Master's thesis

for the acquisition of the academic degree Master of Science

On Landau damping coupled with relaxation for the Vlasov-Poisson-BGK system

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submitted by

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Abstract

In an electronic plasma an initial disturbance from the equilibrium distribution gives rise to an oscillating electromagnetic field. The Soviet physicist Lev Davidovich Landau discovered in the 1940s that for the Vlasov-Poisson system this electric field is actually damped. This phenomenon is referred to as *Landau damping*. In the present work we consider the Vlasov-Poisson-BGK system. It differs from the Vlasov-Poisson system by an additional BGK relaxation operator which takes binary collisions of the plasma particles explicitly into account. Following Landau's approach we show that for this system there is also a damping effect of the electric field. In a first step, we derive the corresponding dispersion relation and show analytically that under the assumption of large wavelengths and small collision frequencies the damping effect can be split up into a Landau damping part and a collisional damping due to the BGK relaxation. In a second step, we solve the dispersion relation numerically and study the behaviour of the zeros of the dispersion relation for different values of the wave vector and the collision frequency.

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Chapter 1

Introduction

More than 99 % of the visible matter of the universe are made up of plasma [24, 29]. Roughly speaking, plasma is an overall electrically neutral substance consisting of ions and free electrons which left the orbit around the atomic nuclei. It is often considered as the fourth state of matter besides solids, liquids and gases that arises from the gaseous state by ionization. The word plasma, which is Greek and means "formed" or "molded", was introduced by the American scientist Irving Langmuir in the 1920s [5, 9]. Together with Lewi Tonks he experimentally produced such an ionized gas by electric discharges in a tube. In his article "Oscillations in ionized gases" [19] he describes his observations:

> "Except near the electrodes, where there are *sheaths* containing very few electrons, the ionized gas contains ions and electrons in about equal numbers so that the resultant space charge is very small. We shall use the name *plasma* to describe this region containing balanced charges of ions and electrons."

Further, he discovered that for an initially disturbed plasma not being in this balanced charge of ions and electrons any more, there are large Coulomb forces. They arise due to the electromagnetic field and lead to collective effects of the plasma. This means that a single plasma particle interacts by long range Coulomb forces with many other ones. In his experiments Langmuir was able to observe that this leads to high-frequency electron oscillations which tend to bring back an overall neutrality. These oscillations are often called "Langmuir oscillations" or "Langmuir waves" [9, 16].

1.1 Characterization of a plasma

Following [9, 10], there are three main properties that characterize a plasma. Before stating them, we first have to introduce another characteristic quantity of plasma, the Debye length. We consider an unperturbed spatially uniform distribution of ions and free electrons such that the resulting overall mixture is neutral. Then, a single additional charged particle, a so-called test particle, is inserted slowly into the distribution. This test particle interacts with the particles of the plasma. Let us assume that the test particle is of positive charge. In this case nearby negatively charged electrons will be attracted and nearby positively charged ions will be repelled by the test particle. The mass of the ions is much larger than the mass of the electrons. Hence, it is assumed that the ions are immobile and only the electrons are moving. The result will be a negatively charged cloud of electrons around the test ion. From a macroscopic point of view the sphere around the test particle then can again be considered neutral of charge. This means that the resulting net charge is approximately equal to zero. It is said that the cloud potential shields the potential of the test particle. For specifying the distance within which this shielding happens we introduce the Debye length. For T being the electron temperature, k_B the Boltzmann constant, ϵ_0 the electric permittivity of vacuum, e the elementary charge and n the electron density, the *Debye length* is defined by

$$\lambda_D = \left(\frac{\epsilon_0 k_B T}{ne^2}\right)^{1/2}. \tag{1.1}$$

The ball around the test particle of radius λ_D is called *Debye sphere* and the region in which the shielding happens is said to be the *plasma sheath*. Its radius is of several Debye lengths. Inside the plasma sheath the condition of macroscopic electrical neutrality may not be fulfilled in contrast to the plasma region outside of it. The ability of cancelling out potentials is referred to as the Debye shielding property of plasma [5, 9].

The first criterion characterizing a plasma is that the Debye length shall be much smaller than the physical dimension L of the system. Then, there is enough space such that Debye shielding can take place and the effects occurring in the plasma sheaths can be neglected. This criterion implicitly contains the condition of macroscopic neutrality for a plasma as deviations from neutrality only occur in length scales comparable to the Debye length. This property is also called the quasi-neutrality of a plasma.

The second criterion concerns the number N_D of electrons inside a Debye sphere. It can be calculated by

$$N_D = \frac{4}{3}\pi\lambda_D^3 n$$

and shall be much larger than one. Then, the huge amount of electrons inside a Debye sphere enables the Debye shielding effect to take place.

The third criterion concerns collisions between the particles. Depending on the degree of ionization some plasmas contain neutral atoms besides the free ions and electrons. In this case it can happen that the oscillating electrons of the plasma collide with neutral atoms and the amplitude of the oscillations is damped. Those collisions shall not appear too often. Let τ denote the mean time that an electron travels between collisions with neutral atoms. Then, the collision frequency $\frac{1}{\tau}$ shall be small compared to the plasma frequency given by

$$\omega_p = \left(\frac{ne^2}{m\epsilon_0}\right)^{1/2}, \tag{1.2}$$

where m denotes the mass of an electron. If this condition is not fulfilled the weakly ionized medium can be considered as a neutral gas rather than a plasma.

In summary, according to [9, 10] a plasma shall exhibit the following properties:

1. $\lambda_D \ll L$,
2. $N_D \gg 1$,
3. $\omega_p \tau > 1$.

1.2 Plasma in nature and in practical applications

Plasma can be found in various forms in nature as well as in practical applications. Stars, interstellar space, galaxies, intergalactic space and gaseous nebulas are all in a plasma state. Also the center of our solar system, the Sun, is made of plasma. It generates energy from thermonuclear fusion processes and sends solar radiation towards the Earth. This is fundamental for human living being possible. Also part of the Earth's atmosphere, the Ionosphere, is in a plasma state. It reaches from an altitude of approximately 60 kilometres up to several thousands of kilometres [9]. Inside our atmosphere natural plasma phenomena are rather limited. Here, polar lights like the aurora borealis or the flash of a lightning bolt are examples. Besides these plasmas occurring in the nature, there is also a lot of research of how plasmas can be artificially generated and used in several applications. For instance, the magnetohydrodynamic generator shall be mentioned. It generates electric energy from the kinetic energy of a dense plasma in a magnetic field. More examples are plasma propulsion systems which are often applied by rocket engineers for long-distance interplanetary space travel missions, particle accelerators or gas lasers. In industry, plasma can be found in the semiconductor fabrication, for coating or the disinfection of medical instruments. In everyday life, neon tubes or plasma displays are very common. For more information on the application of plasma physics the reader is referred to [9, 10].

1.3 The ITER project

Another important field of research in which artificially generated plasmas play an important role is the one of controlled thermonuclear fusion. Here, the project ITER can be considered as a milestone on the way to commercialized energy production by fusion reactions. ITER which is Latin and means "the way" or "the path" is an acronym for **I**nternational **T**hermonuclear **E**xperimental **R**eactor. The idea for this huge research project came up in 1985. Since then the European Union, the United States, Russia, Japan, China, South Korea and India have joined. At the moment the construction of ITER takes place in Cadarache in the south-eastern part of France, before in 2025 the first plasma is scheduled to be created there. The working principle of ITER is based on the fusion of two light nuclei to a heavier atom. At the moment the most accessible approach consists in fusing isotopes of hydrogen, usually deuterium and tritium, under very high temperatures to a heavier helium atom and a loose neutron. In this process energy is set free. For the fusion in the ITER project the concept of magnetic confinement is used. There, the plasma is confined under very powerful magnetic fields in a toroidal chamber, a so-called tokamak. This name comes from a Russian acronym standing for "toroidal chamber with magnetic coils". The tokamak of the ITER project can exemplarily be seen in Figure 1.1.

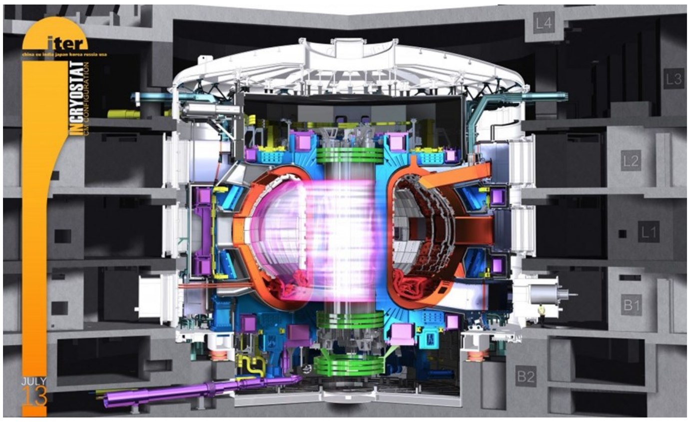


Figure 1.1: Schematic construction of the ITER tokamak.[1]

Using these concepts ITER shall serve as a large science and technology demonstration and the first fusion power plant that generates as much energy as it consumes. A detailed description of ITER and the principles of fusion energy can be found in [24].

1.4 Plasma models

Depending on the considered problem and the underlying physics an adequate description of the plasma model has to be chosen. The microscopic, the kinetic and the macroscopic approach are the three possible main points of view. On a microscopic level the motion of each particle contributing to the plasma is considered individually and their time evolution is assumed to be governed by Newton's laws. This is the most accurate model. As a plasma usually consists of a very large number of particles this approach requires great effort and therefore is hardly used in practice. A less detailed but more feasible formulation is given by kinetic models. This corresponds to a treatment on a mesoscopic level. It is based on a statistical description of the particle distribution in phase space. Concepts like the particle distribution function will be introduced in Chapter 2. Kinetic models are often applied to problems where collective effects dominate over binary collisions. On a macroscopic level the formulation of the considered problem is given in terms of macroscopic quantities like density, mean velocity and energy. This is called a macroscopic or also fluid description. In addition, there are models that specify or combine these three approaches. More information on this topic is given in [27]. In this work, a kinetic approach will be considered.

[1] Source: `https://www.iter.org/img/resize-900-90/www/content/com/Lists/Machine/Attachments/30/tkm_cplx_final_plasma2013-07.jpg`.

1.5 Structure of the thesis

The structure of the thesis is as follows. In Chapter 2 mathematical and physical prerequisites, basics for the kinetic description of plasma and Maxwell's equations will be given.

Chapter 3 is centered around the Vlasov equation which serves as a kinetic model for the time evolution of plasma in certain regimes. It was first derived from the Boltzmann equation by Anatoly Alexandrovich Vlasov in 1938 [32]. For the modeling of a plasma the self-consistent electromagnetic field generated by the plasma particles has to be taken into account. It gives rise to collective effects. This is achieved by coupling the Vlasov equation to Maxwell's equations. Under certain assumptions the magnetic field becomes negligible while the electric field remains. In this case the resulting set of equations is called the Vlasov-Poisson system. One important property of this system is the phenomenon of Landau damping. It describes a damping effect of the electric field that arises due to an initial disturbance and the collective behaviour of plasma. In 1946, when the Soviet physicist Lev Davidovich Landau discovered this phenomenon [18], it was a quite astonishing result that damping occurs even without considering binary collisions of the particles explicitly. Using Landau's approach we will derive this damping effect mathematically and give a physical interpretation.

