}_t^{-1/2}(\partial D).$$

The extension of r is an isomorphism with $r^{-1} = r^\top = -r$, which maps $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ to $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$ and vice versa. For $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, D)$ we have $\gamma_t(\boldsymbol{u}) = r(\pi_t(\boldsymbol{u}))$ and $\pi_t(\boldsymbol{u}) = -r(\gamma_t(\boldsymbol{u}))$. We note that for $\boldsymbol{a} \in \boldsymbol{H}_t^{-1/2}(\partial D)$,

$$\mathrm{div}_{\partial D}\,\boldsymbol{a} = \mathrm{curl}_{\partial D}\, r(\boldsymbol{a}) \quad \text{and} \quad \mathrm{curl}_{\partial D}\,\boldsymbol{a} = -\,\mathrm{div}_{\partial D}\, r(\boldsymbol{a}). \tag{2.1}$$

Furthermore, for $f \in H^1(D)$,

$$\nabla_{\partial D}\gamma_0(f) = \pi_t(\nabla f) \quad \text{and} \quad \mathbf{curl}_{\partial D}\gamma_0(f) = -r(\nabla_{\partial D}\gamma_0(f)) = -\gamma_t(\nabla f). \tag{2.2}$$

Finally, for $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, D)$, it holds that

$$-\,\mathrm{div}_{\partial D}\,\gamma_t(\boldsymbol{u}) = \mathrm{curl}_{\partial D}\,\pi_t(\boldsymbol{u}) = \gamma_n(\mathbf{curl}\,\boldsymbol{u}). \tag{2.3}$$

Throughout we let scalar operators operate on vectors componentwise and vector operators on matrices column by column. For Banach spaces X and Y we denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators on X to Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. Moreover, in our estimates we shall use a generic constant C.

3. The mathematical setting. We decompose the space $\mathbb{R}^3 = \mathbb{R}^3_+ \cup \Sigma_0 \cup \mathbb{R}^3_-$ in a hyperplane $\Sigma_0 := \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_3 = 0\}$ corresponding to the surface of the ground, and the two halfspaces $\mathbb{R}^3_+ := \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_3 > 0\}$ and $\mathbb{R}^3_- := \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_3 < 0\}$

above and below Σ_0 representing air and ground, respectively. For convenience we set $\mathbb{R}^3_0 := \mathbb{R}^3 \setminus \Sigma_0$. We assume that both halfspaces are filled with homogeneous materials with dielectricity ε and permeability μ given by

$$\varepsilon(\boldsymbol{x}) := \begin{cases} \varepsilon_+, & \boldsymbol{x} \in \mathbb{R}^3_+, \\ \varepsilon_-, & \boldsymbol{x} \in \mathbb{R}^3_-, \end{cases} \qquad \mu(\boldsymbol{x}) := \begin{cases} \mu_+, & \boldsymbol{x} \in \mathbb{R}^3_+, \\ \mu_-, & \boldsymbol{x} \in \mathbb{R}^3_-, \end{cases}$$

and we require that ε_+ as well as $\mu_\pm$ are positive numbers, whereas ε_- may be complex with positive real and nonnegative imaginary parts to allow for soil materials that are conducting. The associated (discontinuous) wavenumber is $k := \omega\sqrt{\varepsilon\mu}$, where we assume $\omega > 0$. If $\varepsilon_- \notin \mathbb{R}$, then k is taken to have positive imaginary part. Throughout we investigate radiating solutions of the time harmonic Maxwell system

$$\mathbf{curl}\boldsymbol{H} + \mathrm{i}\,\omega\varepsilon\boldsymbol{E} = 0, \quad \mathbf{curl}\boldsymbol{E} - \mathrm{i}\,\omega\mu\boldsymbol{H} = 0 \tag{3.1}$$

in the exterior of some compact set $C \subset \mathbb{R}^3$. By this we understand, cf., e.g., [16, 31], solutions $\boldsymbol{E}, \boldsymbol{H} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus C)$ which obey the integral radiation condition

$$\int_{\partial B_R(0)} \left| \frac{\boldsymbol{x}}{R} \times \boldsymbol{H}(\boldsymbol{x}) + \Big(\frac{\varepsilon(\boldsymbol{x})}{\mu(\boldsymbol{x})}\Big)^{1/2} \boldsymbol{E}(\boldsymbol{x}) \right|^2 \mathrm{d}s(\boldsymbol{x}) = o(1) \qquad \text{as } R \to \infty, \tag{3.2}$$

where $B_R(0) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| < R\}$ denotes the ball of radius $R > 0$ around the origin.

For layered medium we have to distinguish between the electric and the magnetic dyadic Green's functions. The electric dyadic Green's function $\mathbb{G}^e$ is the radiating (distributional) solution of

$$\mathbf{curl}_x \frac{1}{\mu(\boldsymbol{x})} \mathbf{curl}_x\, \mathbb{G}^e(\boldsymbol{x}, \boldsymbol{y}) - \omega^2 \varepsilon(\boldsymbol{x}) \mathbb{G}^e(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{x})} \delta(\boldsymbol{x} - \boldsymbol{y})\, \mathbb{I}_3, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3,$$

where $\mathbb{I}_3$ denotes the 3×3 identity matrix. Note that we are using $\boldsymbol{x}$ as an independent variable and $\boldsymbol{y}$ denotes the position of the source. The magnetic dyadic Green's function $\mathbb{G}^m$ fulfills the same equation, but ε and μ have to be swapped. From the derivation of these Green's tensors in [31, pp. 318–327] (cf. also [34, 17, 33]), we find that $\mathbb{G}^e$ and $\mathbb{G}^m$ can be written as

$$\mathbb{G}^{e/m}(\boldsymbol{x}, \boldsymbol{y}) = \Pi^{e/m}(\boldsymbol{x}, \boldsymbol{y}) + \frac{1}{k(\boldsymbol{x})^2} \nabla_x \operatorname{div}_x \Pi^{e/m}(\boldsymbol{x}, \boldsymbol{y})$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_0$, $\boldsymbol{x} \neq \boldsymbol{y}$. Here the (matrix valued) functions Π^e and Π^m are given by

$$\Pi^{e/m}(\boldsymbol{x}, \boldsymbol{y}) := \Phi_{k(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{y})\, \mathbb{I}_3 + F^{e/m}(\boldsymbol{x}, \boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_0,\ \boldsymbol{x} \neq \boldsymbol{y}, \tag{3.3}$$

where Φ_{k_+} and Φ_{k_-} denote the fundamental solution for the scalar Helmholtz equation in homogeneous medium with wavenumber k_+ and k_-, respectively; cf. [15, p. 16]. The functions Π^e and Π^m solve

$$(\Delta_x + k(\boldsymbol{x})^2) \Pi^{e/m}(\boldsymbol{x}, \boldsymbol{y}) = -\delta(\boldsymbol{x} - \boldsymbol{y})\, \mathbb{I}_3, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_0,$$

and so F^e and F^m solve

$$(\Delta_x + k(\boldsymbol{x})^2) F^{e/m}(\boldsymbol{x}, \boldsymbol{y}) = 0, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_0.$$

Applying a regularity result due to Weber [35, Thm. 2.9], we find that $\mathbb{G}^{e/m}|_{\mathbb{R}^3_\pm}(\cdot,\boldsymbol{y}) \in C^\infty(\overline{\mathbb{R}^3_\pm} \setminus \{\boldsymbol{y}\})$ for $\boldsymbol{y} \in \mathbb{R}^3_0$. Thus, $F^e(\cdot,\boldsymbol{y})$ and $F^m(\cdot,\boldsymbol{y})$ are smooth functions in $\mathbb{R}^3_0$ for $\boldsymbol{y}$ in any compact subset of $\mathbb{R}^3_0$.

Using Maxwell's equations and integration by parts the following reciprocity relations can be proven; cf. also [16, 24, 13]:

$$\mu(\boldsymbol{y})\mathbb{G}^e(\boldsymbol{x},\boldsymbol{y}) = \mu(\boldsymbol{x})\mathbb{G}^{e\top}(\boldsymbol{y},\boldsymbol{x}) \qquad \text{for } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_0,\ \boldsymbol{x}\neq\boldsymbol{y}, \tag{3.4a}$$

$$\varepsilon(\boldsymbol{y})\mathbb{G}^m(\boldsymbol{x},\boldsymbol{y}) = \varepsilon(\boldsymbol{x})\mathbb{G}^{m\top}(\boldsymbol{y},\boldsymbol{x}) \qquad \text{for } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_0,\ \boldsymbol{x}\neq\boldsymbol{y}, \tag{3.4b}$$

$$k^2(\boldsymbol{y})\mathbf{curl}_x\mathbb{G}^e(\boldsymbol{x},\boldsymbol{y}) = k^2(\boldsymbol{x})(\mathbf{curl}_y\mathbb{G}^m)^\top(\boldsymbol{y},\boldsymbol{x}) \qquad \text{for } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_0,\ \boldsymbol{x}\neq\boldsymbol{y}. \tag{3.4c}$$

We denote by $\Sigma_d := \{\boldsymbol{x} \in \mathbb{R}^3_+ \mid \boldsymbol{x}\cdot\boldsymbol{e}_3 = d\} \subset \mathbb{R}^3_+$ the hyperplane parallel to the surface of the ground at height $d > 0$ and assume that measurements and excitations are restricted to an open bounded sheet $\mathcal{M} \subset \Sigma_d$ supporting the device. A time harmonic excitation, given by a magnetic dipole density $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M}) := L^2(\mathcal{M})^3$ on $\mathcal{M}$, leads to a primary electromagnetic field $(\boldsymbol{E}^i,\boldsymbol{H}^i)$ satisfying (3.1) in $\mathbb{R}^3 \setminus \mathcal{M}$, where the magnetic field has the form

$$\boldsymbol{H}^i = k_+^2 \int_{\mathcal{M}} \mathbb{G}^m(\cdot,\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}); \tag{3.5}$$

cf., e.g., Sommerfeld [34].

We suppose that $\mathbb{R}^3_-$ contains a finite number of perfectly conducting scatterers, each of the form $D_{\delta,j} := \boldsymbol{z}_j + \delta B_j$, where B_j is a bounded domain of class $C^{2,\alpha}$, $0<\alpha<1$, containing the origin, such that all components of B_j are simply connected, and their boundaries are connected, $1 \le j \le m$. The points $\boldsymbol{z}_j \in \mathbb{R}^3_-$, $1 \le j \le m$, that determine the location of the scatterers are assumed to satisfy $|\boldsymbol{z}_j - \boldsymbol{z}_l| \ge c_0$ for $j \neq l$ and $\operatorname{dist}(\boldsymbol{z}_j,\Sigma_0) \ge c_0$ for some constant $c_0 > 0$, $1 \le j,l \le m$. The value of $0 < \delta \le 1$, the common order of magnitude of the diameters of the scatterers, is assumed to be small enough such that the scatterers are disjoint and compactly contained in $\mathbb{R}^3_-$. So the total collection of scatterers takes the form $D_\delta := \bigcup_{j=1}^m (\boldsymbol{z}_j + \delta B_j)$. The perfect conductor sitting in D_δ induces a secondary field $(\boldsymbol{E}^s,\boldsymbol{H}^s)$ which is a radiating solution of (3.1) in $\mathbb{R}^3 \setminus \overline{D_\delta}$ subject to the boundary condition

$$\boldsymbol{\nu} \times \boldsymbol{E}^s = -\boldsymbol{\nu} \times \boldsymbol{E}^i \qquad \text{on } \partial D_\delta. \tag{3.6}$$

For a mathematical treatment of this direct problem we refer the reader to [16, 31, 17]. We define the (measurement) operator G_δ, which maps given excitations $\boldsymbol{\varphi}$ onto the corresponding secondary magnetic field $\boldsymbol{H}^s|_{\mathcal{M}}$ on $\mathcal{M}$, i.e.,

$$G_\delta : \boldsymbol{L}^2(\mathcal{M}) \to \boldsymbol{L}^2(\mathcal{M}), \quad G_\delta\boldsymbol{\varphi} := \boldsymbol{H}^s|_{\mathcal{M}}. \tag{3.7}$$

As in [20, Thm. 2.1] it can be seen that G_δ is a compact operator.

4. The factorization of G_δ. In this section we study a factorization of the measurement operator G_δ from (3.7) similar to the one developed in [20], but here we do not restrict ourselves to tangential excitations and measurements.

Suppose $\boldsymbol{\psi} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$ and denote by $(\boldsymbol{E}^\psi,\boldsymbol{H}^\psi)$ the associated radiating solution of the exterior Maxwell boundary value problem

$$\mathbf{curl}\boldsymbol{H}^\psi + \mathrm{i}\,\omega\varepsilon\boldsymbol{E}^\psi = 0, \quad \mathbf{curl}\boldsymbol{E}^\psi - \mathrm{i}\,\omega\mu\boldsymbol{H}^\psi = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D_\delta}, \tag{4.1a}$$

$$\boldsymbol{\nu} \times \boldsymbol{E}^\psi = \boldsymbol{\psi} \qquad \text{on } \partial D_\delta. \tag{4.1b}$$

Uniqueness of solutions follows for $\Im\varepsilon_- = 0$ from [16, Prop. 2.5]. For $\Im\varepsilon_- > 0$ this was proven in [17, Thm. 2.1]. Existence of solutions will be shown in the next sections by reducing the boundary value problem to an integral equation of the second kind and applying Riesz–Fredholm theory. We define

$$L_\delta : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D_\delta) \to \boldsymbol{L}^2(\mathcal{M}), \quad L_\delta \boldsymbol{\psi} := \boldsymbol{H}^{\psi}|_{\mathcal{M}}. \tag{4.2}$$

Then L_δ is a bounded linear operator. In particular, if $\boldsymbol{E}^i$ and $\boldsymbol{H}^s$ are the primary electric and secondary magnetic fields introduced in section 3, respectively, then $\boldsymbol{\psi} := -\boldsymbol{\nu} \times \boldsymbol{E}^i|_{\partial D_\delta}$ yields $\boldsymbol{H}^{\psi} = \boldsymbol{H}^s$. This means that $L_\delta : -\boldsymbol{\nu} \times \boldsymbol{E}^i|_{\partial D_\delta} \mapsto \boldsymbol{H}^s|_{\mathcal{M}}$.

