Potentialtheorie, Fall 2017/18

Exercise sheet 2

Exercise 1:

(a) Let $x, y \in \mathbb{R}^2$. Show that

$$|x-y| = |x| + \frac{x}{|x|} \cdot y + \mathcal{O}\left(|x|^{-1}\right) ,$$
$$\log|x-y| = \log|x| + \mathcal{O}\left(|x|^{-1}\right) ,$$

for $|x| \to \infty$, uniformly in all directions $x/|x|$.

(b) Polar coordinates in the plane are defined by $x = r\cos\varphi$, $y = r\sin\varphi$, where $r \geq 0$ and $0 \leq \varphi < 2\pi$. The Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ with respect to these coordinates is given by

$$\widetilde{\Delta} u(r,\varphi) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}.$$

Explain what that means exactly, and prove that this is actually true.

Exercise 2:

(a) Give an example of a harmonic function in the strip

$$S := \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}\,, 0 < y < \pi\}$$

that is continuous on $\overline{S}$, vanishes on ∂S and is strictly positive in the interior of S. Does this contradict the maximum principle?

(b) Let $u : \mathbb{R}^2 \setminus \overline{B_1(0)} \to \mathbb{R}$ be harmonic. Define $v : B_1(0) \setminus \{0\} \to \mathbb{R}$ by

$$v(x,y) := u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) .$$

Show that v is harmonic, too.

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