In Chapter 4 we will no longer neglect binary collisions and add a collision term, the BGK relaxation, to the Vlasov equation. The resulting framework is called the Vlasov-Poisson-BGK system. It will be shown that for a small initial disturbance there is also a damping effect of the electric field. To this end, we follow the article by Landau [18] and adapt his approach to the problem considered here. This gives us a complicated analytical equation, called the dispersion relation, for which the zeros shall be determined. First, this is achieved for large wavelengths and small collision frequencies by adapting the methods used in an article by Warren Preston Wood and Barry William Ninham [35]. We shall formulate parts of their argumentation in greater detail and from a more mathematical point of view. This will lead to an expression of the damping coefficient that clearly shows the contribution of Landau damping as well as of the BGK relaxation to the damping effect. Afterwards, the zeros of the dispersion relation are calculated numerically. For this part we make use of a code that was applied in [27] and provided by its author Eric Sonnendrücker such that it could be adapted to the considered Vlasov-Poisson-BGK framework.

Chapter 5 is devoted to the idea of extending the approaches of this thesis to a multi-species BGK model based on the work of Marlies Pirner [25]. Moreover, it contains a short summary and conclusion.

Chapter 2

Basic concepts

Before discussing the main parts of this thesis, some fundamentals from mathematics and physics shall be given. These basic concepts are needed throughout the following chapters. We start by recalling important mathematical quantities and theorems, followed by principles for the kinetic description of plasma and Maxwell's equations.

2.1 Mathematical prerequisites

For the correct treatment of the Vlasov-Poisson and the Vlasov-Poisson-BGK system some integral transforms will be very useful. They will be introduced in the following. After that, the concept of contour integration as well as important theorems from complex analysis are given. Finally, some special functions will be treated.

2.1.1 The Fourier transform

The Fourier transform is a powerful tool in classical as well as in modern analysis. Its basic principles were first presented by Jean Baptiste Joseph Fourier in his book "Théorie analytique de la chaleur" [13] published in 1822. Since then, a whole theory based on his ideas has been developed. Before giving its definition, a suitable function space, namely the Schwartz space, shall be introduced.

Definition 2.1 (Schwartz space, [36])**.** Let $\alpha, \beta \in \mathbb{N}_0^n$ be multi-indices with $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$. The function space

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha f(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^n \right\}$$

with $x^\beta = \prod_{j=1}^n x_j^{\beta_j}$ und $D^\alpha f(x) = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} f(x)$ is called *Schwartz space*. Its elements $f \in \mathcal{S}(\mathbb{R}^n)$ are called *rapidly decreasing*.

On this space the Fourier transform can be defined as follows.

Definition 2.2 (Fourier transform, [36])**.** Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi, x \in \mathbb{R}^n$. Then, the *Fourier transform* of f is given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, \mathrm{d}x.$$

In the next definition the concept of the inverse Fourier transform is introduced.

Definition 2.3 (Inverse Fourier transform, [36])**.** Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi, x \in \mathbb{R}^n$. Then, the *inverse Fourier transform* of f is given by

$$\check{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi) e^{ix \cdot \xi} \, \mathrm{d}\xi.$$

On $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform exhibits the following property.

Lemma 2.4 ([36])**.** The Fourier transform is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself. For $f \in \mathcal{S}(\mathbb{R}^n)$ it holds

$$\check{\hat{f}} = f = \hat{\check{f}}$$

and the notation $\check{f}(x)$ can be identified with $(\hat{f}^{-1})(x)$.

This means that the inverse Fourier transform given in Definition 2.3 exactly describes the corresponding inversion process and the name of the transform is appropriately chosen.

2.1.2 The Laplace transform

Another relevant integral transform is the Laplace transform. It is often used for solving ordinary or partial differential equations. For its definition it is crucial that the contained integral exists.

Theorem 2.5 ([6])**.** Let $f : [0, \infty[\to \mathbb{C}$ be continuous. If there exist constants $a, b \in \mathbb{R}$ such that $|f(t)| \leq ae^{bt}$ for $t \to \infty$ and if $\int_0^T |f(t)| \, \mathrm{d}t < \infty$ for every finite $T > 0$, then the integral

$$\int_0^\infty f(t) e^{-st} \, \mathrm{d}t \quad \text{with } s \in \mathbb{C}$$

exists for $\mathrm{Re}(s) > b$ and converges absolutely and uniformly in the same domain.

With this knowledge we can define the Laplace transform on the corresponding half plane.

Definition 2.6 (Laplace transform, [6])**.** Let f be a function satisfying all assumptions from Theorem 2.5. Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b$ the *(one-sided) Laplace transform* of f is given by

$$\mathcal{L}f(s) = \tilde{f}(s) = \int_0^\infty f(t) e^{-st} \, \mathrm{d}t.$$

As discussed in [15], the Laplace transform exhibits some useful properties. One of these is given in the next lemma.

Lemma 2.7 ([15])**.** The Laplace transform $\mathcal{L} : f \to \mathcal{L}f = \tilde{f}$ is a linear operator, i.e. it holds

$$\mathcal{L}(f + g) = \mathcal{L}f + \mathcal{L}g \quad \text{and} \quad \mathcal{L}(\lambda f) = \lambda\, \mathcal{L}f,$$

with $\lambda \in \mathbb{C}$ and f, g as in Theorem 2.5.

It is desirable to have an inversion formula for the Laplace transform. The existence of the contained integral is treated in the next theorem.

Theorem 2.8 ([6, 27])**.** Let $R, M \in \mathbb{R}$ be constants such that the function $F(s)$ satisfies the following conditions:

(i) $F(s)$ is analytic for $\mathrm{Re}(s) > R$,

(ii) $|sF(s)| \leq M$ for all s such that $|s| > R$.

Then, the integral

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} \,\mathrm{d}s, \quad c > R,$$

exists for all $t > 0$.

Note that in Theorem 2.8 the integral is taken along a vertical line parallel to the imaginary axis such that all singularities of $F(s)$ are located on the left-hand side of it. Knowing about the existence of the integral we can give a proper definition of the inverse Laplace transform.

Definition 2.9 (Inverse Laplace transform, [6, 27])**.** The integral transform

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} \,\mathrm{d}s, \quad c > R,$$

given in Theorem 2.8 is called the *inverse Laplace transform*.

Inserting the Laplace transform of f into the inverse Laplace transform gives us indeed back the original function f.

Lemma 2.10 ([6, 27])**.** Under the assumptions of Theorem 2.8 it holds $F(s) = \mathcal{L}f(s)$, i.e. we obtain the relation

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) e^{st} \,\mathrm{d}s, \quad c > R.$$

Adopting a convention often used by physicists, we write $s = -i\bar{\omega}$ with $\bar{\omega} \in \mathbb{C}$ in what follows. With this notation the Laplace transform of f is given in the form

$$\mathcal{L}f(\bar{\omega}) = \tilde{f}(\bar{\omega}) = \int_0^\infty f(t)e^{i\bar{\omega}t}\,\mathrm{d}t.$$

Using linearity the inverse Laplace transform can be written as

$$f(t) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \tilde{f}(\bar{\omega})e^{-i\bar{\omega}t}\,\mathrm{d}\bar{\omega}.$$

Note that in this setting the integral in the inverse Laplace transform is taken along a horizontal line parallel to the real axis such that all singularities of $\tilde{f}(\bar{\omega})$ are located below it.

2.1.3 Contour integration and the residue theorem

In the definition of the inverse Laplace transform the integral is taken along a straight line in the complex plane. Under certain assumptions, this integration contour can be deformed. We recall important theorems from complex analysis and start with the concept of a path.

Definition 2.11 (Path, [3])**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a curve that is continuously differentiable or piecewise continuously differentiable. Then, γ is called a *path* or a *contour.* If $\gamma(a) = \gamma(b)$, the path is said to be *closed.*

With this definition we can introduce the contour integral of a function f.

Definition 2.12 (Contour integral, [3])**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a path and f a continuous complex-valued function defined on the graph of γ. The *contour integral* of f on γ is defined as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Sometimes the contour integral itself is difficult to evaluate. Then one can make use of helpful theorems from complex analysis.

Theorem 2.13 (Cauchy's integral theorem, [3])**.** Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $f : \Omega \to \mathbb{C}$ an analytic function. If γ is a closed path in Ω, then it holds

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

For stating the next theorem we first have to introduce further quantities.

Definition 2.14 (Winding number, [4])**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a path and $z_0 \in \mathbb{C}$ be a point not lying on the graph of γ. Then,

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

is called the *windung number* of γ around z_0.

In addition to the winding number also the residue is of importance.

Definition 2.15 (Residue, [3])**.** Let $\epsilon > 0$ and f be a function that is analytic in a deleted neighborhood $U_\epsilon(z_0)$ of a point z_0 and has an isolated singularity at z_0. Then, the *residue* of f at z_0 is defined as

$$\mathrm{Res}(f, z_0) = \frac{1}{2\pi i} \int_{C_\delta(z_0)} f(z)\mathrm{d}z,$$

where $C_\delta(z_0)$ denotes a positively oriented circle of radius $0 < \delta < \epsilon$ around z_0.

This allows us to state the residue theorem.

Theorem 2.16 (Residue theorem, [4])**.** Let f be analytic on a simply connected domain $\Omega \subseteq \mathbb{C}$ except for finitely many isolated singularities $z_1, z_2, ..., z_p \in \Omega$ and γ be a closed integration path not intersecting any of the singularities. Then,

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{p} n(\gamma, z_j)\,\mathrm{Res}(f, z_j),$$

where $n(\gamma, z_j)$ denotes the winding number of the path γ around the singularity z_j and $\mathrm{Res}(f, z_j)$ the residue of f at z_j.

The residue theorem will later be used for evaluating the integral in the inverse Laplace transform.

2.1.4 Special functions

We will also be concerned with some special functions throughout this work. One of it, the gamma function, is commonly known from complex analysis. Before giving a definition we state the convergence of the included integral.

Lemma 2.17 ([1])**.** For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ the integral

$$\int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$$

converges absolutely.

We now define the according function.

Definition 2.18 (Gamma function, [1])**.** For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$$

is called *Euler integral of the second kind.* The corresponding function is called the *gamma function.*

For special values, the gamma function can be evaluated easily.

Lemma 2.19 ([26])**.** It holds

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$$

for all $n \in \mathbb{N}_0$.

We will make use of this relationship later on. Another special function is the error function.

Definition 2.20 (Error function, [2])**.** The function $\mathrm{erf} : \mathbb{C} \to \mathbb{C}$ defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, \mathrm{d}t$$

is called the *(Gauss) error function.*

For real arguments $x \in \mathbb{R}$, the function $\mathrm{erf}(x)$ takes real values. For complex arguments we can also work with the imaginary error function.

Definition 2.21 (Imaginary error function, [27])**.** The function $\mathrm{erfi} : \mathbb{C} \to \mathbb{C}$ given by

$$\mathrm{erfi}(z) = -i \, \mathrm{erf}(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} \, \mathrm{d}t$$

is called the *imaginary error function.*

The imaginary error function is a classical special function that can be found in most of the standard numerical software [27]. This will be of advantage when showing the damping effect of the electric field in the Vlasov-Poisson-BGK system numerically.

2.2 Principles for the kinetic description of plasma

The main equation around which this thesis is centered is the Vlasov equation. It offers a kinetic description for the time evolution of plasma. Kinetic models make use of a distribution function describing the density of particles in phase space.

Definition 2.22 (Distribution function, [25])**.** A function $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R},\ (x, v, t) \to f(x, v, t)$ is called a *distribution function* if and only if $f(x, v, t)\,\mathrm{d}x\,\mathrm{d}v$ is the number of particles with velocities in $(v, v + \mathrm{d}v)$ located in the interval $(x, x + \mathrm{d}x)$ at time t.

In this definition the phase space corresponds to a six-dimensional space having three components for the space vector x and three components for the velocity vector v. Note that the space and velocity variables are assumed to be independent. Further, it is assumed that the density of the particles does not vary too fast so that f can be considered as a continuous function [9].

Compared to a microscopic description, kinetic models concentrate on less information and therefore have the advantage of being easier to handle. The concept of using a distribution function hereby corresponds to a statistical approach. If the distribution function is normalized to one, its value can be interpreted as the probability of a particle being at time t at the point (x, v) in phase space [27]. For a fluid description, macroscopic values are of importance. They can be obtained from the distribution function by averaging.