We denote the standard bilinear form on $\boldsymbol{L}^2(\mathcal{M})$ by $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ and the corresponding transpose of L_δ by $L_\delta^\top : \boldsymbol{L}^2(\mathcal{M}) \to \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D_\delta)$.

PROPOSITION 4.1. *Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$. Denote by $\boldsymbol{H}^i$ and $\boldsymbol{H}^s$ the associated primary and secondary magnetic fields introduced in section* 3. *Then*

$$L_\delta^\top \boldsymbol{\varphi} = \frac{1}{\mathrm{i}\,\omega\mu_+}(\boldsymbol{\nu} \times \boldsymbol{H}|_{\partial D_\delta}) \times \boldsymbol{\nu} \qquad \text{on } \partial D_\delta, \tag{4.3}$$

where $\boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s$ is the total magnetic field.

Proof. Given $\boldsymbol{\psi} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D_\delta)$, let $(\boldsymbol{E}^{\boldsymbol{\psi}}, \boldsymbol{H}^{\boldsymbol{\psi}})$ be the radiating solution to (4.1). For any $\boldsymbol{y} \in \mathbb{R}^3 \setminus \overline{D_\delta}$ we have the representation formula

$$\boldsymbol{H}^{\boldsymbol{\psi}}(\boldsymbol{y}) = \int_{\partial D_\delta} \frac{\varepsilon(\boldsymbol{y})}{\varepsilon(\boldsymbol{x})}\Big(\mathbb{G}^{m\top}(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{\nu} \times \mathbf{curl}\boldsymbol{H}^{\boldsymbol{\psi}})(\boldsymbol{x}) + (\mathbf{curl}_x \mathbb{G}^m)^\top(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{\nu} \times \boldsymbol{H}^{\boldsymbol{\psi}})(\boldsymbol{x})\Big)\,\mathrm{d}s(\boldsymbol{x});$$

cf. [16, Prop. A.9]. Using this formula the proposition can be proven by applying (4.1), (3.5), (3.6), two times partial integration as in [31, Thm. 3.31], and (3.2). See also [20] for a corresponding result for tangential densities $\boldsymbol{\varphi}$ on $\mathcal{M}$. ◻

Finally, we consider the diffraction problem

$$\mathbf{curl}\boldsymbol{H}^d + \mathrm{i}\,\omega\varepsilon\boldsymbol{E}^d = 0, \quad \mathbf{curl}\boldsymbol{E}^d - \mathrm{i}\,\omega\mu\boldsymbol{H}^d = 0 \qquad \text{in } \mathbb{R}^3 \setminus \partial D_\delta, \tag{4.4a}$$

with the jump conditions

$$[(\boldsymbol{\nu} \times \boldsymbol{H}^d) \times \boldsymbol{\nu}]_{\partial D_\delta} = \boldsymbol{\chi}, \qquad [\boldsymbol{\nu} \times \boldsymbol{E}^d]_{\partial D_\delta} = 0 \qquad \text{on } \partial D_\delta. \tag{4.4b}$$

Here, $\boldsymbol{\chi} \in \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D_\delta)$ is a given tangential field on ∂D_δ, and the square brackets denote the differences between the respective traces from outside and inside. We are looking for a radiating solution $(\boldsymbol{E}^d, \boldsymbol{H}^d)$ of this problem. Uniqueness of solutions has been stated in [29, Thm. 3.4] for $\Im\varepsilon_- = 0$. If $\Im\varepsilon_- > 0$ this can be shown by the same arguments as used in [33, pp. 61–63]. Existence of solutions will be shown later by writing them in terms of layer potentials. Given the solution, we define

$$F_\delta : \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D_\delta) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D_\delta), \quad F_\delta\boldsymbol{\chi} := \boldsymbol{\nu} \times \boldsymbol{E}^d|_{\partial D_\delta}. \tag{4.5}$$

Then F_δ is a bounded linear operator. For $\boldsymbol{\chi} = (\boldsymbol{\nu} \times \boldsymbol{H}|_{\partial D_\delta}) \times \boldsymbol{\nu}$, i.e., the tangential component of the total magnetic field corresponding to some excitation $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ as described in section 3, the solution of the diffraction problem (4.4) can be constructed from the corresponding primary and secondary fields, namely,

$$\boldsymbol{E}^d = \begin{cases} \boldsymbol{E}^s, & x \in \mathbb{R}^3 \setminus \overline{D_\delta}, \\ -\boldsymbol{E}^i, & x \in D_\delta, \end{cases} \qquad \boldsymbol{H}^d = \begin{cases} \boldsymbol{H}^s, & x \in \mathbb{R}^3 \setminus \overline{D_\delta}, \\ -\boldsymbol{H}^i, & x \in D_\delta. \end{cases}$$

Consequently, we have $F_\delta : (\boldsymbol{\nu}\times\boldsymbol{H}|_{\partial D_\delta})\times\boldsymbol{\nu} \mapsto \boldsymbol{\nu}\times\boldsymbol{E}^s|_{\partial D_\delta} = -\boldsymbol{\nu}\times\boldsymbol{E}^i|_{\partial D_\delta}$. Altogether, we obtain the mapping sequence

$$\boldsymbol{\varphi} \overset{L_\delta^\top}{\longmapsto} \frac{1}{\mathrm{i}\,\omega\mu_+}(\boldsymbol{\nu}\times\boldsymbol{H}|_{\partial D_\delta})\times\boldsymbol{\nu} \overset{F_\delta}{\longmapsto} -\frac{1}{\mathrm{i}\,\omega\mu_+}\boldsymbol{\nu}\times\boldsymbol{E}^i|_{\partial D_\delta} \overset{L_\delta}{\longmapsto} \frac{1}{\mathrm{i}\,\omega\mu_+}\boldsymbol{H}^s|_{\mathcal{M}}.$$

This yields the following theorem; cf. [20] for a corresponding result in case of tangential densities $\boldsymbol{\varphi}$ on $\mathcal{M}$.

THEOREM 4.2. *Given L_δ from* (4.2) *and F_δ from* (4.5) *the measurement operator G_δ from* (3.7) *admits the factorization*

$$G_\delta = \mathrm{i}\,\omega\mu_+ L_\delta F_\delta L_\delta^\top . \tag{4.6}$$

5. Surface potentials. Here, we collect some results concerning boundary integral operators for electromagnetic scattering in two-layered media.

5.1. Surface potentials for homogeneous media. First, we consider a homogeneous medium with wavenumber k_-. If $D\subset\mathbb{R}^3$ is a bounded domain of class $C^{2,\alpha}$, $0<\alpha<1$, the single layer potential with smooth density f is defined by

$$(\mathcal{S}_D^- f)(\boldsymbol{x}) := \int_{\partial D}\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x}\in\mathbb{R}^3\setminus\partial D.$$

Then $\mathcal{S}_D^- f$ and $\boldsymbol{\nu}\times\nabla\mathcal{S}_D^- f$ are continuous across ∂D; cf. [14, Thm. 2.12, Thm. 2.17]. It can be shown [30, Thm. 6.11] that the mapping $\mathcal{S}_D^- : H^{-1/2}(\partial D)\to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is bounded. The jump relations on ∂D remain valid for $f\in H^{-1/2}(\partial D)$, but they have to be interpreted in the sense of trace theorems.

Analogously, the vector potential with smooth tangential density $\boldsymbol{a}$ is given by

$$(\mathcal{A}_D^- \boldsymbol{a})(\boldsymbol{x}) := \int_{\partial D}\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x}\in\mathbb{R}^3\setminus\partial D.$$

Then $\mathcal{A}_D^-\boldsymbol{a}$, $\boldsymbol{\nu}\cdot\mathbf{curl}\mathcal{A}_D^-\boldsymbol{a}$, and $\boldsymbol{\nu}\times\mathbf{curl}\,\mathbf{curl}\mathcal{A}_D^-\boldsymbol{a}$ are continuous across ∂D; cf. [14, Thm. 2.24] and [15, Thm. 6.11]. The tangential components of $\boldsymbol{\nu}\times\mathbf{curl}\mathcal{A}_D^-\boldsymbol{a}$ are discontinuous across ∂D and satisfy the jump relation

$$\boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}\mathcal{A}_D^-\boldsymbol{a}\big|_{\partial D}^{\pm}(\boldsymbol{x}) = \int_{\partial D}\boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}_x(\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y}))\,\mathrm{d}s(\boldsymbol{y}) \pm \frac{1}{2}\boldsymbol{a}(\boldsymbol{x})$$

for $\boldsymbol{x}\in\partial D$. It can be shown [30, Thm. 6.11] that $\mathcal{A}_D^- : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)\to\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl},\mathbb{R}^3)$ is bounded and that the jump relations on ∂D remain valid for $\boldsymbol{a}\in\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$. Furthermore, for smooth tangential densities $\boldsymbol{a}$ we define

$$(M_D^-\boldsymbol{a})(\boldsymbol{x}) := \int_{\partial D}\boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}_{\boldsymbol{x}}(\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y}))\,\mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x}\in\partial D,$$

$$(N_D^-\boldsymbol{a})(\boldsymbol{x}) := \boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}\,\mathbf{curl}\int_{\partial D}\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{a}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \quad \boldsymbol{x}\in\partial D.$$

Combining results from [30, 15] and [27], it can be seen that the operators $M_D^- : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)\to\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $N_D^- : \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)\to\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ are continuous. Moreover, M_D^- is compact, and its transpose with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\partial D}$ is given by ${M_D^-}^\top = rM_D^- r$. The operator N_D^- is symmetric. We also need the following identity (see [15, p. 170]:

$$\operatorname{div}\mathcal{A}_D^-\boldsymbol{a} = \mathcal{S}_D^-\operatorname{div}_{\partial D}\boldsymbol{a}, \qquad \boldsymbol{a}\in\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D). \tag{5.1}$$

5.2. The potential theoretic limit. For $k_- = 0$ the expression Φ_{k_-} reduces to the fundamental solution Φ_0 of Laplace's equation. Substituting Φ_{k_-} by Φ_0 in the definitions above, we obtain integral operators

$$\begin{aligned} \mathcal{S}_D^0 &: H^{-1/2}(\partial D) \to H^1_{\mathrm{loc}}(\mathbb{R}^3), & \mathcal{A}_D^0 &: \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3), \\ M_D^0 &: \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D), & N_D^0 &: \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D). \end{aligned}$$

The corresponding mapping properties and jump relations remain valid for $k = 0$. Suppose that all components of D are simply connected and the complement of D is connected. Then the operator $\frac{1}{2}I + M_D^0$ has trivial nullspace in $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ [14, Thm. 5.5]. Therefore, we can apply Fredholm's alternative and find that $\frac{1}{2}I + M_D^0$ and $\frac{1}{2}I + {M_D^0}^\top$ are invertible on $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$, respectively. From ${M_D^0}^\top = rM_D^0 r$ we observe that for any $\boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ it holds that

$$\left(\frac{1}{2}I \pm {M_D^0}^\top\right) r\boldsymbol{a} = r\left(\frac{1}{2}I \mp M_D^0\right)\boldsymbol{a}. \tag{5.2}$$

Thus, $-\frac{1}{2}I + M_D^0$ and $-\frac{1}{2}I + {M_D^0}^\top$ are invertible on $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$, respectively, too.

Furthermore, given a scalar smooth function f we define

$$(K_D^0 f)(\boldsymbol{x}) := \int_{\partial D} \frac{\partial \Phi_0(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{\nu}(\boldsymbol{y})} f(\boldsymbol{y})\, \mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x} \in \partial D.$$

It can be shown [32, Thm. 4.4.1] that the mapping $K_D^0 : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ is compact. The operator $-\frac{1}{2}I + K_D^0$ has trivial nullspace in $H^{1/2}(\partial D)$ [3, Lem. 2.5]. Hence, by Fredholm's alternative $-\frac{1}{2}I + K_D^0$ and $-\frac{1}{2}I + {K_D^0}^\top$ are invertible on $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$, respectively.

LEMMA 5.1. (a) *The operators* $\pm\frac{1}{2}I + M_D^0$ *are isomorphisms on*

$$\boldsymbol{H}_{\mathrm{div},0}^{-1/2}(\partial D) := \{\boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \mid \mathrm{div}_{\partial D}\, \boldsymbol{a} = 0\}.$$

(b) *For any* $f \in H^{1/2}(\partial D)$,

$$\left(\pm\frac{1}{2}I + {M_D^0}^\top\right)^{-1} \nabla_{\partial D} f = -\nabla_{\partial D}\left(\mp\frac{1}{2}I + K_D^0\right)^{-1} f. \tag{5.3}$$

Proof. Part (a) follows at once from

$$\mathrm{div}_{\partial D}\, M_D^0 \boldsymbol{a} = -{K_D^0}^\top \mathrm{div}_{\partial D}\, \boldsymbol{a}, \qquad \boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D); \tag{5.4}$$

cf. [15, p. 169]. By duality, (5.4) yields ${M_D^0}^\top \nabla_{\partial D} f = -\nabla_{\partial D} K_D^0 f$ for $f \in H^{1/2}(\partial D)$. Thus, we find

$$\left(\pm\frac{1}{2}I + {M_D^0}^\top\right)\nabla_{\partial D} f = -\nabla_{\partial D}\left(\mp\frac{1}{2}I + K_D^0\right) f,$$

which gives (5.3). ∎

5.3. Surface potentials for layered media. Here, we consider again the two-layered medium introduced in section 3. Let $D \subset \mathbb{R}^3_-$ be a bounded domain of class $C^{2,\alpha}$, $0 < \alpha < 1$, such that $\operatorname{dist}(D, \Sigma_0) \geq c_0$ for some constant $c_0 > 0$. We define modified vector potentials with smooth tangential density $\boldsymbol{a}$ by

$$
\begin{aligned}
(\mathcal{A}_D^{e/m}\boldsymbol{a})(\boldsymbol{x}) :=& \int_{\partial D} \Pi^{e/m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}s(\boldsymbol{y}) \\
=&(\mathcal{A}_D^{-}\boldsymbol{a})(\boldsymbol{x}) + \int_{\partial D} F^{e/m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\, \mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \partial D,
\end{aligned}
$$

and boundary integrals

$$
(R_D^{e/m}\boldsymbol{a})(\boldsymbol{x}) := \int_{\partial D} \boldsymbol{\nu}(\boldsymbol{x}) \times \mathbf{curl}_x(F^{e/m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y}))\, \mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{x} \in \partial D, \tag{5.5}
$$

$$
\begin{aligned}
(M_D^{e/m}\boldsymbol{a})(\boldsymbol{x}) :=& \int_{\partial D} \boldsymbol{\nu}(\boldsymbol{x}) \times \mathbf{curl}_x(\Pi^{e/m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y}))\, \mathrm{d}s(\boldsymbol{y}) \\
=&(M_D^{-}\boldsymbol{a})(\boldsymbol{x}) + (R_D^{e/m}\boldsymbol{a})(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D.
\end{aligned} \tag{5.6}
$$