Definition 2.23 (Macroscopic quantities, [20, 25])**.** Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}$ with $\Lambda \subseteq \mathbb{R}^3$ be a distribution function. Assuming that all integrals exist, the *number density* n, the *mean velocity* u and the *temperature* T are given by

$$\begin{pmatrix} n(x,t) \\ n(x,t)u(x,t) \\ 3n(x,t)k_B T(x,t) \end{pmatrix} = \int_{\mathbb{R}^3} f(x,v,t) \begin{pmatrix} 1 \\ v \\ m\left|v - u\left(x,t\right)\right|^2 \end{pmatrix} \mathrm{d}v,$$

where k_B denotes the Boltzmann constant and m the mass of the considered particle species.

It is essential that the fluid model can be derived from the kinetic description by taking moments of the distribution function such that these two perspectives merge into each other. An important distribution is the Maxwell-Boltzmann distribution.

Definition 2.24 (Maxwell-Boltzmann distribution, [25])**.** A distribution of the form

$$M(v) = C \exp\left(-\frac{|v - U|^2}{A}\right)$$

with $C, A \in \mathbb{R}^+$ and $U \in \mathbb{R}^3$ is called a *Maxwell-Boltzmann distribution* or a *Maxwellian velocity distribution.* If the parameters C, A and U depend on x and t the distribution $M(x, v, t)$ is called a *local Maxwell-Boltzmann distribution* or a *local Maxwellian velocity distribution.*

In contrast to fluid models, kinetic ones are more accurate. This is illustrated in the following example.

Example 2.25 ([10, 16])**.** Consider the following one-dimensional distribution functions for which the areas under the curves shall have the same size.

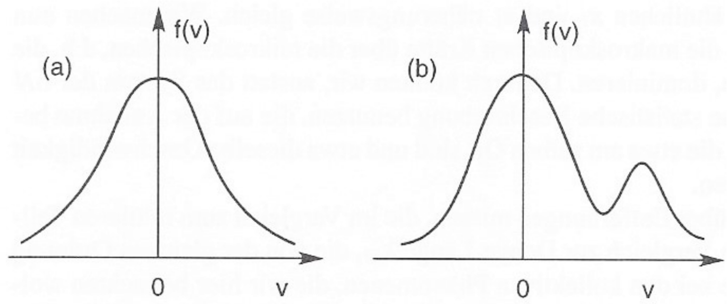


Figure 2.1: (a) Maxwellian distribution. **(b)** Non-Maxwellian distribution.[2]

Then integration with respect to v gives us the same macroscopic quantities for both cases. Hence, a fluid description cannot distinguish between those two distributions whereas from a kinetic point of view they look significantly different.

For the hydrodynamics of liquids and undiluted gases a fluid description often is sufficient whereas for plasmas at high temperatures in most cases a kinetic description is appropriate [16].

2.3 Maxwell's equations

In a plasma we encounter long range Coulomb forces leading to a self-consistent electromagnetic field generated by the plasma particles themselves. This electromagnetic field can be described by Maxwell's equations. Let $E(x,t)$ denote the electric and $B(x,t)$ the magnetic field. Then in a three-dimensional setting Maxwell's equations are given by

$$\begin{aligned} \nabla \cdot E &= \frac{\rho}{\epsilon_0}, && \text{(Gauss's law)} \\ \nabla \cdot B &= 0, && \text{(Gauss's law for magnetism)} \\ \nabla \times E &= -\frac{\partial B}{\partial t}, && \text{(Faraday's law)} \\ \nabla \times B &= \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right). && \text{(Ampère's law)} \end{aligned}$$

Here, ϵ_0 stands for the electric permittivity of vacuum and μ_0 for the magnetic permeability of vacuum. Further, ρ denotes the electric charge density and J the electric current density. These quantities can be calculated from the distribution function by

$$\rho(x,t) = \sum q \int_{\mathbb{R}^3} f(x,v,t)\,\mathrm{d}v \quad \text{and} \quad J(x,t) = \sum q \int_{\mathbb{R}^3} f(x,v,t)v\,\mathrm{d}v,$$

[2]Source: [16] Goldston, R.J. and Rutherford P.H., 1998, p.309.

where q denotes the charge of the particles and the sum is taken over all particle species contributing to the plasma [16, 25].

Chapter 3

The Vlasov-Poisson system

The Vlasov equation is an important partial differential equation for the kinetic modeling of plasma. In this chapter it will be presented and coupled to Maxwell's equations resulting under certain assumptions in the Vlasov-Poisson system. This part is mainly taken from [27]. Then we will study plasma oscillations for a medium that initially was disturbed from equilibrium and follow the approach by Landau [18] which he used to show a damping effect of the electric field. Finally, we will give a physical interpretation of the phenomenon of Landau damping. Note that for simplicity, we will set the physical constants ϵ_0, μ_0 and k_B in the following chapters to one.

3.1 The equations

Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}$ with $\Lambda \subseteq \mathbb{R}^3$ be a distribution function mapping (x, v, t) to $f(x, v, t)$ and $E(x, t)$ and $B(x, t)$ denote the electric and magnetic field. For one species of particles with mass m and charge q and in a non-relativistic setting, the *Vlasov equation* is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{q}{m}(E + v \times B) \cdot \nabla_v f = 0. \tag{3.1}$$

It describes the time evolution of the distribution function of charged particles in an electromagnetic field assuming that binary collisions of the particles can be neglected.

We further assume that the electromagnetic field is self-consistent and not externally applied and that it is not or only very little time-dependent. Its contribution is taken into account by coupling (3.1) to the stationary Maxwell equations

$$\begin{aligned}
\nabla \cdot E &= \rho, \\
\nabla \cdot B &= 0, \\
\nabla \times E &= 0, \\
\nabla \times B &= J.
\end{aligned}$$

Note that in this setting the electric and magnetic field are decoupled.

In many cases, the contribution of the magnetic field is very small. Thus, it will be neglected in further considerations and we are left with the two Maxwell equations

$$\nabla \cdot E = \rho, \tag{3.2}$$

$$\nabla \times E = 0. \tag{3.3}$$

From (3.3) we get that E can be derived from a scalar potential ϕ. Thus, we can write $E = -\nabla\phi$. Inserting this notation in (3.2), we get the Poisson equation

$$-\Delta\phi = \rho.$$

Moreover, we make the assumption that the considered plasma consists only of electrons and ions. This is justifiable for very warm plasmas [9]. As the ions have a much larger mass than the electrons they can be assumed immobile and their contribution can be considered in form of a neutralising background density n_0. Then the Vlasov equation for the electron motion is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{e}{m} E \cdot \nabla_v f = 0, \tag{3.4}$$

where we inserted the negative elementary charge e. The symbol m represents the electron mass. From Maxwell's equations, we get for the electric field

$$E = -\nabla\phi, \quad -\Delta\phi = \rho = e(n_0 - n). \tag{3.5}$$

The macroscopic number density can be calculated from the distribution function by

$$n(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\,\mathrm{d}v.$$

This coupled system is called the *Vlasov-Poisson system.* It will be the basis for further considerations in this chapter.

3.2 On Landau damping

In 1946, Landau published his famous article "On the vibrations of the electronic plasma" [18]. There he treated the Vlasov-Poisson system and showed a damping effect of the electric field that arises due to an initial disturbance from the equilibrium distribution. Unlike from Vlasov in [31], he considered an initial value problem involving a Laplace transform of the distribution function such that an encountered singularity was treated properly. In this way, he predicted the damping phenomenon from a strictly mathematical point of view [23]. In experiments, the effect of Landau damping could be successfully observed only about 20 years later, see [21].

Following Landau's approach we will consider a linearized version of the Vlasov-Poisson system. A correct treatment of the non-linear case was first given in 2010 by Clément Mouhot and Cédric Villani. Their ideas can be found in [22] and [23].

Our study concerns longitudinal plasma oscillations. These plasma waves are characterised by a wave vector k determining the direction of the wave propagation and a wave frequency $\bar{\omega}$. For longitudinal plasma waves it is possible to restrict our studies to a one-dimensional setting. As in [18] we choose the x-axis along the direction of the wave vector k and obtain

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0, \tag{3.6}$$

where $f : \Lambda \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R},\ (x, v, t) \to f(x, v, t)$ with $\Lambda \subseteq \mathbb{R}$ and the electric field $E(x, t)$ can be calculated from

$$\frac{\partial E}{\partial x} = e(n_0 - n). \tag{3.7}$$

3.2.1 Landau's approach

The three main steps Landau performs in order to examine the electric field are a linearization of the Vlasov-Poisson system followed by a Fourier transform in the space variable and a Laplace transform in the time variable. For these parts we shall mainly follow [18].

Let us start with a distribution function f that initially is slightly disturbed from equilibrium. We shall write

$$f(x, v, t) = f^{equ}(v)\ (1 + h(x, v, t)) \tag{3.8}$$

with h being a perturbation that is small compared to the Maxwellian equilibrium distribution

$$f^{equ}(v) = \frac{n_0}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{v^2}{2T_0/m}\right). \tag{3.9}$$

Here, n_0 denotes the constant background density of neutralising ions and T_0 the constant equilibrium temperature.

Linearization of the Vlasov-Poisson equation

Plugging the expression (3.8) for the distribution function f in the Vlasov-Poisson equation (3.6) gives

$$\frac{\partial}{\partial t}(f^{equ} + f^{equ}h) + v\frac{\partial}{\partial x}(f^{equ} + f^{equ}h) - \frac{eE}{m}\frac{\partial}{\partial v}(f^{equ} + f^{equ}h) = 0. \tag{3.10}$$

For the electric field we get from (3.7)

$$\frac{\partial E}{\partial x} = e\left(n_0 - \int_{\mathbb{R}} f\,\mathrm{d}v\right) = e\left(n_0 - \int_{\mathbb{R}} f^{equ}\,\mathrm{d}v - \int_{\mathbb{R}} f^{equ}h\,\mathrm{d}v\right) = -e\int_{\mathbb{R}} f^{equ}h\,\mathrm{d}v. \tag{3.11}$$

As the equilibrium distribution f^{equ} given in (3.9) does not depend on the time variable t and the space variable x, these partial derivatives vanish in (3.10). Further, one can see from (3.11) that the electric field depends on the small perturbation h. Therefore, the last term in (3.10) is non-linear. We obtain the following linearized form

$$\frac{\partial}{\partial t} f^{equ} h + v \frac{\partial}{\partial x} f^{equ} h - \frac{eE}{m} \frac{\partial}{\partial v} f^{equ} = 0.$$

With

$$\frac{\partial}{\partial v} f^{equ} = -\frac{vm}{T_0} f^{equ}$$

and $f^{equ} \neq 0$ this equation simplifies to

$$\frac{\partial}{\partial t} h + v \frac{\partial}{\partial x} h + \frac{evE}{T_0} = 0. \tag{3.12}$$

Fourier expansion in the space variable

In [18] Landau suggests to consider solutions of the form

$$\hat{h}(v,t) e^{ikx} \quad \text{and} \quad \hat{E}(t) e^{ikx} \tag{3.13}$$

for the initial distortion h and the electric field E. The notation $\hat{h}$ and $\hat{E}$ stands for the corresponding Fourier components. They are obtained by performing a Fourier transform or a Fourier series expansion depending on the underlying setting. The latter is commonly used in plasma physics when considering a torus $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$ of side length L [23].

Landau's ansatz for the solution in (3.13) can be considered as a wave propagating in the direction of the wave vector k. Without loss of generality, we assume that the wave propagates in a positive direction and use fixed $k > 0$ throughout this work.

By multiplication with e^{-ikx} and integration with respect to x one gets the following relations for the Fourier components. From (3.12) one obtains

$$\frac{\partial}{\partial t} \hat{h} + ikv\hat{h} + \frac{ev\hat{E}}{T_0} = 0 \tag{3.14}$$

and from (3.11) one gets for the electric field

$$ik\hat{E} = -e \int_{\mathbb{R}} f^{equ} \hat{h} \, \mathrm{d}v. \tag{3.15}$$

Note that for giving expressions (3.14) and (3.15) we have assumed all quantities to be smooth and integrable enough such that one can interchange the order of integration and the process of integration and partial differentiation. Further, h and E shall fulfill appropriate boundary conditions.

Laplace transform in the time variable

In the next step we would like to eliminate the time-dependence in (3.14). For this purpose, we apply a Laplace transform in the time variable t. Let us assume that $\hat{h}$ and $\hat{E}$ fulfill the conditions from Theorem 2.5 such that their Laplace transforms $\tilde{h}$ and $\tilde{E}$ exist and are well-defined. Further, let us use the notation $s = -i\bar{\omega}$ with $\bar{\omega} \in \mathbb{C}$. Multiplication of (3.14) and (3.15) with $e^{i\bar{\omega}t}$ and integration with respect to t results in

$$(-i\bar{\omega} + ikv)\tilde{h} + \frac{ev\tilde{E}}{T_0} = \hat{h}(v,0) \qquad \text{and} \tag{3.16}$$

$$ik\tilde{E} = -e \int_{\mathbb{R}} f^{equ}\tilde{h}\,\mathrm{d}v, \tag{3.17}$$

where $\hat{h}(v,0)$ denotes the initial value of the Fourier transform $\hat{h}$ at time $t = 0$. Moreover, we have assumed that the integration order is exchangeable and that $\hat{h}$ vanishes at very large times.

Algebraic expression for $\tilde{E}$

Solving (3.16) for $\tilde{h}$, inserting this expression into (3.17) and solving for $\tilde{E}$ gives us

$$\tilde{E}(k,\bar{\omega}) = \frac{N(k,\bar{\omega})}{D(k,\bar{\omega})} \tag{3.18}$$

for the electric field with

$$N(k,\bar{\omega}) = \frac{e}{k^2}\int_{\mathbb{R}} \frac{\hat{h}(v,0)f^{equ}}{v - \frac{\bar{\omega}}{k}}\,\mathrm{d}v \tag{3.19}$$

and

$$D(k,\bar{\omega}) = 1 + \frac{e}{k^2}\int_{\mathbb{R}} \frac{\frac{ev}{T_0}f^{equ}}{v - \frac{\bar{\omega}}{k}}\,\mathrm{d}v. \tag{3.20}$$

From equation (3.18) it is possible to draw conclusions for the behaviour of the electric field for an arbitrary initial distribution. To this end, it is desirable to apply the inverse Laplace transform to (3.18). Herein, some difficulties arise.

3.2.2 Application of the inverse Laplace transform

For the application of the inverse Laplace transform

$$\hat{E}(t) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \tilde{E}(k,\bar{\omega})e^{-i\bar{\omega}t}\,\mathrm{d}\bar{\omega}, \tag{3.21}$$

the function $\tilde{E}(k,\cdot)$ has to fulfill the conditions from Theorem 2.8. According to [27], they are satisfied if $\tilde{E}$ is analytic for $\mathrm{Re}(s) = \mathrm{Im}(\bar{\omega}) > R$ with $R \in \mathbb{R}$ being constant. As discussed in the following, this analyticity is given for $R = 0$.

Analyticity of $\tilde{E}$ for $\mathrm{Im}(\bar{\omega}) > 0$

Consider the quotient in (3.18). For $\mathrm{Im}(\bar{\omega}) > 0$ the denominator in the integrals in (3.19) and (3.20) does not vanish. Hence, the expressions $N(k, \bar{\omega})$ and $D(k, \bar{\omega})$ are well-defined. Being a Maxwellian distribution, f^{equ} is analytic. Assume $\hat{h}(v, 0)$ to be also analytic. Then for $\mathrm{Im}(\bar{\omega}) > 0$ the expression $\tilde{E}$ as in (3.18) is the quotient of analytic functions and hence analytic apart from zeros of the denominator which do not appear in this case. This will be clear from later considerations. For the moment, $\tilde{E}$ can be assumed analytic for $\mathrm{Im}(\bar{\omega}) > 0$, and in this case the inverse Laplace transform is well-defined [27].

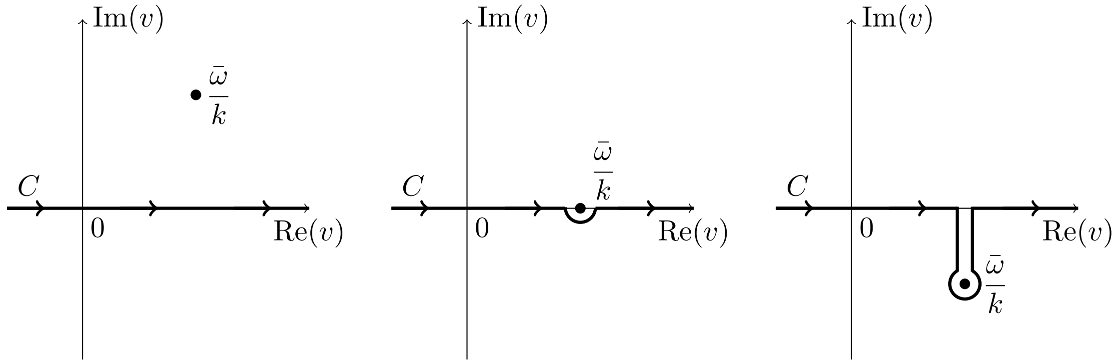
Analytical continuation of $N(k, \bar{\omega})$ and $D(k, \bar{\omega})$ for $\mathrm{Im}(\bar{\omega}) \leq 0$

In the next step we would like to extend the idea of the inverse Laplace transform and define an analytical continuation for $\mathrm{Im}(\bar{\omega}) \leq 0$. For this purpose, we first analyse the behaviour of the integrals in (3.19) and (3.20) for all possible cases of $\mathrm{Im}(\bar{\omega})$ as done in [5] and [27].

Let us consider a function of the form

$$G(\bar{\omega}) = \int_{\mathbb{R}} \frac{g(v)}{v - \frac{\bar{\omega}}{k}} \, \mathrm{d}v \tag{3.22}$$

with g being an analytic function. We also want $G(\bar{\omega})$ to be analytic. Depending on the values of $\bar{\omega}$ and k there appears a zero in the denominator of (3.22) at $v = \frac{\bar{\omega}}{k}$. As before let us assume $k > 0$ is fixed. For $\mathrm{Im}(\bar{\omega}) > 0$ the pole is situated above the real line. Hence, the integration in (3.22) can be evaluated without any problems along the real line. This case is illustrated in Figure 3.1a. For decreasing values of $\mathrm{Im}(\bar{\omega})$ the pole at $v = \frac{\bar{\omega}}{k}$ starts to drop downwards. As long as the pole does not cross the integration contour this does not affect the analyticity of $G(\bar{\omega})$. Otherwise, there would be a discontinuous jump. Hence, the integration contour has to be chosen such that the pole at $v = \frac{\bar{\omega}}{k}$ always lies above it. For $\mathrm{Im}(\bar{\omega}) = 0$ the considered pole lies on the real axis. In this case, an analytical continuation can be given by integrating along a small semicircle that passes below the pole. This is displayed in Figure 3.1b. For $\mathrm{Im}(\bar{\omega}) < 0$ the pole can be found below the real axis. In this case Landau proposed to integrate around the pole as shown in Figure 3.1c.



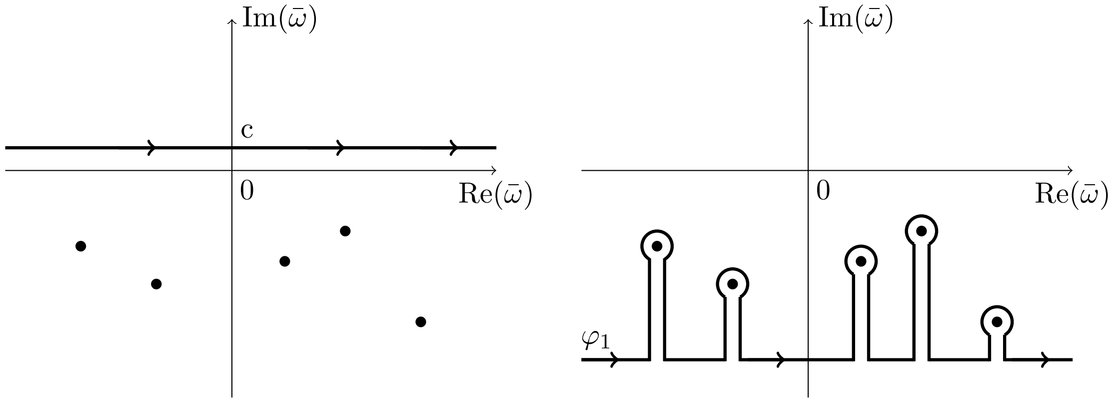
(a) $\mathrm{Im}(\bar{\omega}) > 0$ **(b)** $\mathrm{Im}(\bar{\omega}) = 0$ **(c)** $\mathrm{Im}(\bar{\omega}) < 0$

Figure 3.1: Position of the pole $v = \frac{\bar{\omega}}{k}$ and adequate integration contours for different values of $\mathrm{Im}(\bar{\omega})$.

Let C denote the corresponding integration contour. Then integrating in (3.22) along C instead of the real axis gives an entire function $G(\bar{\omega})$. Hence, by adapting the integration contour, we are able to give an analytical continuation for $N(k, \bar{\omega})$ as in (3.19) and for $D(k, \bar{\omega})$ as in (3.20).

Analytical continuation of the inverse Laplace transform for $\mathrm{Im}(\bar{\omega}) \leq 0$

Remember that for the application of the inverse Laplace transform as in (3.21), it is crucial that the integral is taken along a horizontal line parallel to the real axis such that all singularities lie below the contour of integration. This situation is illustrated in Figure 3.2a. The poles are depicted by the black points. Using the analytical continuations for $N(k, \bar{\omega})$ and $D(k, \bar{\omega})$ we can shift the integration contour in the inverse Laplace transform downwards and deform it continuously in the following way such that the poles of $\tilde{E}(k, \bar{\omega})$ are still situated below the path of integration.



(a) Classical integration contour in the inverse Laplace transform along a horizontal line at the value c.

(b) Deformed integration contour for the analytical continuation of the inverse Laplace transform.

Figure 3.2: Integration contours for the inverse Laplace transform and its analytical continuation.

Using the deformed integration contour φ_1 from Figure 3.2b we are able to give an analytical continuation of the inverse Laplace transform for $\mathrm{Im}(\bar{\omega}) \leq 0$ [5, 16].

Evaluation of the inverse Laplace transform using the residue theorem

The integral along the deformed integration path

$$\int_{\varphi_1} \tilde{E}(k,\bar{\omega}) e^{-i\bar{\omega}t} \,\mathrm{d}\bar{\omega}$$

appearing in the inverse Laplace transform with φ_1 as in Figure 3.2b shall now be evaluated. A convenient method giving the value of the integral for an arbitrarily deformed integration contour consists in using Theorem 2.16, the residue theorem. For its application, the integration contour has to be closed. This is done by adding a semicircle of radius going to infinity parametrized by φ_2 to the integration path φ_1. This closed new contour shall be denoted by $\varphi = \varphi_1 + \varphi_2$. It can be seen in Figure 3.3.