Because $F^{e/m}(\cdot,\boldsymbol{y})$ is smooth in $\mathbb{R}^3_0$ for $\boldsymbol{y}$ in any compact subset of $\mathbb{R}^3_0$, we find that $\mathcal{A}_D^{e/m}\boldsymbol{a}$ and $\boldsymbol{\nu} \times \mathbf{curl}\,\mathbf{curl}\mathcal{A}_D^{e/m}\boldsymbol{a}$ are continuous across ∂D. Furthermore,

$$
\boldsymbol{\nu} \times \mathbf{curl}\mathcal{A}_D^{e/m}\boldsymbol{a}\big|^{\pm}_{\partial D} = \Big(\pm\frac{1}{2}I + M_D^{e/m}\Big)\boldsymbol{a} \qquad \text{on } \partial D.
$$

The mapping $\mathcal{A}_D^{e/m} : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3_0)$ is continuous, and the operators $R_D^{e/m} : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $M_D^{e/m} : \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ are compact. Assume that the exterior of D is connected and that the wavenumber k_- is not an interior Maxwell eigenvalue for D. Then the operator $\frac{1}{2}I + M_D^m$ has trivial nullspace in $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$. This can be proven in essentially the same way as [14, Thm. 4.23] for homogeneous medium and continuous densities. So we can apply Fredholm's alternative to obtain that $\frac{1}{2}I + M_D^m$ and $\frac{1}{2}I + M_D^{m\top}$ are invertible on $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$, respectively.

6. First estimates.

In the following two sections we consider the case of a single scatterer, i.e., $D_\delta = \boldsymbol{z} + \delta B$. Multiple scatterers will be studied in section 8.

We often have to deal with changes of coordinates; thus we introduce the following notation. Given $\boldsymbol{a} \in C(\partial D_\delta)^3$ and $\boldsymbol{b} \in C(\partial B)^3$ we define $\hat{\boldsymbol{a}}, (\boldsymbol{a})^\wedge \in C(\partial B)^3$ and $\check{\boldsymbol{b}}, (\boldsymbol{b})^\vee \in C(\partial D_\delta)^3$ by

$$
(\boldsymbol{a})^\wedge(\boldsymbol{\xi}) := \hat{\boldsymbol{a}}(\boldsymbol{\xi}) := \boldsymbol{a}(\delta\boldsymbol{\xi} + \boldsymbol{z}) \quad \text{and} \quad (\boldsymbol{b})^\vee(\boldsymbol{x}) := \check{\boldsymbol{b}}(\boldsymbol{x}) := \boldsymbol{b}\Big(\frac{\boldsymbol{x} - \boldsymbol{z}}{\delta}\Big) \tag{6.1}
$$

for $\boldsymbol{\xi} \in \partial B$ and $\boldsymbol{x} \in \partial D_\delta$, respectively. This notation is also applied to functions from Sobolev spaces.

For arbitrary bounded domains $D \subset \mathbb{R}^3$ of class $C^{2,\alpha}$, $0 < \alpha < 1$, we use the following norms on $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$:

$$
\begin{aligned}
\|\boldsymbol{a}\|_{\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)} &:= \inf_{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},D),\, \gamma_t(\boldsymbol{u}) = \boldsymbol{a}} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D)} \qquad \text{for } \boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D), \\
\|\boldsymbol{b}\|_{\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)} &:= \inf_{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},D),\, \pi_t(\boldsymbol{u}) = \boldsymbol{b}} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D)} \qquad \text{for } \boldsymbol{b} \in \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D).
\end{aligned}
$$

A simple calculation (cf. [1, Lem. 4.1] for a similar result) yields the following scaling properties of these norms under changes of coordinates as in (6.1). Suppose $\boldsymbol{a} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$, $\boldsymbol{b} \in \boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)$, and assume $0 < \delta \le 1$. Then

$$\delta^{\frac{3}{2}} \|\hat{\boldsymbol{a}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)} \le \|\boldsymbol{a}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)} \le \delta^{\frac{1}{2}} \|\hat{\boldsymbol{a}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)}, \tag{6.2a}$$

$$\delta^{\frac{3}{2}} \|\hat{\boldsymbol{b}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial B)} \le \|\boldsymbol{b}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)} \le \delta^{\frac{1}{2}} \|\hat{\boldsymbol{b}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial B)}. \tag{6.2b}$$

In order to derive the asymptotic expansion in section 7 we need to expand the fundamental solution $\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y}) = \Phi_{k_-}(\delta(\boldsymbol{\xi}-\boldsymbol{\eta}))$ for $\boldsymbol{x} = \delta\boldsymbol{\xi}+\boldsymbol{z} \neq \delta\boldsymbol{\eta}+\boldsymbol{z} = \boldsymbol{y} \in \partial D_\delta$, i.e., $\boldsymbol{\xi} \neq \boldsymbol{\eta} \in \partial B$, as $\delta \to 0$. Expanding $e^{\mathrm{i}\,k_-\delta|\boldsymbol{\xi}-\boldsymbol{\eta}|}$ in a power series we obtain the following formulas:

$$\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\delta}\left(\Phi_0(\boldsymbol{\xi}-\boldsymbol{\eta}) + \frac{\mathrm{i}\,k_-\delta}{4\pi} + \mathcal{O}(\delta^2)\right) \qquad \text{as } \delta \to 0, \tag{6.3}$$

$$\nabla_x \Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\delta^2}\left(\nabla_\xi \Phi_0(\boldsymbol{\xi}-\boldsymbol{\eta}) - \frac{k_-^2\delta^2}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|} + \mathcal{O}(\delta^3)\right) \qquad \text{as } \delta \to 0. \tag{6.4}$$

Remark 6.1 (eigenvalues). In section 5.3 we had to assume that the wavenumber k_- is not an interior Maxwell eigenvalue for the bounded domain D to obtain invertibility of the operators $\frac{1}{2}I + M_D^m$ and $\frac{1}{2}I + M_D^{m\top}$. Interior Maxwell eigenvalues for D are wavenumbers κ such that Maxwell's equations (3.1) in D with homogeneous boundary condition $\boldsymbol{\nu} \times \boldsymbol{E}|_{\partial D} = 0$ on ∂D have a nontrivial solution. If $\Im\kappa > 0$, it is well known that solutions to the interior Maxwell boundary value problem are unique (cf. [31, Thm. 4.17]), and thus κ is no eigenvalue. On the other hand, there is a discrete set of real eigenvalues $\kappa_j > 0$, $j \in \mathbb{N}$, for D accumulating only at infinity; cf. [31, Thm. 4.18].

Let $\{k_j\}_{j\in\mathbb{N}}$ be the set of interior Maxwell eigenvalues corresponding to the reference domain B. By a change of coordinates in the variational formulation of the eigenvalue problem (see [31, p. 96]), we find that $\{\delta^{-1}k_j\}_{j\in\mathbb{N}}$ is the set of eigenvalues corresponding to the domain $D_\delta = z + \delta B$, $0 < \delta \le 1$. Therefore, we can assume henceforth in the derivation of the asymptotic expansion without loss of generality that δ is small enough so that $k_- \notin \{\delta^{-1}k_j\}_{j\in\mathbb{N}}$, i.e., that k_- is no interior Maxwell eigenvalue for the domains D_δ considered hereafter.

In the next lemma we investigate the scaling properties of the operator $M^m_{D_\delta}$.

LEMMA 6.2. *For $\boldsymbol{a} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$ we have*

$$M^m_{D_\delta}\boldsymbol{a} = (M^0_B\hat{\boldsymbol{a}})^\vee + (E^m_M\hat{\boldsymbol{a}})^\vee.$$

Here E^m_M is a bounded linear operator, which is $\mathcal{O}(\delta^2)$ in $\mathcal{L}(\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B))$ as $\delta \to 0$, independent of $\boldsymbol{a}$.

Proof. Let $\boldsymbol{a} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$ and $\boldsymbol{a}_j$, $j \in \mathbb{N}$, be smooth tangential vector fields with smooth surface divergence on ∂D_δ so that $\boldsymbol{a}_j$ converges to $\boldsymbol{a}$ in $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$. For fixed $j \in \mathbb{N}$ and $\boldsymbol{x} \in D_\delta$ we observe by a change of variables $\boldsymbol{\xi} := \frac{\boldsymbol{x}-\boldsymbol{z}}{\delta}$ and $\boldsymbol{\eta} := \frac{\boldsymbol{y}-\boldsymbol{z}}{\delta}$ that

$$(M^0_{D_\delta}\boldsymbol{a}_j)(\boldsymbol{x}) = \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{\xi}) \times \frac{1}{\delta}\mathbf{curl}_\xi\left(\hat{\boldsymbol{a}}_j(\boldsymbol{\eta})\frac{1}{4\pi\delta|\boldsymbol{\xi}-\boldsymbol{\eta}|}\right)\delta^2\,\mathrm{d}s(\boldsymbol{\eta}) = (M^0_B\hat{\boldsymbol{a}}_j)(\boldsymbol{\xi}),$$

i.e., $M^0_{D_\delta}\boldsymbol{a}_j = (M^0_B\hat{\boldsymbol{a}}_j)^\vee$. From (6.4) we find

$$\nabla_x(\Phi_{k_-} - \Phi_0)(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{\delta^2}\left(-\frac{k_-^2\delta^2}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|} + \mathcal{O}(\delta^3)\right)$$

for $\boldsymbol{x} \neq \boldsymbol{y}$ as $\delta \to 0$. So, again by a change of coordinates we obtain for $\boldsymbol{x} \in \partial D_\delta$

$$\begin{aligned}\big((M^-_{D_\delta} - M^0_{D_\delta})\boldsymbol{a}_j\big)(\boldsymbol{x}) &= \int_{\partial D_\delta} \boldsymbol{\nu}(\boldsymbol{x}) \times \big(\nabla_x(\Phi_{k_-} - \Phi_0)(\boldsymbol{x}-\boldsymbol{y}) \times \boldsymbol{a}_j(\boldsymbol{y})\big)\,\mathrm{d}s(\boldsymbol{y})\\ &= \delta^2 \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{\xi}) \times \left(\left(-\frac{k_-^2}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|} + \mathcal{O}(\delta)\right) \times \hat{\boldsymbol{a}}_j(\boldsymbol{\eta})\right)\mathrm{d}s(\boldsymbol{\eta}) =: (E^-_M \hat{\boldsymbol{a}}_j)(\boldsymbol{\xi}).\end{aligned}$$

The kernel of E^-_M is pseudohomogeneous of class -2 (cf. [32, pp. 168–175]), and hence E^-_M is continuous from $H^{-1/2}(\partial B)^3$ into $H^{3/2}(\partial B)^3$; cf. [32, Thm. 4.3.2]. So E^-_M is also continuous from $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)$ to $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)$; in particular, it is $\mathcal{O}(\delta^2)$ in $\mathcal{L}(\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B))$ as $\delta \to 0$. Thus, recalling the continuity properties of the operators $M^-_{D_\delta}$, M^0_B, and E^-_M and letting $j \to \infty$, we obtain

$$M^-_{D_\delta}\boldsymbol{a} = (M^0_B \hat{\boldsymbol{a}})^\vee + (E^-_M \hat{\boldsymbol{a}})^\vee.$$

Recalling (5.6) it remains to estimate the norm of $R^m_{D_\delta}\boldsymbol{a}$. For this purpose, we denote by $\tilde{R}^m_{D_\delta}\boldsymbol{a}$ the extension of $R^m_{D_\delta}$ to $\boldsymbol{H}(\mathbf{curl}, D_\delta)$ (with respect to the trace operator γ_t), which is obtained canonically from (5.5) via

$$\tilde{R}^m_{D_\delta}\boldsymbol{a} := \int_{\partial D_\delta} \mathbf{curl}_x F^m(\cdot, \boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) \qquad \text{in } D_\delta.$$

Then, because F^m is smooth near the center of the scatterer,

$$\begin{aligned}\|R^m_{D_\delta}\boldsymbol{a}\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)} &= \inf_{\boldsymbol{u}\in \boldsymbol{H}(\mathbf{curl},D_\delta),\, \gamma_t(\boldsymbol{u}) = R^m_{D_\delta}\boldsymbol{a}} \|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl},D_\delta)} \leq \|\tilde{R}^m_{D_\delta}\boldsymbol{a}\|^2_{\boldsymbol{H}(\mathbf{curl},D_\delta)}\\ &= \int_{D_\delta}\Big|\int_{\partial D_\delta} \mathbf{curl}_x F^m(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\Big|^2\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_{D_\delta}\Big|\mathbf{curl}_x \int_{\partial D_\delta} \mathbf{curl}_x F^m(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\Big|^2\,\mathrm{d}\boldsymbol{x}\\ &\leq \int_{D_\delta}\Big(\|\mathbf{curl}_x F^m(\boldsymbol{x},\cdot)\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)}\\ &\qquad + \|\mathbf{curl}_x\,\mathbf{curl}_x F^m(\boldsymbol{x},\cdot)\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)}\Big)\|\boldsymbol{a}\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)}\,\mathrm{d}\boldsymbol{x}\\ &\leq C\delta^3\|\boldsymbol{a}\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)}\int_{D_\delta} 1\,\mathrm{d}\boldsymbol{x} \leq C\delta^6\|\boldsymbol{a}\|^2_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)}.\end{aligned}$$

Using (6.2) we find

$$\|(R^m_{D_\delta}\boldsymbol{a})^\wedge\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)} \leq \delta^{-\frac{3}{2}}\|R^m_{D_\delta}\boldsymbol{a}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)} \leq C\delta^2\|\hat{\boldsymbol{a}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)}.$$

Thus, we define

$$E^m_M\boldsymbol{b} := E^-_M\boldsymbol{b} + (R^m_{D_\delta}\check{\boldsymbol{b}})^\wedge, \qquad \boldsymbol{b} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B),$$

and obtain the desired result. ∎

For $\boldsymbol{a} \in \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$ Lemma 6.2 yields

$$\Big(\frac{1}{2}I + M^m_{D_\delta}\Big)\boldsymbol{a} = \Big(\Big(\frac{1}{2}I + M^0_B + E^m_M\Big)\hat{\boldsymbol{a}}\Big)^\vee.$$

Thus,

$$\left(\frac{1}{2}I+M_{D_\delta}^m\right)^{-1}\boldsymbol{a}=\left(\left(\frac{1}{2}I+M_B^0+E_M^m\right)^{-1}\hat{\boldsymbol{a}}\right)^\vee=\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\hat{\boldsymbol{a}}\right)^\vee+(\tilde{E}_M^m\hat{\boldsymbol{a}})^\vee, \tag{6.5}$$

where $\tilde{E}_M^m$ is a bounded linear operator, which is $\mathcal{O}(\delta^2)$ in $\mathcal{L}(\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B))$ as $\delta\to 0$, independent of $\boldsymbol{a}$.