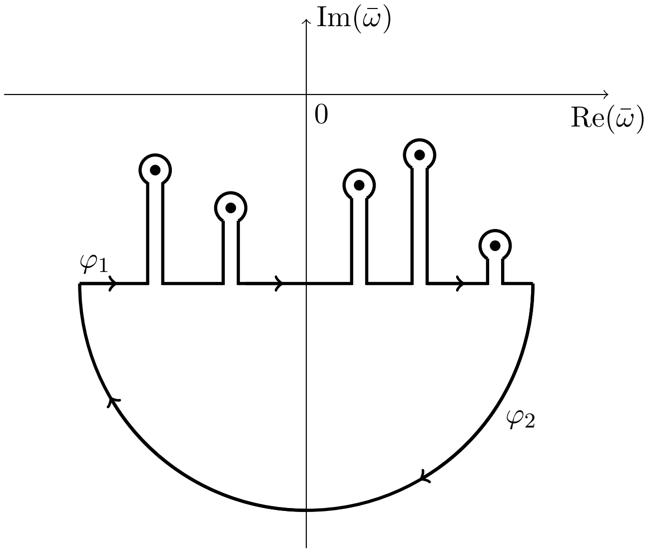


Figure 3.3: Closed integration contour for the application of the residue theorem.

Then it holds by additivity of the integral

$$\int_{\varphi} \tilde{E}(k,\bar{\omega}) e^{-i\bar{\omega}t} \,\mathrm{d}\bar{\omega} = \int_{\varphi_1} \tilde{E}(k,\bar{\omega}) e^{-i\bar{\omega}t} \,\mathrm{d}\bar{\omega} + \int_{\varphi_2} \tilde{E}(k,\bar{\omega}) e^{-i\bar{\omega}t} \,\mathrm{d}\bar{\omega}$$

and the residue theorem gives the following relation for the integral along φ

$$\int_{\varphi} \tilde{E}(k,\bar{\omega}) e^{-i\bar{\omega}t} \,\mathrm{d}\bar{\omega} = 2\pi i \sum_{j=1}^{p} n(\varphi, \bar{\omega}_j) \,\mathrm{Res}(\tilde{E}, \bar{\omega}_j) e^{-i\bar{\omega}_j t}.$$

Here, the poles of $\tilde{E}$ are denoted by $\bar{\omega}_j$ with $j = 1, ..., p$ for a finite number $p \in \mathbb{N}$. They are all surrounded once by the integration contour φ. Hence, for their winding number it holds $n(\varphi, \bar{\omega}_j) = 1$. Further, for the radius of the semicircle going to infinity the contribution of the integral along φ_2 becomes negligible [27]. We approximatively get

$$\int_{\varphi_1} \tilde{E}(k,\bar{\omega})e^{-i\bar{\omega}t}\,\mathrm{d}\bar{\omega} \approx 2\pi i \sum_{j=1}^{p} \mathrm{Res}(\tilde{E},\bar{\omega}_j)e^{-i\bar{\omega}_j t}.$$

Plugging this expression in (3.21), we get for the Fourier components of the electric field

$$\hat{E}(t) \approx \sum_{j=1}^{p} \mathrm{Res}(\tilde{E},\bar{\omega}_j)e^{-i\bar{\omega}_j t}. \tag{3.23}$$

3.2.3 Dispersion relation of the Vlasov-Poisson system

We see from (3.23) that for determining the temporal behaviour of the electric field the poles $\bar{\omega}_j$ of $\tilde{E}$ play an important role. Let us first make some considerations how these poles can appear.

From (3.18) we know that $\tilde{E}(k,\bar{\omega})$ can be written as the quotient of $N(k,\bar{\omega})$ and $D(k,\bar{\omega})$. Using the discussed analytical continuations for $\mathrm{Im}(\bar{\omega}) \leq 0$ and assuming that $\hat{h}(v,0)$ is analytic, the expressions $N(k,\bar{\omega})$ and $D(k,\bar{\omega})$ can be treated as entire functions. This implies that the only poles of $\tilde{E}(k,\bar{\omega})$ are the zeros of the denominator $D(k,\bar{\omega})$, i.e. the values $\bar{\omega}_j$ for which

$$D(k,\bar{\omega}) = 1 + \frac{e}{k^2}\int_C \frac{\frac{ev}{T_0}f^{equ}}{v-\frac{\bar{\omega}}{k}}\,\mathrm{d}v = 0 \tag{3.24}$$

holds. This equation is called the corresponding *dispersion relation* as it can theoretically be rewritten in the form $\bar{\omega} = \bar{\omega}(k)$ [16].

Relation of the zeros of the dispersion relation and the damping effect

It is now a challenging task to analytically determine the zeros of the dispersion relation (3.24). In the linear setting the main focus is on the zero with the largest imaginary part as this one is dominating for large times [18, 27]. Let $\bar{\omega}_j$ be this zero. We shall denote its real and imaginary part by

$$\bar{\omega}_j = \omega + i\gamma$$

with $\omega, \gamma \in \mathbb{R}$. Hence, for large times the Fourier components of the electric field behave like

$$\hat{E}(t) \to \mathrm{Res}(\tilde{E},\bar{\omega}_j)e^{-i\omega t}e^{\gamma t}.$$

The growth of this expression is determined by the factor $e^{\gamma t}$. To show a damping effect of the electric field, it is therefore crucial that the imaginary part of the largest zero of the dispersion relation is smaller than zero, i.e. $\gamma < 0$.

Landau's solution for large wavelengths

In the limiting case of long waves ($k \to 0$), Landau was able to show such a damping effect in [18] under the assumption that $|\gamma| \ll |\omega|$. For further details the reader is referred to [18]. He found for the real part ω of the plasma wave frequency the relation

$$\omega^2 \approx \omega_p^2 + 3\left(kv_{th}\right)^2, \tag{3.25}$$

where ω_p is the electron plasma frequency introduced in (1.2) and v_{th} stands for the thermal velocity of the electrons given by

$$v_{th} = \left(\frac{T_0}{m}\right)^{1/2}. \tag{3.26}$$

Equation (3.25) is called the *Bohm-Gross dispersion relation* for longitudinal electron plasma waves [9]. For the imaginary part γ of the plasma frequency, Landau obtained the expression

$$\gamma \approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_p^4}{\left(kv_{th}\right)^3} \exp\left(-\frac{\omega^2}{2k^2 v_{th}^2}\right) = -\gamma_L. \tag{3.27}$$

Its value is strictly less than zero. Hence, the desired damping effect named after Landau is proven.

For a detailed description and explanation of how to obtain the above solution of the dispersion relation (3.24) by analytical approximations, we refer to several plasma physics textbooks such as [10], [12], [16] or [29].

3.3 Physical interpretation of Landau damping

Until now the considerations on Landau damping were quite theoretical. The aim of this section is to give a more descriptive physical interpretation of this damping phenomenon. The most common approach is the one from an energetic point of view. We shall follow the explanations from [9, 10, 12, 16].

When dealing with the dispersion relation, it can be shown that the appearance of the damping coefficient γ arises from the pole of (3.24) at the point $v = \frac{\bar{\omega}}{k}$. Let us introduce the notation

$$v_\phi = \frac{\bar{\omega}}{k}$$

and call this quantity the *phase velocity* of the wave of the electric field. Among the plasma electrons there are particles that move with a velocity that is nearly equal to the phase velocity of the wave, i.e. with $v \approx v_\phi$. These particles shall be called *resonant particles*. They are subject to an almost constant electric field. For this reason they are able to interchange energy with the wave by wave-particle interaction.

For a better understanding of the process of energy exchange, we shall make use of the following picture from [10]. Note that this does not faultlessly explain the phenomenon of Landau damping and shall only serve as a visualization. Let us imagine a surfer trying to catch a wave in the ocean. If the surfer simply stood motionless in the water, his surfboard would get up and down as the waves do. In this process there is no energy exchange between the surfboard and the wave. If the surfer had a little motorboat instead of a board he would be able to move with a speed much larger than the wave velocity and just ride over the waves. There would be no significant exchange of energy in this process either. A situation different from that is given if the surfer is moving with a velocity that is almost equal to the wave velocity. In this case he is able to interact with the ocean wave. If the surfboard has a velocity slightly less than the wave velocity it can be accelerated by the wave. In this process the wave loses energy to the surfer. If the surfboard moves a little bit faster than the wave, the surfer would kind of push on the wave. In this setting the wave would gain energy from the surfer. This thought experiment is illustrated in Figure 3.4.

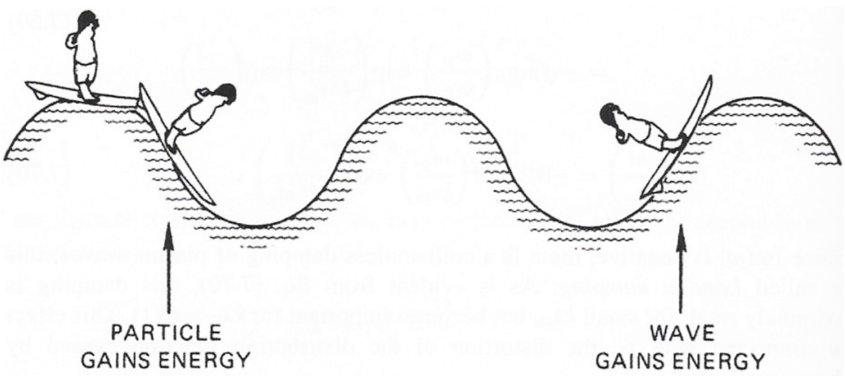


Figure 3.4: Descriptive picture of Landau damping using a surfer riding on a surfboard.[3]

In a plasma there are both resonant particles moving slower as well as faster than the electric field wave. On average, due to the Maxwellian distribution function, there are more resonant electrons with a velocity slightly smaller than the phase velocity.

[3]Source: [10] Chen, F.F., 2016, p.230.

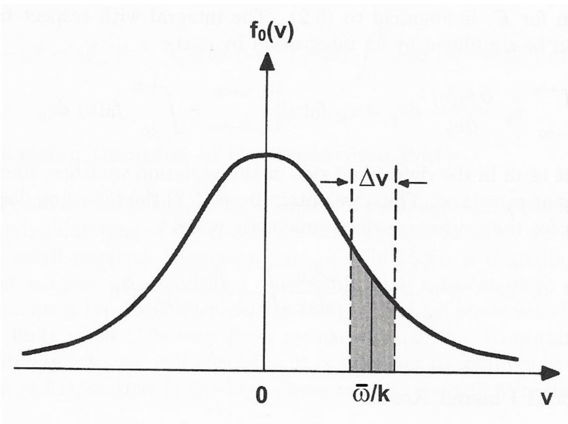


Figure 3.5: Resonant particles in the equilibrium Maxwellian distribution function.[4]

In Figure 3.5 one can see that the equilibrium distribution, denoted by $f_0(v)$, is a decreasing function of $|v|$. In the strip of width Δv around $\frac{\bar{\omega}}{k}$ the resonant particles can be found. In this area there are more electrons moving slower than the phase velocity than faster ones. Hence, in total the wave loses more energy to the plasma particles as it gains from them. This leads to the observed damping effect of the electric field.

As pointed out in [29], this interpretation is not completely consistent. Another possibility consists in treating Landau damping as the result of phase mixing. This approach was studied intensively by Nico van Kampen in [30]. Yet another possibility is given by Clément Mouhot and Cédric Villani in [23].

[4]Source: [9] Bittencourt, J.A., 2004, p.503.

Chapter 4

The Vlasov-Poisson-BGK system

Up to this point the Vlasov-Poisson system consisting of the Vlasov equation (3.4) coupled to the electric field (3.5) was in the focus of our considerations. It is appropriate when binary collisions between the particles are neglected. If indeed binary collisions are taken into account one has to extend the Vlasov equation by a collision operator added to the right-hand side of (3.4). Thereby, the Boltzmann collision operator is a common choice. Since it is very complex a simplified version, the BGK operator, shall be used in this work. It will be introduced in the next section. The Vlasov-Poisson-BGK system will also be given there. Then the corresponding dispersion relation for this system will be derived. As before, the zeros of the dispersion relation play a crucial role for determining the damping effect of the electric field. Hence, the dispersion relation will first be solved analytically, followed by a numerical treatment that qualitatively confirms the analytical results.

4.1 The equations

As before let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R},\ (x, v, t) \to f(x, v, t)$ with $\Lambda \subseteq \mathbb{R}^3$ be a distribution function. The function $M : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R},\ (x, v, t) \to M(x, v, t)$ shall be a Maxwellian distribution of the form

$$M(x, v, t) = \frac{n(x, t)}{(2\pi T(x, t)/m)^{3/2}} \ \exp\left(-\frac{|v - u(x, t)|^2}{2T(x, t)/m}\right)$$

with $n(x, t), u(x, t)$ and $T(x, t)$ being the macroscopic quantities derived from the function f as described in Definition 2.23. Using these functions we can define the BGK operator.

Definition 4.1 (BGK operator, [27])**.** Let ν be a given constant collision frequency. Then, the operator

$$Q_{BGK} = \nu(M - f)$$

is called the *BGK collision operator.*

The BGK operator is named after the scientists Prabhu L. Bhatnagar, Eugene P. Gross, and Max Krook who introduced it in 1954 [8]. It is also known as the Krook operator [27]. It describes a relaxation of f towards the Maxwellian M and can therefore be interpreted as a relaxation time approximation [28]. This relaxation process is sketched in Figure 4.1. The BGK model conserves mass, momentum and energy. Further, the H-theorem which allows to define the entropy of a system can be proven. In addition, the BGK model exhibits the same Maxwellian structure as the Boltzmann equation in equilibrium [25].

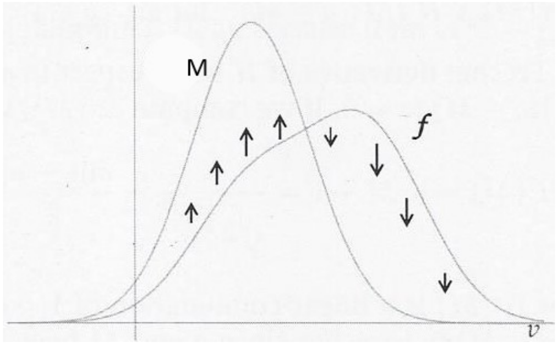


Figure 4.1: Relaxation of f towards the Maxwellian distribution M.[5]

Putting the BGK collision operator on the right-hand side of the Vlasov equation (3.4) gives us

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{e}{m} E \cdot \nabla_v f = \nu(M - f),$$

where ν shall be a small constant collision frequency. For the electric field, we still have from Maxwell's equations

$$E = -\nabla\phi, \quad -\Delta\phi = \rho = e(n_0 - n).$$

This coupled system is called the *Vlasov-Poisson-BGK system.*

As before, we choose the x-axis along the direction of the wave vector k and assume that the wave propagates in a positive direction such that $k > 0$. Under these conditions, we can restrict our considerations to a one-dimensional setting. Further, we assume that the system is in equilibrium temperature such that we can make an isothermal approximation with T_0 denoting the constant equilibrium temperature. This is a reasonable assumption when considering a dilute system with a small collision frequency [34]. The Vlasov-Poisson-BGK equation then writes

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = \nu(M - f) \tag{4.1}$$

with $f : \Lambda \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R},\ (x, v, t) \to f(x, v, t)$ with $\Lambda \subseteq \mathbb{R}$ and the Maxwellian distribution

$$M(x, v, t) = \frac{n(x, t)}{(2\pi T_0/m)^{1/2}} \ \exp\left(-\frac{|v - u(x, t)|^2}{2T_0/m}\right), \tag{4.2}$$

[5]Source: Adapted from [25] Pirner, M., 2018, p.31.

where the macroscopic quantities $n(x,t)$ and $u(x,t)$ are determined by

$$\begin{pmatrix} n(x,t) \\ n(x,t)u(x,t) \end{pmatrix} = \int_{\mathbb{R}} f(x,v,t) \begin{pmatrix} 1 \\ v \end{pmatrix} \mathrm{d}v.$$

For the electric field we have the relation

$$\frac{\partial E}{\partial x} = e(n_0 - n). \tag{4.3}$$

4.2 On Landau damping coupled with relaxation

The system consisting of (4.1) and (4.3) will be the subject of further considerations. In detail, we will show a damping effect of the electric field that is partly due to Landau damping as well as to the BGK relaxation. To this end, we will derive an algebraic expression related to the electric field from which we can get the corresponding dispersion relation of the Vlasov-Poisson-BGK system.

4.2.1 Adaption of Landau's approach

In order to determine the damping phenomenon, we will adapt Landau's approach from [18] to the Vlasov-Poisson-BGK system. It consists in linearizing the equations, performing a Fourier expansion in the space variable and a Laplace transform in the time variable.

Consider a distribution function

$$f(x,v,t) = f^{equ}(v)\ (1 + h(x,v,t)) \tag{4.4}$$

that initially is slightly disturbed from equilibrium. The function h denotes a perturbation that shall be small compared to the Maxwellian equilibrium distribution

$$f^{equ}(v) = \frac{n_0}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{v^2}{2T_0/m}\right), \tag{4.5}$$

where n_0 stands for the constant background density of neutralising ions and T_0 for the constant equilibrium temperature.

For the macroscopic quantities such as density $n(x,t)$ and mean velocity $u(x,t)$ we get with the expression from (4.4)

$$\begin{aligned} n(x,t) &= \int_{\mathbb{R}} f\,\mathrm{d}v &&= \int_{\mathbb{R}} f^{equ}\,\mathrm{d}v + \int_{\mathbb{R}} f^{equ} h\,\mathrm{d}v &&= n_0 + \sigma(x,t), && (4.6) \\ n(x,t)u(x,t) &= \int_{\mathbb{R}} f v\,\mathrm{d}v &&= \int_{\mathbb{R}} f^{equ} v\,\mathrm{d}v + \int_{\mathbb{R}} f^{equ} h v\,\mathrm{d}v &&= \mu(x,t), && (4.7) \end{aligned}$$

where we introduced the notations $\sigma(x,t)$ for the perturbed density and $\mu(x,t)$ for the perturbed mean velocity as in [20]. Note that the macroscopic velocity of the equilibrium function given by $\int_{\mathbb{R}} f^{equ} v \,\mathrm{d}v$ is zero in this setting.

Linearization of the Vlasov-Poisson-BGK equation

We start by plugging the expression for the distribution function f given in (4.4) in the Vlasov-Poisson-BGK system. The Vlasov-Poisson-BGK equation (4.1) then becomes

$$\begin{aligned}\frac{\partial}{\partial t}(f^{equ} + f^{equ}h) + v\frac{\partial}{\partial x}(f^{equ} + f^{equ}h) - \frac{eE}{m}\frac{\partial}{\partial v}(f^{equ} + f^{equ}h)&\\ = \nu(M - f^{equ} - f^{equ}h).&\end{aligned} \tag{4.8}$$

For the electric field we get from (4.3) the same relation as in the Vlasov-Poisson case. With the notation from (4.6) we can write

$$\frac{\partial E}{\partial x} = -e\int_{\mathbb{R}} f^{equ} h \,\mathrm{d}v = -e\sigma. \tag{4.9}$$

The linearization of the left-hand side of (4.8) is done analogously as in Chapter 3. For the right-hand side of (4.8) we want to perform a first-order Taylor expansion of M with respect to the small quantities σ and μ around 0 as done in [20]. So let us consider the Maxwellian M from (4.2). Using the relations from (4.6) and (4.7), we can write it as

$$M(x,v,t) = \frac{n_0 + \sigma(x,t)}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{\left|v - \frac{\mu(x,t)}{n_0+\sigma(x,t)}\right|^2}{2T_0/m}\right).$$

The zeroth order of the expansion is given by

$$M_{|\sigma=\mu=0} = \frac{n_0}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{v^2}{2T_0/m}\right) = f^{equ}.$$

For the first order we determine the partial derivatives with respect to σ and μ and evaluate them for $\sigma = \mu = 0$ which gives

$$\begin{aligned}\frac{\partial}{\partial\sigma}M_{|\sigma=\mu=0} &= \frac{1}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{v^2}{2T_0/m}\right) = \frac{1}{n_0}f^{equ},\\ \frac{\partial}{\partial\mu}M_{|\sigma=\mu=0} &= \frac{n_0}{(2\pi T_0/m)^{1/2}}\ \exp\left(-\frac{v^2}{2T_0/m}\right)\frac{vm}{n_0T_0} = \frac{vm}{n_0T_0}f^{equ}.\end{aligned}$$

Hence, we obtain in a neighborhood of $\sigma = \mu = 0$

$$M - f^{equ} \approx f^{equ} + \frac{1}{n_0}f^{equ}\sigma + \frac{vm}{n_0T_0}f^{equ}\mu - f^{equ} = f^{equ}\left(\frac{\sigma}{n_0} + \frac{vm\mu}{n_0T_0}\right).$$

Inserting these results in (4.8), we get in combination with $f^{equ} \neq 0$ the following linearized form of the Vlasov-Poisson-BGK equation

$$\frac{\partial}{\partial t}h + v\frac{\partial}{\partial x}h + \frac{evE}{T_0} = \nu\left(\frac{\sigma}{n_0} + \frac{vm\mu}{n_0T_0} - h\right). \tag{4.10}$$

Fourier expansion in the space variable

In the next step, we seek for a relation concerning the Fourier components of the linearized Vlasov-Poisson-BGK system. Multiplication of (4.10) and (4.9) with e^{-ikx} and integration with respect to x gives us

$$\frac{\partial}{\partial t}\hat{h} + ikv\hat{h} + \frac{ev\hat{E}}{T_0} = \nu\left(\frac{\hat{\sigma}}{n_0} + \frac{vm\hat{\mu}}{n_0T_0} - \hat{h}\right) \qquad \text{and} \tag{4.11}$$

$$ik\hat{E} = -e\hat{\sigma}, \tag{4.12}$$

where

$$\hat{\sigma} = \int_{\mathbb{R}} f^{equ}\hat{h}\,\mathrm{d}v,$$
$$\hat{\mu} = \int_{\mathbb{R}} f^{equ}\hat{h}v\,\mathrm{d}v$$

denote the corresponding Fourier components of σ and μ. Note that for determining the system consisting of (4.11) and (4.12) we have again assumed all quantities to be smooth and integrable enough such that one can interchange the order of integration and the process of integration and partial differentiation. In addition, it is necessary that h and E fulfill appropriate boundary conditions.

Laplace transform in the time variable

We notice that in (4.11) a partial derivative with respect to t still appears. This one shall be eliminated by applying a Laplace transform in the time variable. Hence, let us assume that the functions $\hat{h}, \hat{E}, \hat{\sigma}$ and $\hat{\mu}$ satisfy the conditions from Theorem 2.5. Multiplication of (4.11) and (4.12) with $e^{i\bar{\omega}t}$ and integration with respect to t leads to the well-defined system

$$(-i\bar{\omega} + ikv)\tilde{h} + \frac{ev\tilde{E}}{T_0} - \hat{h}(v,0) = \nu\left(\frac{\tilde{\sigma}}{n_0} + \frac{vm\tilde{\mu}}{n_0T_0} - \tilde{h}\right) \qquad \text{and} \tag{4.13}$$

$$ik\tilde{E} = -e\tilde{\sigma}, \tag{4.14}$$

where the Laplace transforms of $\hat{\sigma}$ and $\hat{\mu}$ are given by

$$\tilde{\sigma} = \int_{\mathbb{R}} f^{equ}\tilde{h}\,\mathrm{d}v, \tag{4.15}$$

$$\tilde{\mu} = \int_{\mathbb{R}} f^{equ}\tilde{h}v\,\mathrm{d}v. \tag{4.16}$$

Note that $\hat{h}(v,0)$ in (4.13) denotes the initial value of the Fourier transform $\hat{h}$ at time $t=0$. Further, we have assumed interchangeability of the order of integration and appropriate boundary conditions with vanishing $\hat{h}$ for $t \to \infty$.