7. Asymptotic expansion. In this section we expand the three operators L_δ, $L_\delta^\top$, and F_δ occurring in the factorization (4.6) of the measurement operator G_δ separately as the inhomogeneity size δ tends to zero. Then, we use these expansions to calculate the leading order term in the asymptotic expansion of G_δ.

First, we consider the exterior Maxwell boundary value problem (4.1) and study the asymptotic behavior of the operator L_δ from (4.2). A radiating solution of this problem is given by

$$\begin{aligned}\boldsymbol{E}^\psi &:= \frac{\varepsilon_-}{\varepsilon}\mathbf{curl}\mathcal{A}_{D_\delta}^m\left(\frac{1}{2}I+M_{D_\delta}^m\right)^{-1}\boldsymbol{\psi} && \text{in } \mathbb{R}_0^3\setminus\overline{D_\delta},\\ \boldsymbol{H}^\psi &:= -\,\mathrm{i}\,\omega\varepsilon_-\int_{\partial D_\delta}\mathbb{G}^m(\cdot,\boldsymbol{y})\left(\left(\frac{1}{2}I+M_{D_\delta}^m\right)^{-1}\boldsymbol{\psi}\right)(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) && \text{in } \mathbb{R}_0^3\setminus\overline{D_\delta}.\end{aligned}$$

By Taylor expansion we obtain for $\boldsymbol{x}\in\mathcal{M}$, $\boldsymbol{z}\in\mathbb{R}_-^3$ with $\mathrm{dist}(\boldsymbol{z},\Sigma_0)\geq c_0$ for some constant $c_0>0$, and $\boldsymbol{\eta}\in\partial B$ as $\delta\to 0$ that

$$\mathbb{G}^m(\boldsymbol{x},\delta\boldsymbol{\eta}+\boldsymbol{z})=\mathbb{G}^m(\boldsymbol{x},\boldsymbol{z})+\delta\sum_{l=1}^3\frac{\partial\mathbb{G}^m}{\partial y_l}(\boldsymbol{x},\boldsymbol{z})\eta_l+\mathcal{O}(\delta^2).$$

Thus, by a change of coordinates, applying (6.5) we have

$$\begin{aligned}\boldsymbol{H}^\psi(\boldsymbol{x}) &= -\,\mathrm{i}\,\omega\varepsilon_-\delta^2\int_{\partial B}\mathbb{G}^m(\boldsymbol{x},\delta\boldsymbol{\eta}+\boldsymbol{z})\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\hat{\boldsymbol{\psi}}\right)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})+\mathcal{O}(\delta^4)\\ &= -\,\mathrm{i}\,\omega\varepsilon_-\delta^2\mathbb{G}^m(\boldsymbol{x},\boldsymbol{z})\int_{\partial B}\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\hat{\boldsymbol{\psi}}\right)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\ &\quad -\,\mathrm{i}\,\omega\varepsilon_-\delta^3\int_{\partial B}\sum_{l=1}^3\eta_l\frac{\partial\mathbb{G}^m}{\partial y_l}(\boldsymbol{x},\boldsymbol{z})\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\hat{\boldsymbol{\psi}}\right)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})+\mathcal{O}(\delta^4)\end{aligned}$$

for $\boldsymbol{x}\in\mathcal{M}$ as $\delta\to 0$. The last term on the right-hand side is bounded by $C\delta^4\|\hat{\boldsymbol{\psi}}\|_{\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)}$, where the constant C is independent of δ and $\boldsymbol{\psi}$, uniformly for $\boldsymbol{x}\in\mathcal{M}$. We define $L_0:\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)\to\boldsymbol{L}^2(\mathcal{M})$,

$$L_0\boldsymbol{a}:=-\,\mathrm{i}\,\omega\varepsilon_-\mathbb{G}^m(\cdot,\boldsymbol{z})\int_{\partial B}\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\boldsymbol{a}\right)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}), \tag{7.1}$$

and $L_1:\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)\to\boldsymbol{L}^2(\mathcal{M})$,

$$L_1\boldsymbol{a}:=-\,\mathrm{i}\,\omega\varepsilon_-\int_{\partial B}\sum_{l=1}^3\eta_l\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\left(\left(\frac{1}{2}I+M_B^0\right)^{-1}\boldsymbol{a}\right)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}). \tag{7.2}$$

Then L_0 and L_1 are bounded linear operators, and we have the following asymptotic behavior.

PROPOSITION 7.1. *For all* $\boldsymbol{\psi} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D_\delta)$,

$$L_\delta \boldsymbol{\psi} = \delta^2 L_0 \hat{\boldsymbol{\psi}} + \delta^3 L_1 \hat{\boldsymbol{\psi}} + E_L \hat{\boldsymbol{\psi}}, \tag{7.3}$$

where E_L *is a bounded linear operator, which is* $\mathcal{O}(\delta^4)$ *in* $\mathcal{L}(\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B), \boldsymbol{L}^2(\mathcal{M}))$ *as* $\delta \to 0$, *independent of* $\boldsymbol{\psi}$.

Next we consider the asymptotic behavior of the operator $L_\delta^\top$ from (4.3). Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ and $\boldsymbol{\psi} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D_\delta)$. For $X \in \{B, D_\delta\}$ we denote by $\langle \cdot, \cdot \rangle_{\partial X}$ the duality pairing between $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial X)$ and $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial X)$. Using (7.3) we obtain

$$\begin{aligned} \langle L_\delta^\top \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\partial D_\delta} &= \langle \boldsymbol{\varphi}, L_\delta \boldsymbol{\psi} \rangle_{\mathcal{M}} = \langle \delta^2 L_0^\top \boldsymbol{\varphi} + \delta^3 L_1^\top \boldsymbol{\varphi} + E_L^\top \boldsymbol{\varphi}, \hat{\boldsymbol{\psi}} \rangle_{\partial B} \\ &= \langle (L_0^\top \boldsymbol{\varphi})^\vee + \delta (L_1^\top \boldsymbol{\varphi})^\vee + \delta^{-2} (E_L^\top \boldsymbol{\varphi})^\vee, \boldsymbol{\psi} \rangle_{\partial D_\delta}, \end{aligned}$$

where $L_0^\top, L_1^\top, E_L^\top : \boldsymbol{L}^2(\mathcal{M}) \to \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B)$ are the dual operators of L_0, L_1, and E_L, respectively. Because by duality $E_L^\top$ is $\mathcal{O}(\delta^4)$ in $\mathcal{L}(\boldsymbol{L}^2(\mathcal{M}), \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B))$, we obtain the following asymptotic behavior.

PROPOSITION 7.2. *For all* $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$,

$$L_\delta^\top \boldsymbol{\varphi} = (L_0^\top \boldsymbol{\varphi})^\vee + \delta (L_1^\top \boldsymbol{\varphi})^\vee + \delta^{-2} (E_L^\top \boldsymbol{\varphi})^\vee,$$

where $E_L^\top$ *is a bounded linear operator, which is* $\mathcal{O}(\delta^4)$ *in* $\mathcal{L}(\boldsymbol{L}^2(\mathcal{M}), \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B))$ *as* $\delta \to 0$, *independent of* $\boldsymbol{\varphi}$.

Now we calculate the operators $L_0^\top$ and $L_1^\top$ explicitly. Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ and $\boldsymbol{a} \in \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)$. Recalling the definition of the operator L_0 from (7.1) we find

$$\begin{aligned} \langle \boldsymbol{\varphi}, L_0 \boldsymbol{a} \rangle_{\mathcal{M}} &= \int_{\mathcal{M}} \Big(-\mathrm{i}\,\omega \varepsilon_- \mathbb{G}^m(\boldsymbol{x}, \boldsymbol{z}) \int_{\partial B} \Big(\Big(\frac{1}{2} I + M_B^0 \Big)^{-1} \boldsymbol{a} \Big)(\boldsymbol{\eta}) \,\mathrm{d}s(\boldsymbol{\eta}) \Big) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) \,\mathrm{d}s(\boldsymbol{x}) \\ &= \Big(-\mathrm{i}\,\omega \varepsilon_- \int_{\mathcal{M}} \mathbb{G}^{m\top}(\boldsymbol{x}, \boldsymbol{z}) \boldsymbol{\varphi}(\boldsymbol{x}) \,\mathrm{d}s(\boldsymbol{x}) \Big) \cdot \int_{\partial B} \Big(\Big(\frac{1}{2} I + M_B^0 \Big)^{-1} \boldsymbol{a} \Big)(\boldsymbol{\eta}) \,\mathrm{d}s(\boldsymbol{\eta}). \end{aligned}$$

Recalling (3.4b) and (3.5), we obtain

$$\begin{aligned} \langle \boldsymbol{\varphi}, L_0 \boldsymbol{a} \rangle_{\mathcal{M}} &= \frac{1}{\mathrm{i}\,\omega \mu_+} \boldsymbol{H}^i(\boldsymbol{z}) \cdot \int_{\partial B} \Big(\Big(\frac{1}{2} I + M_B^0 \Big)^{-1} \boldsymbol{a} \Big)(\boldsymbol{\eta}) \,\mathrm{d}s(\boldsymbol{\eta}) \\ &= \int_{\partial B} \frac{1}{\mathrm{i}\,\omega \mu_+} \Big(\Big(\frac{1}{2} I + M_B^{0\,\top} \Big)^{-1} \pi_t \big(\boldsymbol{H}^i(\boldsymbol{z}) \big) \Big)(\boldsymbol{\xi}) \cdot \boldsymbol{a}(\boldsymbol{\xi}) \,\mathrm{d}s(\boldsymbol{\xi}), \end{aligned}$$

where π_t denotes the projection on the tangent plane to ∂B. Therefore, we have

$$L_0^\top \boldsymbol{\varphi} = \frac{1}{\mathrm{i}\,\omega \mu_+} \Big(\frac{1}{2} I + M_B^{0\,\top} \Big)^{-1} \pi_t (\boldsymbol{H}^i(\boldsymbol{z})). \tag{7.4}$$

In the same way we obtain from (7.2) that

$$\langle \boldsymbol{\varphi}, L_1 \boldsymbol{a} \rangle_{\mathcal{M}} = \int_{\partial B} \frac{1}{\mathrm{i}\,\omega \mu_+} \Big(\Big(\frac{1}{2} I + M_B^{0\,\top} \Big)^{-1} \pi_t \Big(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z}) \Big) \Big)(\boldsymbol{\xi}) \cdot \boldsymbol{a}(\boldsymbol{\xi}) \,\mathrm{d}s(\boldsymbol{\xi}).$$

Here, η_l denotes the lth component of the surface variable on ∂B. Thus we have

$$L_1^\top \boldsymbol{\varphi} = \frac{1}{\mathrm{i}\,\omega \mu_+} \Big(\frac{1}{2} I + M_B^{0\,\top} \Big)^{-1} \pi_t \Big(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z}) \Big). \tag{7.5}$$

We return to the diffraction problem (4.4) and the operator F_δ from (4.5). Given $\boldsymbol{\chi} \in \boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)$, we define

$$\boldsymbol{E}^d := -\frac{1}{\mathrm{i}\,\omega\varepsilon}\mathbf{curl}\frac{\mu_-}{\mu}\mathbf{curl}\mathcal{A}^e_{D_\delta}(\boldsymbol{\nu}\times\boldsymbol{\chi}) \qquad \text{in } \mathbb{R}^3_0 \setminus \partial D_\delta,$$

$$\boldsymbol{H}^d := \frac{\mu_-}{\mu}\mathbf{curl}\mathcal{A}^e_{D_\delta}(\boldsymbol{\nu}\times\boldsymbol{\chi}) \qquad \text{in } \mathbb{R}^3_0 \setminus \partial D_\delta.$$

Then $(\boldsymbol{E}^d, \boldsymbol{H}^d)$ is a radiating solution to (4.4), and recalling (3.3), (5.1), and (2.1) we find

$$\begin{aligned}
\boldsymbol{\nu}\times\boldsymbol{E}^d\big|_{\partial D_\delta}(\boldsymbol{x}) &= -\frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}_x\,\mathbf{curl}_x\int_{\partial D_\delta}\Pi^e(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{\nu}\times\boldsymbol{\chi})(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\\
&= \mathrm{i}\,\omega\mu_-\boldsymbol{\nu}(\boldsymbol{x})\times\int_{\partial D_\delta}\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\nu}\times\boldsymbol{\chi})(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\\
&\quad+\frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}(\boldsymbol{x})\times\int_{\partial D_\delta}\nabla_x\Phi_{k_-}(\boldsymbol{x}-\boldsymbol{y})(\mathrm{curl}_{\partial D_\delta}\boldsymbol{\chi})(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\\
&\quad-\frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}(\boldsymbol{x})\times\mathbf{curl}_x\,\mathbf{curl}_x\int_{\partial D_\delta}F^e(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{\nu}\times\boldsymbol{\chi})(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})\,.
\end{aligned}$$

Remark 7.3. The previous formula employs a slight abuse of notation, because pointwise evaluation is not defined for elements of $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$. However, we included the argument for better readability.