Algebraic expression for $\tilde{\sigma}$

We recognize that the system consisting of (4.13) and (4.14) is given in terms of $\tilde{h}, \tilde{E}, \tilde{\sigma}$ and $\tilde{\mu}$. These quantities do all depend explicitly or implicitly on $\tilde{h}$. Following [8], we would like to eliminate two of those parameters from the Vlasov-Poisson-BGK system. The easiest way consists in replacing $\tilde{E}$ and $\tilde{\mu}$ in terms of $\tilde{\sigma}$. This enables us to finally obtain an algebraic expression for the Laplace transform $\tilde{\sigma}$ of the density. Since the Laplace transform of the density $\tilde{\sigma}$ is related to the Laplace transform $\tilde{E}$ of the electric field by (4.14), we can draw conclusions on the qualitative behaviour of $\tilde{E}$ from examining $\tilde{\sigma}$. This will make up the main part of the following considerations.

In order to eliminate $\tilde{\mu}$ from the Vlasov-Poisson-BGK equation, we start by rearranging (4.13) as follows

$$(-i\bar{\omega} + ikv + \nu)\tilde{h} = \nu \left(\frac{\tilde{\sigma}}{n_0} + \frac{vm\tilde{\mu}}{n_0 T_0} \right) - \frac{ev\tilde{E}}{T_0} + \hat{h}(v,0).$$

In the next step we multiply this equation with f^{equ} and integrate over the velocity v. With $\int_{\mathbb{R}} f^{equ} v \,\mathrm{d}v = 0$ and the notations from (4.15) and (4.16) we get the relation

$$\tilde{\mu} = \frac{\hat{\sigma}(0) + i\bar{\omega}\tilde{\sigma}}{ik}, \tag{4.17}$$

where $\hat{\sigma}(0) = \int_{\mathbb{R}} f^{equ} \hat{h}(v,0) \,\mathrm{d}v$ denotes the initial value of the Fourier transform $\hat{\sigma}$ at time $t=0$. For the Laplace transform $\tilde{E}$ of the electric field we get from (4.14)

$$\tilde{E} = \frac{-e\tilde{\sigma}}{ik}. \tag{4.18}$$

Inserting (4.17) and (4.18) into (4.13), we get after rearranging

$$(-i\bar{\omega} + ikv + \nu)\tilde{h} = \tilde{\sigma} \left(\frac{\nu}{n_0} + \frac{\nu v m i\bar{\omega}}{ikn_0 T_0} + \frac{e^2 v}{ikT_0} \right) + \frac{\nu v m}{ikn_0 T_0} \hat{\sigma}(0) + \hat{h}(v,0).$$

This expression for the Vlasov-Poisson-BGK equation only depends on $\tilde{h}$ and $\tilde{\sigma}$. These quantities are related by (4.15). Division by $(-i\bar{\omega} + ikv + \nu)$, multiplication with f^{equ} and integration with respect to v leads to

$$\tilde{\sigma} = \tilde{\sigma} \int_{\mathbb{R}} \frac{\left(\frac{\nu}{n_0} + \frac{\nu v m i\bar{\omega}}{ikn_0 T_0} + \frac{e^2 v}{ikT_0} \right) f^{equ}}{(-i\bar{\omega} + ikv + \nu)} \,\mathrm{d}v + \int_{\mathbb{R}} \frac{\frac{\nu v m}{ikn_0 T_0} \hat{\sigma}(0) f^{equ}}{(-i\bar{\omega} + ikv + \nu)} \,\mathrm{d}v + \int_{\mathbb{R}} \frac{\hat{h}\,(v,0)\, f^{equ}}{(-i\bar{\omega} + ikv + \nu)} \,\mathrm{d}v.$$

This equation can be solved for $\tilde{\sigma}$. One obtains the following algebraic expression

$$\tilde{\sigma}(k, \bar{\omega}) = \frac{N(k, \bar{\omega})}{D(k, \bar{\omega})}$$

with

$$N(k, \bar{\omega}) = \int_{\mathbb{R}} \frac{\left(\frac{\nu v m}{i k n_0 T_0} \hat{\sigma}(0) + \hat{h}(v, 0)\right) f^{equ}}{ikv - i\bar{\omega} + \nu} \, \mathrm{d}v$$

and

$$D(k, \bar{\omega}) = 1 - \int_{\mathbb{R}} \frac{\left(\frac{\nu}{n_0} + \frac{\nu v m \bar{\omega}}{k n_0 T_0} - \frac{e^2 i v}{k T_0}\right) f^{equ}}{ikv - i\bar{\omega} + \nu} \, \mathrm{d}v.$$

4.2.2 Dispersion relation of the Vlasov-Poisson-BGK system

As in Chapter 3, one would like to apply the inverse Laplace transform to $\tilde{\sigma}$ and evaluate it using the residue theorem. Hence, one has to choose adequate integration contours such that $N(k, \bar{\omega})$ and $D(k, \bar{\omega})$ as well as the integral in the inverse Laplace transform are well-defined and analytically continued for $\mathrm{Im}(\bar{\omega}) \leq 0$. This shall be kept in mind when considering the dispersion relation

$$D(k, \bar{\omega}) = 1 - \int_{\mathbb{R}} \frac{\left(\frac{\nu}{n_0} + \frac{\nu v m \bar{\omega}}{k n_0 T_0} - \frac{e^2 i v}{k T_0}\right) f^{equ}}{ikv - i\bar{\omega} + \nu} \, \mathrm{d}v = 0. \tag{4.19}$$

It can be shown analogously to the Vlasov-Poisson system that the poles of $\tilde{\sigma}$ are exactly the zeros of the dispersion relation (4.19) and that the sign of the imaginary part of the largest such zero determines the long-time behaviour of the electric field. For this reason, the dispersion relation (4.19) and its zeros will be treated in the following sections.

4.3 Analytical treatment of the dispersion relation

The first aim of this analytical treatment of the dispersion relation of the Vlasov-Poisson-BGK system consists in showing a damping effect of the electric field. Secondly, the contribution of Landau damping as well as of the BGK relaxation to the total damping effect shall be worked out separately. Here, we follow and adapt the proceedings from [34] and [35].

4.3.1 Rewriting the dispersion relation

We start by considering the dispersion relation (4.19). Inserting the definition of the thermal velocity (3.26) and the expression (4.5) for the equilibrium distribution function f^{equ} we get

$$1 - \frac{1}{\sqrt{2\pi} v_{th}} \int_{\mathbb{R}} \frac{\left(\nu + \frac{\nu v \bar{\omega}}{k v_{th}^2} - \frac{n_0 e^2 i v}{m k v_{th}^2}\right) \exp\left(-\frac{v^2}{2 v_{th}^2}\right)}{ikv - i\bar{\omega} + \nu} \, \mathrm{d}v = 0.$$

Next, we make the substitution

$$v = \sqrt{2} v_{th} u$$

and use the notation

$$X = \frac{i\bar{\omega} - \nu}{\sqrt{2} i k v_{th}} \tag{4.20}$$

to obtain

$$1 + \frac{1}{\sqrt{2\pi} v_{th}} \frac{i}{k} \int_{\mathbb{R}} \left(\nu + \frac{\sqrt{2}\nu u \bar{\omega}}{k v_{th}} - \frac{\sqrt{2} n_0 e^2 i u}{m k v_{th}}\right) \frac{\exp(-u^2)}{u - X} \, \mathrm{d}u = 0. \tag{4.21}$$

Conditions imposed on the system

As for the Vlasov-Poisson system, we seek for a solution of the dispersion relation in the limiting case of long waves, i.e. for $k \to 0$. Further, we assume the collision frequency ν to be sufficiently small such that the restriction

$$\frac{\nu}{\sqrt{2} k v_{th}} \ll 1$$

is satisfied. For the zero of the dispersion relation with the largest imaginary part we shall again denote $\bar{\omega} = \omega + i\gamma$. Then we can rewrite X from (4.20) as

$$X = \frac{\omega}{\sqrt{2} k v_{th}} + i \frac{\nu + \gamma}{\sqrt{2} k v_{th}} = \eta + i\xi \tag{4.22}$$

with $\eta, \xi \in \mathbb{R}$. The absolute value of the damping coefficient γ shall be very small such that ξ in (4.22) can be considered a small positive quantity. More precisely, $|\gamma| \ll |\omega|$ shall hold. This condition can be justified a posteriori by taking into account that k is very small.

Expansion of the dispersion relation in terms of ξ

In the next step we want to expand the integral in (4.21) in terms of ξ. For this purpose, we first notice that it is of the form

$$I(X) = \int_{\mathbb{R}} \frac{g(u)}{u - X} \, \mathrm{d}u \tag{4.23}$$

with $g(u)$ being an analytic function. For $u = X$ this integral is not defined. We have to consider its Cauchy principal value.

Definition 4.2 (Cauchy principal value, [33])**.** For an integral as in (4.23) its *Cauchy principal value* is defined as

$$P\int_{\mathbb{R}} \frac{g(u)}{u-X}\,\mathrm{d}u = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{X-\epsilon} \frac{g(u)}{u-X}\,\mathrm{d}u + \int_{X+\epsilon}^{\infty} \frac{g(u)}{u-X}\,\mathrm{d}u\right).$$

With this definition we can evaluate (4.23) as follows.

Lemma 4.3 (Jackson's identity, [17])**.** Let $I(X)$ be an integral as given in (4.23). Then, for small $\xi > 0$ it holds

$$I(X) = I(\eta + i\xi) = \sum_{j=0}^{\infty} \frac{(i\xi)^j}{j!}\left(P\int_{\mathbb{R}} \frac{g^{(j)}(u)}{u-\eta}\,\mathrm{d}u + \pi i g^{(j)}(\eta)\right), \tag{4.24}$$

where P denotes the Cauchy principal value of the integral.

In the following, we will refer to this representation as *Jackson's identity* as it was proven by John David Jackson in [17]. An advantage of this expansion is that the real and the imaginary parts of the argument appear separated from each other.