Define $P_{D_\delta} : \boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta) \to \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial D_\delta)$ by

$$P_{D_\delta}\boldsymbol{a} := -\frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}\times\mathbf{curl}\,\mathbf{curl}\int_{\partial D_\delta}F^e(\cdot,\boldsymbol{y})(\boldsymbol{\nu}\times\boldsymbol{a})(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y})$$

for $\boldsymbol{a} \in \boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial D_\delta)$. Then we can see as in the proof of Lemma 6.2 that

$$\|(P_{D_\delta}\boldsymbol{a})^\wedge\|_{\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)} \le C\delta^2\|\hat{\boldsymbol{a}}\|_{\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial B)}.$$

Therefore, by change of coordinates, applying (6.3) and (6.4), we obtain

$$\begin{aligned}
(\boldsymbol{\nu}\times\boldsymbol{E}^d\big|_{\partial D_\delta})^\wedge(\boldsymbol{\xi}) &= \delta^{-1}\frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}(\boldsymbol{\xi})\times\int_{\partial B}\nabla_{\boldsymbol{\xi}}\Phi_0(\boldsymbol{\xi}-\boldsymbol{\eta})(\mathrm{curl}_{\partial B}\,\hat{\boldsymbol{\chi}})(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\
&\quad+\delta\,\mathrm{i}\,\omega\mu_-\boldsymbol{\nu}(\boldsymbol{\xi})\times\int_{\partial B}\Phi_0(\boldsymbol{\xi}-\boldsymbol{\eta})(\boldsymbol{\nu}\times\hat{\boldsymbol{\chi}})(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\
&\quad+\delta\,\mathrm{i}\,\omega\mu_-\boldsymbol{\nu}(\boldsymbol{\xi})\times\int_{\partial B}\frac{1}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}(\mathrm{curl}_{\partial B}\,\hat{\boldsymbol{\chi}})(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})+\mathcal{O}(\delta^2)
\end{aligned}$$

as $\delta \to 0$. The $\mathcal{O}(\delta^2)$-term in (6.3) and the $\mathcal{O}(\delta^3)$-term in (6.4) define pseudo-homogeneous kernels of class -3 (cf. [32, pp. 168–174]); i.e., the corresponding integral operators are continuous from $H^{-1/2}(\partial B)^3$ into $H^{5/2}(\partial B)^3$ (cf. [32, Thm. 4.4.1]). Thus, these operators are also continuous from $\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial B)$ into $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)$, and together with the (constant) $\mathcal{O}(\delta)$-term in (6.3) they lead to terms of order $\mathcal{O}(\delta^2)$ in $\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)$ in the asymptotic expansion of $\boldsymbol{\nu}\times\boldsymbol{E}^d|_{\partial D_\delta}$ as $\delta \to 0$.

We define $F_0 : \boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\partial B) \to \boldsymbol{H}^{-1/2}_{\mathrm{div}}(\partial B)$,

$$(F_0\boldsymbol{a})(\boldsymbol{\xi}) := \frac{1}{\mathrm{i}\,\omega\varepsilon_-}\boldsymbol{\nu}(\boldsymbol{\xi})\times\int_{\partial B}\nabla_{\boldsymbol{\xi}}\Phi_0(\boldsymbol{\xi}-\boldsymbol{\eta})(\mathrm{curl}_{\partial B}\,\boldsymbol{a})(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}), \tag{7.6}$$

and $F_1:\ \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B) \to \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)$,

$$(7.7)\quad (F_1\boldsymbol{a})(\boldsymbol{\xi}) := \mathrm{i}\,\omega\mu_-\big(\boldsymbol{\nu}\times\mathcal{A}_B^0(\boldsymbol{\nu}\times\boldsymbol{a})\big|_{\partial B}\big)(\boldsymbol{\xi}) + \mathrm{i}\,\omega\mu_-\boldsymbol{\nu}(\boldsymbol{\xi})\times\int_{\partial B}\frac{1}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}(\mathrm{curl}_{\partial B}\,\boldsymbol{a})(\boldsymbol{\eta})\ \mathrm{d}s(\boldsymbol{\eta}).$$

Note that $-\mathrm{i}\,\omega\varepsilon_- F_0 = N_B^0$. Thus, F_0 and the first part of F_1 are bounded. Because the kernel of the second part of F_1 is homogeneous of class -2 (cf. [32, sec. 4.3.2]), the second part of F_1 is continuous also. We obtain the following asymptotic behavior.

PROPOSITION 7.4. *For all $\boldsymbol{\chi}\in\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D_\delta)$,*

$$F_\delta\boldsymbol{\chi} = \delta^{-1}(F_0\hat{\boldsymbol{\chi}})^\vee + \delta(F_1\hat{\boldsymbol{\chi}})^\vee + (E_F\hat{\boldsymbol{\chi}})^\vee,$$

where E_F is a bounded linear operator, which is $\mathcal{O}(\delta^2)$ in $\mathcal{L}(\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B), \boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B))$ as $\delta\to 0$, independent of $\boldsymbol{\chi}$.

Next we consider the boundary value problem of finding $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl}, B)$ such that

$$(7.8a)\qquad \mathbf{curl}\,\mathbf{curl}\boldsymbol{u} = 0 \qquad \text{in } B,$$

$$(7.8b)\qquad \mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in } B,$$

$$(7.8c)\qquad \boldsymbol{\nu}\times\boldsymbol{u} = \boldsymbol{c} \qquad \text{on } \partial B,$$

where $\boldsymbol{c}\in\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)$ is a given tangential function. We show that (7.8) has at most one solution and use this fact to prove that $F_0 L_0^\top = 0$ on $\boldsymbol{L}^2(\mathcal{M})$ and $L_0 F_0 = 0$ on $\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B)$.

LEMMA 7.5. *Let $\boldsymbol{c}\in\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B)$. Then the boundary value problem* (7.8) *has at most one solution in $\boldsymbol{H}(\mathbf{curl}, B)$.*

Proof. Using integration by parts we find for any solution $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl}, B)$ of (7.8) with homogeneous boundary condition $\boldsymbol{c}=0$ that

$$0 = \int_B \mathbf{curl}\,\mathbf{curl}\boldsymbol{u}(\boldsymbol{x})\cdot\overline{\boldsymbol{u}(\boldsymbol{x})}\ \mathrm{d}\boldsymbol{x} = \int_B |\mathbf{curl}\boldsymbol{u}(\boldsymbol{x})|^2\ \mathrm{d}\boldsymbol{x} + \big\langle\gamma_t(\mathbf{curl}\boldsymbol{u}), \overline{\pi_t(\boldsymbol{u})}\big\rangle_{\partial B} = \int_B |\mathbf{curl}\boldsymbol{u}(\boldsymbol{x})|^2\ \mathrm{d}\boldsymbol{x}.$$

Hence, $\mathbf{curl}\boldsymbol{u} = 0$ in B, and because the boundaries of all components of B are assumed to be connected, we obtain from [31, Thms. 3.41 and 3.42] a scalar potential $p\in H^1(B)$ with $\gamma_0(p)=0$ on ∂B such that $\boldsymbol{u}=\nabla p$. Finally, because $\Delta p = \mathrm{div}\,\boldsymbol{u} = 0$ in B by (7.8b), we have $p=0$ in B. Hence, $\boldsymbol{u}=\nabla p = 0$ in B. ◻

PROPOSITION 7.6. *Let $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$ and $\boldsymbol{a}\in\boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B)$. Then $F_0 L_0^\top\boldsymbol{\varphi} = 0$ and $L_0 F_0\boldsymbol{a} = 0$.*

Proof. Given $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$, by (7.4) and (7.6) we find that on ∂B

$$F_0 L_0^\top\boldsymbol{\varphi} = \frac{1}{\omega^2\varepsilon_-\mu_+}\boldsymbol{\nu}\times\mathbf{curl}\,\mathbf{curl}\mathcal{A}_B^0\Big(\boldsymbol{\nu}\times\Big(\frac{1}{2}I + {M_B^0}^\top\Big)^{-1}\pi_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big),$$

where $\boldsymbol{H}^i(\boldsymbol{z})$ is given by (3.5). An easy computation applying (5.2) shows that

$$(7.9)\qquad \pm\boldsymbol{\nu}\times\Big(\pm\frac{1}{2}I + {M_B^0}^\top\Big)^{-1}\pi_t(\cdot) = \mp\Big(\mp\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t(\cdot)\,.$$

Therefore,

$$F_0 L_0^\top \boldsymbol{\varphi} = -\frac{1}{\omega^2 \varepsilon_- \mu_+} \boldsymbol{\nu} \times \mathbf{curl}\,\mathbf{curl} \mathcal{A}_B^0 \Big(-\frac{1}{2} I + M_B^0\Big)^{-1} \gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big).$$

Now let

$$\boldsymbol{u} := \mathbf{curl} \mathcal{A}_B^0 \Big(-\frac{1}{2} I + M_B^0\Big)^{-1} \gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big) \qquad \text{in } B;$$

then $\boldsymbol{u}$ is a solution to (7.8) with $\boldsymbol{c} = \gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)$. From Lemma 7.5 we obtain that $\boldsymbol{u} = \boldsymbol{H}^i(\boldsymbol{z})$ is constant in B. Hence,

$$F_0 L_0^\top \boldsymbol{\varphi} = -\frac{1}{\omega^2 \varepsilon_- \mu_+} \gamma_t(\mathbf{curl} \boldsymbol{u}) = 0 \qquad \text{on } \partial B.$$

Because F_0 is symmetric, also $L_0 F_0 \boldsymbol{a} = 0$ for each $\boldsymbol{a} \in \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial B)$. ∎

Recalling Theorem 4.2, we can we put our results together and obtain the following asymptotic expansion of the measurement operator G_δ.

THEOREM 7.7. *Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$; then*

$$G_\delta \boldsymbol{\varphi} = \mathrm{i}\,\omega \mu_+ \delta^3 \left(L_0 F_1 L_0^\top \boldsymbol{\varphi} + L_1 F_0 L_1^\top \boldsymbol{\varphi}\right) + \mathcal{O}(\delta^4)$$

in $\boldsymbol{L}^2(\mathcal{M})$ as $\delta \to 0$. More precisely, the last term on the right-hand side is bounded by $C\delta^4 \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\mathcal{M})}$, where the constant C is independent of δ and $\boldsymbol{\varphi}$.

The proof of this theorem follows straightforwardly from the previous propositions and Theorem 4.2. We refer the reader to [1, Thm. 5.9] for a similar proof in the electrostatic case.

Finally, for $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$, we are going to calculate $L_0 F_1 L_0^\top \boldsymbol{\varphi}$ and $L_1 F_0 L_1^\top \boldsymbol{\varphi}$ explicitly.

LEMMA 7.8. *For each $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ we have*

$$F_0 L_1^\top \boldsymbol{\varphi} = -\frac{1}{\omega^2 \varepsilon_- \mu_+} \gamma_t\big(\mathbf{curl} \boldsymbol{H}^i(\boldsymbol{z})\big) \qquad \text{on } \partial B. \tag{7.10}$$

Proof. Given $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$, by (7.5) and (7.6), applying (7.9) we find

$$\begin{aligned} F_0 L_1^\top \boldsymbol{\varphi} &= \frac{1}{\omega^2 \varepsilon_- \mu_+} \boldsymbol{\nu} \times \mathbf{curl}\,\mathbf{curl} \mathcal{A}_B^0 \left(\boldsymbol{\nu} \times \Big(\frac{1}{2} I + M_B^{0\,\top}\Big)^{-1} \pi_t \left(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z})\right)\right) \\ &= -\frac{1}{\omega^2 \varepsilon_- \mu_+} \boldsymbol{\nu} \times \mathbf{curl}\,\mathbf{curl} \mathcal{A}_B^0 \Big(-\frac{1}{2} I + M_B^0\Big)^{-1} \gamma_t \left(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z})\right). \end{aligned}$$

Let

$$\boldsymbol{u} := \mathbf{curl} \mathcal{A}_B^0 \Big(-\frac{1}{2} I + M_B^0\Big)^{-1} \gamma_t \left(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z})\right) \qquad \text{in } B;$$

then $\boldsymbol{u}$ is a solution to (7.8) with $\boldsymbol{c} = \gamma_t(\sum_{l=1}^3 \eta_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z}))$. From Lemma 7.5 we obtain $\boldsymbol{u}(\boldsymbol{\xi}) = \sum_{l=1}^3 \xi_l \frac{\partial \boldsymbol{H}^i}{\partial y_l}(\boldsymbol{z})$ for a.e. $\boldsymbol{\xi} \in B$. An easy calculation shows that therefore $\mathbf{curl} \boldsymbol{u} = \mathbf{curl} \boldsymbol{H}^i(\boldsymbol{z})$ in B, which ends the proof. ∎

LEMMA 7.9. *For each $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ we have*

$$F_1 L_0^\top \boldsymbol{\varphi} = -\frac{\mu_-}{\mu_+}\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big) \qquad \text{on } \partial B. \tag{7.11}$$

Proof. Given $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$, by (7.4) and (7.7), applying (2.1) and (7.9) we find that on ∂B

$$\begin{aligned}(F_1 L_0^\top \boldsymbol{\varphi})(\boldsymbol{\xi}) &= -\frac{\mu_-}{\mu_+}\Big(\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\xi})\\ &+\frac{\mu_-}{\mu_+}\boldsymbol{\nu}(\boldsymbol{\xi}) \times \int_{\partial B}\frac{1}{8\pi}\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}\Big(\operatorname{div}_{\partial B}\Big(\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}).\end{aligned} \tag{7.12}$$

By Lemma 5.1(a), $-\frac{1}{2}I + M_B^0$ is an isomorphism on $\boldsymbol{H}^{-1/2}_{\mathrm{div},0}(\partial B)$. Therefore, because by (2.3) it holds that $\operatorname{div}_{\partial B}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big) = -\gamma_n\big(\mathbf{curl}\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\big) = 0$, we find

$$\operatorname{div}_{\partial B}\Big(\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big) = 0.$$

Hence, the second term on the right-hand side of (7.12) vanishes, and we obtain the desired result. □

DEFINITION 7.10. *For a bounded $C^{2,\alpha}$ domain $D \subset \mathbb{R}^3$ we define the magnetic polarizability tensor $\mathbb{M}_D^0 \in \mathbb{R}^{3\times 3}$ by $\mathbb{M}_D^0 := (m_{ij}^0)_{i,j=1}^3$ with*

$$m_{ij}^0 := -\int_{\partial D}\eta_j\Big(\Big(-\frac{1}{2}I + {K_D^0}^\top\Big)^{-1}\nu_i\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}), \qquad 1 \le i,j \le 3.$$

The electric polarizability tensor $\mathbb{M}_D^\infty \in \mathbb{R}^{3\times 3}$ corresponding to the domain D is given by $\mathbb{M}_D^\infty := (m_{ij}^\infty)_{i,j=1}^3$ with

$$m_{ij}^\infty := \int_{\partial D}\eta_j\Big(\Big(\frac{1}{2}I + {K_D^0}^\top\Big)^{-1}\nu_i\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}), \qquad 1 \le i,j \le 3.$$

The magnetic and the electric polarizability tensor are symmetric and positive definite matrices; cf. [3, 19].

PROPOSITION 7.11. *For each $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$ we have*

$$L_1 F_0 L_1^\top \boldsymbol{\varphi} = \frac{1}{\mathrm{i}\,\omega\mu_+}\frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z})\mathbb{M}_B^\infty\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}) \qquad \text{on } \mathcal{M}.$$

Proof. Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$. By (7.10) and (7.2),

$$L_1 F_0 L_1^\top \boldsymbol{\varphi} = -\frac{1}{\mathrm{i}\,\omega\mu_+}\int_{\partial B}\sum_{l=1}^3\eta_l\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\Big(\Big(\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$

on $\mathcal{M}$. Applying (7.9), we find

$$\begin{aligned}&-\int_{\partial B}\sum_{l=1}^3\eta_l\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\Big(\Big(\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\ &\quad= \sum_{l=1}^3\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\int_{\partial B}\eta_l\,\mathbb{I}_3\Big(\boldsymbol{\nu}\times\Big(-\frac{1}{2}I + {M_B^0}^\top\Big)^{-1}\pi_t\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\end{aligned}$$

on $\mathcal{M}$. Because by (2.2)

$$\pi_t\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\big) = \pi_t\big(\nabla_\eta\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\cdot\boldsymbol{\eta}\big)\big) = \nabla_{\partial B}\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\cdot\boldsymbol{\eta}\big)$$

on ∂B, where $\boldsymbol{\eta}$ denotes the coordinate function in a neighborhood of ∂B, we can apply (5.3) and (2.2) to obtain for $1 \le l \le 3$ that

$$\begin{aligned}\int_{\partial B} \eta_l\, \mathbb{I}_3\Big(\boldsymbol{\nu}\times\Big(-\frac{1}{2}I + M_B^{0\,\top}\Big)^{-1}\pi_t\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\ = \int_{\partial B}\pi_t(\eta_l\,\mathbb{I}_3)^\top\Big(\mathbf{curl}_{\partial B}\Big(\frac{1}{2}I + K_B^0\Big)^{-1}\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\cdot\boldsymbol{\eta}\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}).\end{aligned}$$

From the duality of $\mathbf{curl}_{\partial B}$ and $\mathrm{curl}_{\partial B}$, and from (2.3), we find

$$\begin{aligned}\int_{\partial B}\pi_t(\eta_l\,\mathbb{I}_3)^\top\Big(\mathbf{curl}_{\partial B}\Big(\frac{1}{2}I + K_B^0\Big)^{-1}\big(\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\cdot\boldsymbol{\eta}\big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})\\ = \int_{\partial B}\big(\boldsymbol{\nu}\cdot\mathbf{curl}(\eta_l\,\mathbb{I}_3)\big)^\top(\boldsymbol{\xi})\left(\Big(\Big(\frac{1}{2}I + K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\xi})\cdot\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\right)\mathrm{d}s(\boldsymbol{\xi}).\end{aligned}$$

An easy calculation reveals

$$\sum_{l=1}^3\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\mathbf{curl}_\eta(\eta_l\,\mathbb{I}_3)^\top = \big(\mathbf{curl}_y\mathbb{G}^{m\top}\big)^\top(\cdot,\boldsymbol{z}).$$

Applying (3.4b) and (3.4c), we observe

$$\big(\mathbf{curl}_y\mathbb{G}^{m\top}\big)^\top(\cdot,\boldsymbol{z}) = \frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z}).$$

So, we find

$$\begin{aligned}\sum_{l=1}^3\frac{\partial\mathbb{G}^m}{\partial y_l}(\cdot,\boldsymbol{z})\int_{\partial B}\big(\boldsymbol{\nu}\cdot\mathbf{curl}(\eta_l\,\mathbb{I}_3)\big)^\top(\boldsymbol{\xi})\left(\Big(\Big(\frac{1}{2}I + K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\xi})\cdot\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\right)\mathrm{d}s(\boldsymbol{\xi})\\ =\frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z})\int_{\partial B}\boldsymbol{\nu}(\boldsymbol{\xi})\left(\Big(\Big(\frac{1}{2}I + K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\xi})\cdot\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\right)\mathrm{d}s(\boldsymbol{\xi})\\ =\frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z})\mathbb{M}_B^\infty\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}). \qquad \square\end{aligned}$$

PROPOSITION 7.12. *For each $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$ we have*

$$L_0F_1L_0^\top\boldsymbol{\varphi} = \mathrm{i}\,\omega\varepsilon_-\frac{\mu_-}{\mu_+}\mathbb{G}^m(\cdot,\boldsymbol{z})\mathbb{M}_B^0\boldsymbol{H}^i(\boldsymbol{z}).$$

Proof. Let $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$ and set

$$\boldsymbol{u} := \mathbf{curl}\mathcal{A}_B^0\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(z)\big) \qquad \text{in } B.$$

As in the proof of Proposition 7.6 we find that $\boldsymbol{u} = \boldsymbol{H}^i(\boldsymbol{z})$ in B. So we obtain

$$\gamma_n\Big(\mathbf{curl}\mathcal{A}_B^0\Big(-\frac{1}{2}I + M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(z)\big)\Big) = \gamma_n\big(\boldsymbol{H}^i(\boldsymbol{z})\big) \qquad \text{on } \partial B. \tag{7.13}$$

By (7.11) and (7.1),

$$L_0 F_1 L_0^\top \boldsymbol{\varphi} = \mathrm{i}\,\omega\varepsilon_- \frac{\mu_-}{\mu_+}\mathbb{G}^m(\cdot,\boldsymbol{z})\int_{\partial B}\Big(\Big(\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$

on $\mathcal{M}$. Observing that $\pi_t(\mathbb{I}_3) = \nabla_{\partial B}\boldsymbol{\eta}$ on ∂B, where $\boldsymbol{\eta}$ again denotes the surface variable on ∂B, and applying (2.2), we can calculate

$$\int_{\partial B}\Big(\Big(\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$
$$=\int_{\partial B}\Big(\Big(\frac{1}{2}I+{M_B^0}^\top\Big)^{-1}\nabla_{\partial B}\boldsymbol{\eta}\Big)^\top(\boldsymbol{\xi})\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\xi})\,\mathrm{d}s(\boldsymbol{\xi}).$$

Applying (5.3), the duality of $-\nabla_{\partial B}$ and $\mathrm{div}_{\partial B}$, and (2.3), we have

$$\int_{\partial B}\Big(\Big(\frac{1}{2}I+{M_B^0}^\top\Big)^{-1}\nabla_{\partial B}\boldsymbol{\eta}\Big)^\top(\boldsymbol{\xi})\gamma_t\Big(\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)(\boldsymbol{\xi})\,\mathrm{d}s(\boldsymbol{\xi})$$
$$=\int_{\partial B}\Big(\Big(-\frac{1}{2}I+K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\eta})\Big(-\gamma_n\Big(\mathbf{curl}\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}).$$

Finally, recalling (7.13), we obtain

$$\int_{\partial B}\Big(\Big(-\frac{1}{2}I+K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\eta})\Big(-\gamma_n\Big(\mathbf{curl}\mathcal{A}_B^0\Big(-\frac{1}{2}I+M_B^0\Big)^{-1}\gamma_t\big(\boldsymbol{H}^i(\boldsymbol{z})\big)\Big)\Big)(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$
$$=-\int_{\partial B}\Big(\Big(-\frac{1}{2}I+K_B^0\Big)^{-1}\boldsymbol{\eta}\Big)(\boldsymbol{\eta})\big(\boldsymbol{\nu}(\boldsymbol{\eta})\cdot\boldsymbol{H}^i(\boldsymbol{z})\big)\,\mathrm{d}s(\boldsymbol{\eta}) = \mathbb{M}_B^0\boldsymbol{H}^i(\boldsymbol{z}). \qquad \square$$

From Theorem 7.7 and Propositions 7.11 and 7.12 we obtain the following corollary.

Corollary 7.13. *Let* $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$, *and let* $\boldsymbol{H}^i$ *be the corresponding incident field from* (3.5). *Then*

$$G_\delta\boldsymbol{\varphi} = \delta^3\Big(-k_-^2\mathbb{G}^m(\cdot,\boldsymbol{z})\mathbb{M}_B^0\boldsymbol{H}^i(\boldsymbol{z}) + \frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z})\mathbb{M}_B^\infty\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z})\Big) + \mathcal{O}(\delta^4)$$

in $\boldsymbol{L}^2(\mathcal{M})$, *as* $\delta\to 0$. *More precisely, the last term on the right-hand side is bounded by* $C\delta^4\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\mathcal{M})}$, *where the constant* C *is independent of* δ *and* $\boldsymbol{\varphi}$.

8. Multiple scatterers. The results of the previous sections can be extended to the practically important case of finitely many well-separated small scatterers as introduced in section 3. This generalization works in the same way as we did in [1, section 6] for the electrostatic case. Because the calculations are rather technical and no new ideas are needed, we just mention the final result and leave the details to the reader.

Let $\mathbb{M}_{B_1}^0,\dots,\mathbb{M}_{B_m}^0$ and $\mathbb{M}_{B_1}^\infty,\dots,\mathbb{M}_{B_m}^\infty$ denote the magnetic and electric polarizability tensors corresponding to $B_1,\dots,B_m$, respectively. In case of multiple scatterers Corollary 7.13 reads as follows.

Corollary 8.1. *Let* $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\mathcal{M})$, *and let* $\boldsymbol{H}^i$ *be the corresponding incident field from* (3.5). *Then*

(8.1)
$$G_\delta\boldsymbol{\varphi} = \delta^3\sum_{l=1}^m\Big(-k_-^2\mathbb{G}^m(\cdot,\boldsymbol{z}_l)\mathbb{M}_{B_l}^0\boldsymbol{H}^i(\boldsymbol{z}_l) + \frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot,\boldsymbol{z}_l)\mathbb{M}_{B_l}^\infty\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}_l)\Big) + \mathcal{O}(\delta^4)$$

in $\boldsymbol{L}^2(\mathcal{M})$*, as* $\delta \to 0$*. More precisely, the last term on the right-hand side is bounded by* $C\delta^4\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\mathcal{M})}$*, where the constant* C *is independent of* δ *and* $\boldsymbol{\varphi}$*.*

9. A characterization of the scatterers. Using the asymptotic formula (8.1) we can now derive a characterization of the centers of the scatterers $\boldsymbol{z}_1, \dots, \boldsymbol{z}_l$ using a range criterion. For this purpose we introduce the operator $T : \boldsymbol{L}^2(\mathcal{M}) \to \boldsymbol{L}^2(\mathcal{M})$ describing the leading order term in the asymptotic expansion (8.1), given by

$$T\boldsymbol{\varphi} := \sum_{l=1}^{m} \Big(-k_-^2 \,\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\mathbb{M}^0_{B_l}\boldsymbol{H}^i(\boldsymbol{z}_l) + \frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_l)\mathbb{M}^\infty_{B_l}\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}_l)\Big). \tag{9.1}$$

Because (3.5) implies that $\boldsymbol{H}^i$ depends linearly on $\boldsymbol{\varphi}$, it follows that T is linear. From Corollary 8.1 we obtain

$$G_\delta = \delta^3 T + \mathcal{O}(\delta^4) \tag{9.2}$$

as $\delta \to 0$ in $\mathcal{L}(\boldsymbol{L}^2(\mathcal{M}))$. Next we define the operator $R:\ \mathbb{C}^{3\times 2m} \to \boldsymbol{L}^2(\mathcal{M})$:

$$R\boldsymbol{a} := k_-^2 \sum_{l=1}^{m} \Big(\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_l + \frac{\mu_-}{\mu_+}\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_{m+l}\Big) \tag{9.3}$$

for $\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2m}) \in \mathbb{C}^{3\times 2m}$, $\boldsymbol{a}_l \in \mathbb{C}^3$. Endowing $\mathbb{C}^{3\times 2m}$ with the bilinear form $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathbb{C}^{3\times 2m}} := \sum_{l=1}^{2m} \boldsymbol{a}_l \cdot \boldsymbol{b}_l$ for $\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2m})$ $\boldsymbol{b} = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_{2m}) \in \mathbb{C}^{3\times 2m}$ with $\boldsymbol{a}_l,\ \boldsymbol{b}_l \in \mathbb{C}^3$, using (3.5), (3.4b), and (3.4c), we obtain

$$\begin{aligned}
\langle R\boldsymbol{a}, \boldsymbol{\varphi}\rangle_{\mathcal{M}} &= \sum_{l=1}^{m} \boldsymbol{a}_l \cdot k_-^2 \int_{\mathcal{M}} \mathbb{G}^{m\top}(\boldsymbol{x}, \boldsymbol{z}_l)\boldsymbol{\varphi}(\boldsymbol{x})\ \mathrm{d}s(\boldsymbol{x}) \\
&\quad + \sum_{l=1}^{m} \boldsymbol{a}_{l+m} \cdot k_-^2 \frac{\mu_-}{\mu_+} \int_{\mathcal{M}} (\mathbf{curl}_x\mathbb{G}^e)^\top(\boldsymbol{x}, \boldsymbol{z}_l)\boldsymbol{\varphi}(\boldsymbol{x})\ \mathrm{d}s(\boldsymbol{x}) \\
&= \sum_{l=1}^{m} \Big(\boldsymbol{a}_l \cdot \frac{\mu_-}{\mu_+}\boldsymbol{H}^i(\boldsymbol{z}_l) + \boldsymbol{a}_{l+m} \cdot \frac{\mu_-}{\mu_+}\mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}_l)\Big)
\end{aligned}$$

for any $\boldsymbol{a} \in \mathbb{C}^{3\times 2m}$ and $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M})$. So, $R^\top : \boldsymbol{L}^2(\mathcal{M}) \to \mathbb{C}^{3\times 2m}$ is given by

$$R^\top\boldsymbol{\varphi} = \frac{\mu_-}{\mu_+}\big(\boldsymbol{H}^i(\boldsymbol{z}_1), \dots, \boldsymbol{H}^i(\boldsymbol{z}_m), \mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}_1), \dots, \mathbf{curl}\boldsymbol{H}^i(\boldsymbol{z}_m)\big). \tag{9.4}$$

LEMMA 9.1. (a) *R is injective.* (b) *$R^\top$ is surjective.*

Proof. (a) Suppose $\boldsymbol{a} \in \mathbb{C}^{3\times 2m}$ such that $R\boldsymbol{a} = 0$. Then

$$\tilde{\boldsymbol{H}} := k_-^2 \sum_{l=1}^{m} \Big(\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_l + \frac{\mu_-}{\mu}\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_{m+l}\Big)$$

together with the associated electric field $\tilde{\boldsymbol{E}} := -1/(\mathrm{i}\,\omega\varepsilon)\mathbf{curl}\tilde{\boldsymbol{H}}$ is a radiating solution of Maxwell's equations (3.1) in $\mathbb{R}^3 \setminus \bigcup_{l=1}^m \{\boldsymbol{z}_l\}$ that satisfies $\tilde{\boldsymbol{H}}|_{\mathcal{M}} = 0$. Now we can follow the proof of [20, Thm. 3.2] and obtain $\tilde{\boldsymbol{H}} = 0$ in $\mathbb{R}^3 \setminus \bigcup_{l=1}^m \{\boldsymbol{z}_l\}$.

Let $l \in \{1, \dots, m\}$; then of course $\lim_{t\to 0} \tilde{\boldsymbol{H}}(\boldsymbol{z}_l + t\boldsymbol{b}) = 0$ for any $\boldsymbol{b} \in \mathbb{R}^3$. A short calculation shows that the singularity of $\mathbb{G}^m(\cdot, \boldsymbol{z}_l)$ in $\boldsymbol{z}_l$ is of order 3, while the singularity of $\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_l)$ in $\boldsymbol{z}_l$ is of order 2. So, from $\lim_{t\to 0} \mathbb{G}^m(\boldsymbol{z}_l + t\boldsymbol{e}_3, \boldsymbol{z}_l)\boldsymbol{a}_l = 0$ it follows that $\boldsymbol{a}_l = 0$. Indeed, otherwise the singularity of $\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_l$ at $\boldsymbol{z}_l$ would imply that $\lim_{t\to 0} |\tilde{\boldsymbol{H}}(\boldsymbol{z}_l + t\boldsymbol{e}_3)| = \infty$. Accordingly, $\lim_{t\to 0} \mathbf{curl}_x \mathbb{G}^e(\boldsymbol{z}_l + t\boldsymbol{e}_1, \boldsymbol{z}_l)\boldsymbol{a}_{m+l} = 0$ and $\lim_{t\to 0} \mathbf{curl}_x \mathbb{G}^e(\boldsymbol{z}_l + t\boldsymbol{e}_2, \boldsymbol{z}_l)\boldsymbol{a}_{m+l} = 0$ yield $\boldsymbol{a}_{m+l} = 0$. Because $l \in \{1, \dots, m\}$ was arbitrary, we are done.

(b) This part of the proof follows from part (a) and the well-known relation $\mathcal{R}(R^\top) = \mathcal{N}(R)^a$ between ranges and null spaces of dual operators with finite rank. Here, $\mathcal{N}(R)^a$ denotes the annihilator of $\mathcal{N}(R)$ in $\mathbb{C}^{3\times 2m}$. ∎

Comparing the formulas (9.3) and (9.4) for R and $R^\top$ and the definition (9.1) of T, we find that these operators are related by $T = RMR^\top$, where the operator $M : \mathbb{C}^{3\times 2m} \to \mathbb{C}^{3\times 2m}$ is given by

$$M\boldsymbol{a} := \frac{\mu_+}{\mu_-}\left(-\mathbb{M}^0_{B_1}\boldsymbol{a}_1, \dots, -\mathbb{M}^0_{B_m}\boldsymbol{a}_m, \frac{1}{k_-^2}\mathbb{M}^\infty_{B_1}\boldsymbol{a}_{m+1}, \dots, \frac{1}{k_-^2}\mathbb{M}^\infty_{B_m}\boldsymbol{a}_{2m}\right).$$

From the positive definiteness of the magnetic and electric polarizability tensors $\mathbb{M}^0_{B_1}, \dots, \mathbb{M}^0_{B_m}$ and $\mathbb{M}^\infty_{B_1}, \dots, \mathbb{M}^\infty_{B_m}$ we conclude that M is invertible. Taking a closer look at the range of T, we first observe that $\mathcal{R}(T) \subset \mathcal{R}(R)$. We show that this inclusion is actually an equality.

PROPOSITION 9.2. *The range of* T *has dimension* $6m$ *and is given by*

$$\mathcal{R}(T) = \operatorname{span}_{\mathbb{C}}\left\{\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{e}_j\,,\, \mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_l)\boldsymbol{e}_j \;\middle|\; j = 1, 2, 3;\ l = 1, \dots, m\right\}.$$

Proof. The surjectivity of $R^\top$ and M implies $\mathcal{R}(T) = \mathcal{R}(RMR^\top) = \mathcal{R}(R)$. The proposition is then an immediate consequence of (9.3) and Lemma 9.1(a). ∎

Now we present the main tool for the identification of the positions $\boldsymbol{z}_l$: the characterization of the centers of the scatterers in terms of the range of the leading order term T of the asymptotic expansion of the measurement operator G_δ.

PROPOSITION 9.3. *Let* $\boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2) \in (\mathbb{C}^3 \times \mathbb{C}^3) \setminus \{(0, 0)\}$, $\boldsymbol{z} \in \mathbb{R}^3_-$, *and*

$$\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} := \big(\mathbb{G}^m(\cdot, \boldsymbol{z})\boldsymbol{d}_1 + \mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z})\boldsymbol{d}_2\big)|_{\mathcal{M}}.$$

Then, $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} \in \mathcal{R}(T)$ *if and only if* $\boldsymbol{z} \in \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$.

Proof. Assume that $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} \in \mathcal{R}(T)$. As a consequence of Proposition 9.2, $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$ may be represented as

$$\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} = \sum_{l=1}^m \big(\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_l + \mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_{l+m})\boldsymbol{a}_{l+m}\big) \qquad \text{on } \mathcal{M},$$

with $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2m} \in \mathbb{C}^3$. But then both

$$\boldsymbol{H}^a := \sum_{l=1}^m \Big(\mathbb{G}^m(\cdot, \boldsymbol{z}_l)\boldsymbol{a}_l + \frac{\mu_+}{\mu}\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z}_{l+m})\boldsymbol{a}_{l+m}\Big)$$

and

$$\boldsymbol{H}^b := \mathbb{G}^m(\cdot, \boldsymbol{z})\boldsymbol{d}_1 + \frac{\mu_+}{\mu}\mathbf{curl}_x\mathbb{G}^e(\cdot, \boldsymbol{z})\boldsymbol{d}_2,$$

together with their associated electric fields, are radiating solutions of Maxwell's equations (3.1) in $\mathbb{R}^3 \setminus (\bigcup_{l=1}^m\{\boldsymbol{z}_l\} \cup \{\boldsymbol{z}\})$ that coincide on $\mathcal{M}$. So, $\tilde{\boldsymbol{H}} := \boldsymbol{H}^a - \boldsymbol{H}^b$ together with its electric field is a radiating solution of (3.1) in $\mathbb{R}^3 \setminus (\bigcup_{l=1}^m\{\boldsymbol{z}_l\} \cup \{\boldsymbol{z}\})$ that satisfies $\tilde{\boldsymbol{H}}|_{\mathcal{M}} = 0$. Following the proof of [20, Thm. 3.2] we conclude that $\tilde{\boldsymbol{H}} = 0$ everywhere in $\mathbb{R}^3 \setminus (\bigcup_{l=1}^m\{\boldsymbol{z}_l\} \cup \{\boldsymbol{z}\})$. Thus $\boldsymbol{H}^a = \boldsymbol{H}^b$ in $\mathbb{R}^3 \setminus (\bigcup_{l=1}^m\{\boldsymbol{z}_l\} \cup \{\boldsymbol{z}\})$. This is only possible if $\boldsymbol{z} \in \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$, and we have established the necessity of this condition. The sufficiency follows from Proposition 9.2. ∎

10. Determining the position of the inhomogeneities. Let $(\cdot,\cdot)_{\boldsymbol{L}^2(\mathcal{M})}$ denote the (complex) scalar product on $\boldsymbol{L}^2(\mathcal{M})$. Because G_δ is a compact operator on $\boldsymbol{L}^2(\mathcal{M})$, it admits a singular value decomposition

$$G_\delta \boldsymbol{\varphi} = \sum_{j=1}^{\infty} \sigma_j^\delta (\boldsymbol{\varphi}, \boldsymbol{v}_j^\delta)_{\boldsymbol{L}^2(\mathcal{M})} \boldsymbol{u}_j^\delta, \qquad \boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M}),$$

where $((\sigma_j^\delta)^2)_{j\in\mathbb{N}}$ are the eigenvalues of $G_\delta^* G_\delta$, written in decreasing order with multiplicity, $\sigma_j^\delta \geq 0$. Similarly, the finite rank operator T can be decomposed as

$$T\boldsymbol{\varphi} = \sum_{l=1}^{6m} \sigma_l (\boldsymbol{\varphi}, \boldsymbol{v}_l)_{\boldsymbol{L}^2(\mathcal{M})} \boldsymbol{u}_l, \qquad \boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M}),$$

with $s_1 \geq s_2 \geq \cdots \geq s_{6m} > 0$. From (9.2) we obtain

$$G_\delta^* G_\delta = \delta^6 T^* T + \mathcal{O}(\delta^7)$$

in $\mathcal{L}(\boldsymbol{L}^2(\mathcal{M}))$ as $\delta \to 0$. So, applying [26, Thm. V.4.10] we get the following aymptotic formula for the singular values as $\delta \to 0$:

$$(\sigma_j^\delta)^2 = \delta^6 \sigma_j^2 + \mathcal{O}(\delta^7), \qquad j \in \mathbb{N}, \tag{10.1}$$

where we have set $\sigma_l = 0$ for $l \geq 6m$. Next, for $j \in \mathbb{N}$ and $l = 1, \ldots, 6m$, let

$$P_j^\delta : \boldsymbol{L}^2(\mathcal{M}) \to \operatorname{span}_{\mathbb{C}}\{\boldsymbol{u}_1^\delta, \ldots, \boldsymbol{u}_j^\delta\} \qquad \text{and} \qquad P_l : \boldsymbol{L}^2(\mathcal{M}) \to \operatorname{span}_{\mathbb{C}}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_l\}$$

denote the orthogonal projections onto these subspaces, respectively. We can write these projections as line integrals of the resolvent of $G_\delta G_\delta^*$ and $\delta^6 TT^*$, respectively; see [26, III-(6.19)]. Then, a short calculation shows that

$$P_l^\delta = P_l + \mathcal{O}(\delta), \qquad l = 1, \ldots, 6m, \tag{10.2}$$

as $\delta \to 0$ in $\mathcal{L}(\boldsymbol{L}^2(\mathcal{M}))$, provided that we make appropriate choices of eigenvectors $\boldsymbol{u}_l^\delta$ and $\boldsymbol{u}_l$, $l = 1, \ldots, 6m$.

In Proposition 9.3 we have seen that a point $\boldsymbol{z} \in \mathbb{R}^3_-$ coincides with one of the positions $\boldsymbol{z}_l$, $l = 1, \ldots, m$, if and only if $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} \in \mathcal{R}(T)$ or, equivalently, $(I - P_{6m})\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} = 0$. If we decompose the test function orthogonally as $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} = P_{6m}\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} + (I - P_{6m})\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$ and define the angle $\beta(\boldsymbol{z}) \in [0, \pi/2]$ by

$$\cot \beta(\boldsymbol{z}) := \frac{\|P_{6m}\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}\|_{\boldsymbol{L}^2(\mathcal{M})}}{\|(I - P_{6m})\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}\|_{\boldsymbol{L}^2(\mathcal{M})}},$$

then we have

$$\boldsymbol{z} \in \{\boldsymbol{z}_l \mid l = 1, \ldots, m\} \iff \beta(\boldsymbol{z}) = 0 \iff \cot \beta(\boldsymbol{z}) = \infty.$$

Unfortunately, we cannot compute $\beta(\boldsymbol{z})$, because P_{6m} corresponds to the leading order term T of the asymptotic expansion (9.2), but what we measure is the full measurement operator G_δ. However, in view of (10.2), for small values of δ the projected test function $P_{6m}\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$ is well approximated by $P_{6m}^\delta \boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$, and the projectors

P_p^δ can be computed for each $p \in \mathbb{N}$ by means of the singular value expansion of the measurement operator G_δ. Hence, for $p \in \mathbb{N}$, we define the angle $\beta_p^\delta(\boldsymbol{z}) \in [0, \pi/2]$ by

$$\cot \beta_p^\delta(\boldsymbol{z}) := \frac{\|P_p^\delta \boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}\|_{\boldsymbol{L}^2(\mathcal{M})}}{\|(I - P_p^\delta)\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}\|_{\boldsymbol{L}^2(\mathcal{M})}} = \left(\frac{\sum_{j \le p} |(\boldsymbol{u}_j^\delta, \boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}})_{\boldsymbol{L}^2(\mathcal{M})}|^2}{\sum_{j > p} |(\boldsymbol{u}_j^\delta, \boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}})_{\boldsymbol{L}^2(\mathcal{M})}|^2} \right)^{1/2} .$$

If we plot $\cot \beta_{6m}^\delta(\boldsymbol{z})$, we expect to see large values for points $\boldsymbol{z}$ which are close to the positions $\boldsymbol{z}_l$, $l = 1, \dots, m$.

Because the number m of unknown scatterers is usually not known a priori, it has to be estimated somehow. Two different strategies are available: On the one hand, recalling (10.1), m may be estimated by looking for a gap in the set of singular values σ_l^δ of G_δ. This works if δ is small enough and the noise level is not too high. Otherwise it may give misleading results. On the other hand, we can plot $\cot \beta_p^\delta(\boldsymbol{z})$ for increasing values of p, until the number of reconstructed scatterers does not increase any more. This is reasonable, because for any subspace $U \subset \mathcal{R}(T)$ Proposition 9.3 reduces to

$$\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} \in U \quad \Longrightarrow \quad \boldsymbol{z} \in \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}.$$

So, testing whether $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$ is contained in a subspace $U \subset \mathcal{R}(T)$, we can only expect to reconstruct a (possibly empty) subset of $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$. The number of reconstructed scatterers is monotonically increasing as $\dim(U)$ increases until all m scatterers are reconstructed for $\dim(U) = 6m$. Because none of the singular vectors of G_δ corresponding to singular values σ_j^δ, $j > 6m$, is expected to be exactly of the form $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$, $\boldsymbol{z} \notin \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$, the number of reconstructed scatterers should be constant for moderately sized $j > 6m$. Both strategies have been successfully tested in [8].

Finally, we show numerical results to illustrate the feasibility of the reconstruction method. We consider a two-layered background medium; the upper layer is empty ($\varepsilon_+ = \varepsilon_0 = 8.85 \cdot 10^{-12}$ Fm^{-1}, $\mu_+ = \mu_0 = 1.25 \cdot 10^{-6}$ Hm^{-1}), while the lower halfspace is filled with soil ($\varepsilon_- = \varepsilon_0(\varepsilon_r + \mathrm{i}\frac{\sigma}{\omega\varepsilon_0}) = 8.67 \cdot 10^{-11} + \mathrm{i}\, 5.95 \cdot 10^{-9}$ Fm^{-1}, $\mu_- = (1 + \chi)\mu_0 = 1.25 \cdot 10^{-6}$ Hm^{-1}, i.e., $\sigma = 7.5 \cdot 10^{-4}$ Sm^{-1}, $\chi = 1.9 \cdot 10^{-5}$, and $\varepsilon_r = 9.8$). The parameters for the lower halfspace are measurement data taken by Igel and Preetz [25] in the course of the project [23].

The measurement device operates on a square of size 50×50 cm^2 parallel to the surface of ground centered at $(0, 0, 10)$ cm. We simulate the measurement operator G_δ as done in [20]. For this purpose we impose magnetic dipoles with three linearly independent polarizations and a frequency of 20 kHz on a 6×6 equidistant grid on the measurement device. Then we approximate the corresponding scattered fields on the same grid using a boundary element method. The scatterers are two ellipsoids with semiaxes $(0.1, 0.2, 0.3)$ cm and $(2, 3, 1)$ cm buried at position $(-15, 15, -10)$ cm and $(15, -15, -40)$ cm, respectively. The simulated forward data contain an estimated numerical error of 4%. Additionally, we perturb the simulated scattered field by a uniformly distributed relative error of 3%.

The values of $\cot \beta_{12}^\delta(\boldsymbol{z})$ for $\boldsymbol{z} \in [-25, 25]^2 \times [-50, 0]$ cm^3 are used to visualize the location of the scatterers. The numerical implementation is essentially the same that has been used in [20] for a linear sampling method. Concerning implementation details, we refer the reader to this work; see also [8]. Figure 10.1 shows the first 20 singular values of the measurement operator G_δ and horizontal cross sections of $\cot \beta_{12}^\delta(\boldsymbol{z})$ for $z_3 = -10$ cm and $z_3 = -40$ cm, respectively. There is no distinct gap after the first 12 singurar values. One reason for this is the (numerical) error

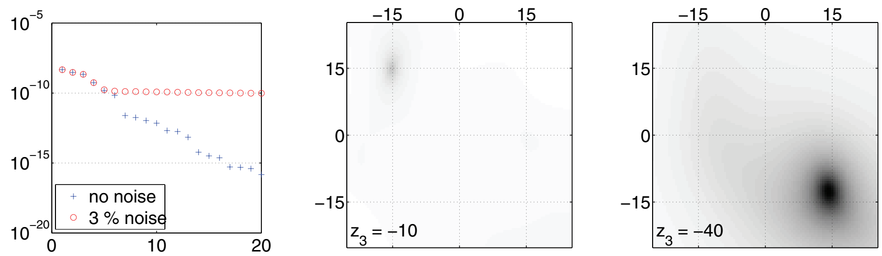

FIG. 10.1. *Singular values of* G_δ *and cross-sectional plots of* $\cot\beta_{12}^{\delta}(z)$ *for* $z_3 = -10$ *cm and* $z_3 = -40$ *cm with* 3% *noise.*

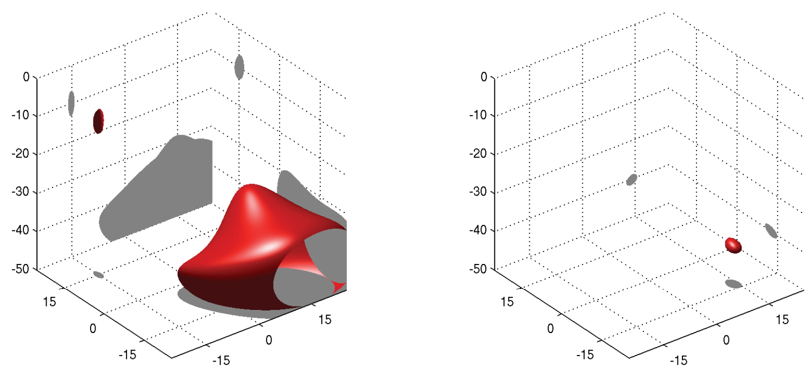

FIG. 10.2. *Isosurface plots* $\cot\beta_{12}^{\delta}(\boldsymbol{z}) = 25$ *and* $\cot\beta_{12}^{\delta}(\boldsymbol{z}) = 125$ *with* 3% *noise.*

in the forward data. Here the iterative procedure described above can be used to estimate the number of the scatterers. In the cross sectional plots the centers of the scatterers are clearly determined. Figure 10.2 shows isosurface plots $\cot\beta_{12}^{\delta}(\boldsymbol{z}) = 20$ and $\cot\beta_{12}^{\delta}(\boldsymbol{z}) = 200$. We emphasize that these visualizations should not be mistaken as reconstructions of the shape of the scatterers. These plots give just an idea of possible positions of buried scatterers; they can be expected to be inside the (red) surfaces. Our method does not allow a binary test for whether some point belongs to a scatterer or not. If we perturb the simulated forward data in this example with 5% equally distributed noise, the reconstructions of the positions of the scatterers get worse, but still two scatterers are reconstructed. For higher amounts of noise the method fails.

Note that this is only one particular numerical example that by no means covers all possible situations of interest. Comparing the method proposed here with the linear sampling method from [20], using (among others) the example above, we found that the linear sampling method is more sensitive to uncorrelated noise. Using the unperturbed simulated data, the position of the scatterers has been reconstructed by the linear sampling method. But with 3% noise in the data the linear sampling method failed. The MUSIC-type reconstruction method studied in [24] gives numerical results comparable to the results presented here, although we mention that much higher frequencies have been used in [24]. In their final implementation both methods are quite similar. Our analysis from sections 9 and 10 is meant to be an extension of [24] and a rigorous justification of both methods.

11. Conclusions. We have considered an inverse scattering problem for small scatterers in a two-layered background medium which originated in the project [23] on humanitarian demining. An asymptotic expansion of the near field measurement operator as the size of the scatterers tends to zero has been proven. We used the asymptotic formula to justify a noniterative reconstruction method that can be interpreted as an asymptotic version of a factorization method, or as a MUSIC-type method. First numerical results indicate that this method may be appropriate to detect small buried objects from sufficiently accurate measurements of the scattered field above the surface of ground. Although our results have been derived for an idealized setting, we expect that the asymptotic expansion as well as the reconstruction method can be applied to more realistic models for measurement devices used for humanitarian demining, including special coil geometries such as, e.g., the double D design considered in [17]. We intend to address this in the future.

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REFERENCES

[1] H. Ammari, R. Griesmaier, and M. Hanke, *Identification of small inhomogeneities: Asymptotic factorization*, Math. Comput., 76 (2007), pp. 1425–1448.

[2] H. Ammari, E. Iakovleva, D. Lesselier, and G. Perrusson, *MUSIC-type electromagnetic imaging of a collection of small three-dimensional bounded inclusions*, SIAM J. Sci. Comput., 29 (2007), pp. 674–709.

[3] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math. 1846, Springer-Verlag, Berlin, 2004.

[4] H. Ammari and A. Khelifi, *Electromagnetic scattering by small dielectric inhomogeneities*, J. Math. Pures Appl., 82 (2003), pp. 749–842.

[5] H. Ammari, S. Moskow, and M. S. Vogelius, *Boundary integral formulae for the reconstruction of electric and electromagnetic inhomogeneities of small volume*, ESAIM Control Optim. Calc. Var., 9 (2003), pp. 49–66.

[6] H. Ammari, M. S. Vogelius, and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter* II. *The full Maxwell equations*, J. Math. Pures Appl., 80 (2001), pp. 769–814.

[7] H. Ammari and D. Volkov, *The leading order term in the asymptotic expansion of the scattering amplitude of a collection of finite number of dielectric inhomogeneities of small diameter*, Multiscale Computational Engineering, 3 (2005), pp. 149–160.

[8] M. Brühl, M. Hanke, and M. S. Vogelius, *A direct impedance tomography algorithm for locating small inhomogeneities*, Numer. Math., 93 (2003), pp. 635–654.

[9] A. Buffa, M. Costabel, and D. Sheen, *On traces of $H(\mathrm{curl}, \Omega)$ in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[10] F. Cakoni and D. Colton, *Qualitative methods in inverse scattering theory. An introduction*, Interaction of Mechanics and Mathematics, Springer-Verlag, Berlin, 2006.

[11] F. Cakoni, M'B. Fares, and H. Haddar, *Analysis of two linear sampling methods applied to electromagnetic imaging of buried objects*, Inverse Problems, 22 (2006), pp. 845–867.

[12] M. Cheney, *The linear sampling method and the MUSIC algorithm*, Inverse Problems, 17 (2001), pp. 591–595.

[13] W. C. Chew, *Waves and Fields in Inhomogeneous Media*, Van Nostrand Reinhold, New York, 1990.

[14] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, John Wiley & Sons, New York, 1983.
[15] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Appl. Math. Sci. 93, Springer-Verlag, Berlin, 1998.
[16] P.-M. Cutzach and C. Hazard, *Existence, uniqueness and analyticity properties for electromagnetic scattering in a two-layered medium*, Math. Methods Appl. Sci., 21 (1998), pp. 433–461.
[17] F. Delbary, K. Erhard, R. Kress, R. Potthast, and J. Schulz, *Inverse electromagnetic scattering in a two-layered medium with an application to mine detection*, Inverse Problems, 24 (2008), 015002.
[18] A. J. Devaney, *Super-resolution processing of multi-static data using time reversal and MUSIC*, preprint, Department of Electrical Engineering, Northeastern University, Boston, MA, 1999.
[19] A. Friedman and M. S. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: A theorem on continuous dependence*, Arch. Ration. Mech. Anal., 105 (1989), pp. 299–326.
[20] B. Gebauer, M. Hanke, A. Kirsch, W. Muniz, and C. Schneider, *A sampling method for detecting buried objects using electromagnetic scattering*, Inverse Problems, 21 (2005), pp. 2035–2050.
[21] B. Gebauer, M. Hanke, and C. Schneider, *Sampling methods for low-frequency electromagnetic imaging*, Inverse Problems, 24 (2008), 015007.
[22] D. Guelle, A. Smith, A. Lewis, and T. Bloodworth, *EUR* 20837 *Metal Detector Handbook for Humanitarian Demining*, Office for Official Publications of the European Communities, Luxembourg, 2003.
[23] *HuMin/MD—Metal Detectors for Humanitarian Demining—Development Potentials in Data Analysis Methodology and Measurement,* Project Network, available online at http://www.humin-md.de/.
[24] E. Iakovleva, S. Gdoura, D. Lesselier, and G. Perrusson, *Multi-static response matrix of a 3-D inclusion in half space and MUSIC imaging*, IEEE Trans. Antennas Propagat., 55 (2007), pp. 2598–2609.
[25] J. Igel and H. Preetz, *Elektromagnetische Bodenparameter und ihre Abhängigkeit von den Bodeneigenschaften.—Zwischenbericht Projektverbund Humanitäres Minenräumen*, Technical report, Leibniz Institute of Applied Geosciences, Hannover, Germany, 2005.
[26] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren Math. Wiss. 132, Springer-Verlag, Berlin, 1966.
[27] A. Kirsch, *Surface gradients and continuity properties for some integral operators in classical scattering theory*, Math. Methods Appl. Sci., 11 (1989), pp. 789–804.
[28] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems, 14 (1998), pp. 1489–1512.
[29] A. Kirsch, *An integral equation for Maxwell's equations in a layered medium with an application to the factorization method*, J. Integral Equations Appl., 19 (2007), pp. 333–359.
[30] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, UK, 2000.
[31] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford, UK, 2003.
[32] J.-C. Nédélec, *Acoustic and Electromagnetic Equations. Integral Representations for Harmonic Problems*, Appl. Math. Sci. 144, Springer-Verlag, New York, 2001.
[33] M. Petry, *Über die Streuung Zeitharmonischer Wellen im Geschichteten Raum*, Ph.D. thesis, Georg-August-Universität zu Göttingen, Göttingen, Germany, 1993.
[34] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, New York, 1949.
[35] C. Weber, *Regularity theorems for Maxwell's equations*, Math. Methods Appl. Sci., 3 (1981), pp. 523–536.