Our next goal is to apply Jackson's identity (4.24) to the dispersion relation. First, we define

$$A = \frac{\sqrt{2}n_0e^2i}{mkv_{th}} \quad \text{and} \quad B = \frac{\sqrt{2}\nu\bar{\omega}}{kv_{th}} \tag{4.25}$$

and rewrite (4.21) in the form

$$1 + \frac{1}{\sqrt{2\pi}v_{th}}\frac{i}{k}\int_{\mathbb{R}} (\nu + Bu - Au)\frac{\exp(-u^2)}{u-X}\,\mathrm{d}u = 0. \tag{4.26}$$

With the notations

$$g(u) = \exp(-u^2) \quad \text{and} \quad G(u) = u\exp(-u^2) \tag{4.27}$$

we get from (4.26) the representation

$$1 + \frac{1}{\sqrt{2\pi}v_{th}}\frac{i}{k}\left(\nu\int_{\mathbb{R}} \frac{g(u)}{u-X}\,\mathrm{d}u + (B-A)\int_{\mathbb{R}} \frac{G(u)}{u-X}\,\mathrm{d}u\right) = 0,$$

to which we now want to apply Jackson's identity (4.24). To first order in ξ we obtain

$$\begin{aligned} 1 + \frac{1}{\sqrt{2\pi}v_{th}}\frac{i}{k}\Bigg[&\nu\left(P\int_{\mathbb{R}} \frac{g(u)}{u-\eta}\,\mathrm{d}u + \pi i g(\eta) + i\xi\,P\int_{\mathbb{R}} \frac{g'(u)}{u-\eta}\,\mathrm{d}u - \pi\xi g'(\eta)\right) \\ &+ (B-A)\left(P\int_{\mathbb{R}} \frac{G(u)}{u-\eta}\,\mathrm{d}u + \pi i G(\eta) + i\xi\,P\int_{\mathbb{R}} \frac{G'(u)}{u-\eta}\,\mathrm{d}u - \pi\xi G'(\eta)\right)\Bigg] = 0. \end{aligned} \tag{4.28}$$

Evaluation of the principal value integrals

Equation (4.28) contains four principal value integrals. We need to evaluate them for drawing further conclusions. Since the absolute value of ω is comparably large, the real number η defined in (4.20) becomes huge for the limiting case of large wavelengths ($k \to 0$). We would like to obtain an asymptotic expansion of the principal value integrals in terms of the small quantity η^{-1}. We first notice that with the functions $g(u)$ and $G(u)$ defined in (4.27) we have

$$g'(u) = -2u\exp(-u^2) = -2G(u) \quad \text{and} \quad G'(u) = \exp(-u^2)(1-2u^2)$$

and start by considering the first principal value integral from (4.28). We get

$$\begin{aligned} P\int_{\mathbb{R}} \frac{g(u)}{u-\eta}\,\mathrm{d}u &= P\int_{\mathbb{R}} \frac{\exp(-u^2)}{u-\eta}\,\mathrm{d}u = -\frac{1}{\eta}\,P\int_{-\infty}^{\infty} \frac{\exp(-u^2)}{1-\frac{u}{\eta}}\,\mathrm{d}u \\ &= -\frac{1}{\eta}\int_{-\infty}^{\infty} \exp(-u^2)\sum_{n=0}^{\infty}\left(\frac{u}{\eta}\right)^n \mathrm{d}u. \end{aligned}$$

In the last step we used that $\frac{1}{1-\frac{u}{\eta}}$ defines the limit of the geometric series. Note that the considered principal value integral is assumed to have a finite value for η sufficiently large. Further, the P denoting the principal value integral has been dropped as the integral no longer has a singularity. Next, we naively interchange summation and integration and obtain

$$\begin{aligned} P\int_{\mathbb{R}} \frac{g(u)}{u-\eta}\,\mathrm{d}u &= -\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\int_{-\infty}^{\infty} u^n\exp(-u^2)\,\mathrm{d}u \\ &= -\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\left[(-1)^n\int_0^{\infty} u^n\exp(-u^2)\,\mathrm{d}u + \int_0^{\infty} u^n\exp(-u^2)\,\mathrm{d}u\right] \\ &= -\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\left[\frac{1}{2}(-1)^n\int_0^{\infty} v^{\frac{n-1}{2}}\exp(-v)\,\mathrm{d}v + \frac{1}{2}\int_0^{\infty} v^{\frac{n-1}{2}}\exp(-v)\,\mathrm{d}v\right] \\ &= -\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\left[\frac{1}{2}(-1)^n\int_0^{\infty} v^{\frac{n+1}{2}-1}\exp(-v)\,\mathrm{d}v + \frac{1}{2}\int_0^{\infty} v^{\frac{n+1}{2}-1}\exp(-v)\,\mathrm{d}v\right] \\ &= -\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\left[\frac{1}{2}(-1)^n\Gamma\left(\frac{n+1}{2}\right) + \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\right] \\ &= \begin{cases} -\displaystyle\sum_{n=0}^{\infty}\frac{1}{\eta^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right) & \text{for } n \in 2\mathbb{N}_0, \\ 0 & \text{for } n \in 2\mathbb{N}_0+1. \end{cases} \end{aligned}$$

Here, we made use of the substitution $v = u^2$. Further, we inserted the gamma function which was introduced in Definition 2.18. Evaluating the gamma function using the relations from Lemma 2.19 leads to

$$P\int_{\mathbb{R}} \frac{g(u)}{u-\eta}\,\mathrm{d}u = -\frac{\sqrt{\pi}}{\eta}\left(1 + \frac{1}{2\eta^2} + \frac{3}{4\eta^4} + \mathcal{O}\left(\frac{1}{\eta^6}\right)\right).$$

Equivalently, we obtain for the remaining principal value integrals

$$
\begin{aligned}
P\int_{\mathbb{R}} \frac{G(u)}{u-\eta}\,\mathrm{d}u &= -\frac{\sqrt{\pi}}{\eta}\left(\frac{1}{2\eta}+\frac{3}{4\eta^3}+\frac{15}{8\eta^5}+\mathcal{O}\left(\frac{1}{\eta^7}\right)\right),\\
P\int_{\mathbb{R}} \frac{g'(u)}{u-\eta}\,\mathrm{d}u &= \frac{\sqrt{\pi}}{\eta}\left(\frac{1}{\eta}+\frac{3}{2\eta^3}+\frac{15}{4\eta^5}+\mathcal{O}\left(\frac{1}{\eta^7}\right)\right),\\
P\int_{\mathbb{R}} \frac{G'(u)}{u-\eta}\,\mathrm{d}u &= \frac{\sqrt{\pi}}{\eta}\left(\frac{1}{\eta^2}+\frac{3}{\eta^4}+\mathcal{O}\left(\frac{1}{\eta^6}\right)\right).
\end{aligned}
$$

Inserting these expansions to order η^{-4} in the dispersion relation (4.28) gives us

$$
\begin{aligned}
1+\frac{1}{\sqrt{2\pi}v_{th}}\frac{i}{k}\Bigg[&\nu\left(-\frac{\sqrt{\pi}}{\eta}\left(1+\frac{1}{2\eta^2}\right)+\pi i\exp(-\eta^2)\right)\\
&+\nu\left(i\xi\frac{\sqrt{\pi}}{\eta}\left(\frac{1}{\eta}+\frac{3}{2\eta^3}\right)+\pi\xi 2\eta\exp(-\eta^2)\right)\\
&+(B-A)\left(-\frac{\sqrt{\pi}}{\eta}\left(\frac{1}{2\eta}+\frac{3}{4\eta^3}\right)+\pi i\eta\exp(-\eta^2)\right)\\
&+(B-A)\left(i\xi\frac{\sqrt{\pi}}{\eta}\left(\frac{1}{\eta^2}\right)-\pi\xi(1-2\eta^2)\exp(-\eta^2)\right)\Bigg]=0.
\end{aligned}
$$

With the expressions from (4.25) for A and B and by inserting $\bar{\omega}=\omega+i\gamma$ we get

$$
\begin{aligned}
1&-\frac{i\nu}{\sqrt{2}kv_{th}\eta}-\frac{i\nu}{2\sqrt{2}kv_{th}\eta^3}-\frac{\nu\sqrt{\pi}\exp(-\eta^2)}{\sqrt{2}kv_{th}}-\frac{\xi\nu}{\sqrt{2}kv_{th}\eta^2}-\frac{3\xi\nu}{2\sqrt{2}kv_{th}\eta^4}\\
&+\frac{i\nu\sqrt{2\pi}\xi\eta\exp(-\eta^2)}{kv_{th}}-\frac{i\nu\omega}{2k^2v_{th}^2\eta^2}-\frac{3i\nu\omega}{4k^2v_{th}^2\eta^4}-\frac{\sqrt{\pi}\eta\exp(-\eta^2)}{\sqrt{2}k^2v_{th}^2}-\frac{\nu\omega\xi}{k^2v_{th}^2\eta^3}\\
&-\frac{i\sqrt{\pi}\nu\omega\xi\exp(-\eta^2)}{k^2v_{th}^2}+\frac{i\sqrt{\pi}\nu\omega\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2}+\frac{\nu\gamma}{2k^2v_{th}^2\eta^2}+\frac{3\nu\gamma}{4k^2v_{th}^2\eta^4}\\
&-\frac{i\sqrt{\pi}\nu\gamma\eta\exp(-\eta^2)}{k^2v_{th}^2}-\frac{i\nu\gamma\xi}{k^2v_{th}^2\eta^3}+\frac{\sqrt{\pi}\nu\gamma\xi\exp(-\eta^2)}{k^2v_{th}^2}-\frac{\sqrt{\pi}\nu\gamma\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2}\\
&-\frac{n_0e^2}{2k^2v_{th}^2m\eta^2}-\frac{3n_0e^2}{4k^2v_{th}^2m\eta^4}+\frac{i\sqrt{\pi}n_0e^2\eta\exp(-\eta^2)}{k^2v_{th}^2m}+\frac{in_0e^2\xi}{k^2v_{th}^2m\eta^3}\\
&-\frac{\sqrt{\pi}n_0e^2\xi\exp(-\eta^2)}{k^2v_{th}^2m}+\frac{\sqrt{\pi}n_0e^2\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2m}=0.
\end{aligned}
\tag{4.29}
$$

We can clearly split this equation into its real and imaginary part. They are studied in the next sections.

4.3.2 Solution for the real part ω of the plasma wave frequency

The real part of equation (4.29) is given as follows

$$\begin{aligned}1 &- \frac{\nu\sqrt{\pi}\exp(-\eta^2)}{\sqrt{2}kv_{th}} - \frac{\xi\nu}{\sqrt{2}kv_{th}\eta^2} - \frac{3\xi\nu}{2\sqrt{2}kv_{th}\eta^4} - \frac{\sqrt{\pi}\eta\exp(-\eta^2)}{\sqrt{2}k^2v_{th}^2} - \frac{\nu\omega\xi}{k^2v_{th}^2\eta^3}\\ &+ \frac{\nu\gamma}{2k^2v_{th}^2\eta^2} + \frac{3\nu\gamma}{4k^2v_{th}^2\eta^4} + \frac{\sqrt{\pi}\nu\gamma\xi\exp(-\eta^2)}{k^2v_{th}^2} - \frac{\sqrt{\pi}\nu\gamma\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2}\\ &- \frac{n_0e^2}{2k^2v_{th}^2m\eta^2} - \frac{3n_0e^2}{4k^2v_{th}^2m\eta^4} - \frac{\sqrt{\pi}n_0e^2\xi\exp(-\eta^2)}{k^2v_{th}^2m} + \frac{\sqrt{\pi}n_0e^2\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2m} = 0.\end{aligned}$$

In the next step we insert the definition of ξ from (4.22) back into the equation. We obtain

$$\begin{aligned}1 &- \frac{\nu\sqrt{\pi}\exp(-\eta^2)}{\sqrt{2}kv_{th}} - \frac{(\nu+\gamma)\,\nu}{2k^2v_{th}^2\eta^2} - \frac{3\,(\nu+\gamma)\,\nu}{4k^2v_{th}^2\eta^4} - \frac{\sqrt{\pi}\eta\exp(-\eta^2)}{\sqrt{2}k^2v_{th}^2} - \frac{\nu\omega\,(\nu+\gamma)}{\sqrt{2}k^3v_{th}^3\eta^3} + \frac{\nu\gamma}{2k^2v_{th}^2\eta^2}\\ &+ \frac{3\nu\gamma}{4k^2v_{th}^2\eta^4} + \frac{\sqrt{\pi}\nu\gamma\,(\nu+\gamma)\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3} - \frac{\sqrt{\pi}\nu\gamma\,(\nu+\gamma)\,2\eta^2\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3} - \frac{n_0e^2}{2k^2v_{th}^2m\eta^2}\\ &- \frac{3n_0e^2}{4k^2v_{th}^2m\eta^4} - \frac{\sqrt{\pi}n_0e^2\,(\nu+\gamma)\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3m} + \frac{\sqrt{\pi}n_0e^2\,(\nu+\gamma)\,2\eta^2\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3m} = 0.\end{aligned}$$

Since ν and γ are small quantities, we can neglect terms in ν^2 and in $\nu\gamma$. Further, for $\frac{\omega^2}{k^2v_{th}^2} \gg 1$ it is possible to approximate $\omega^2 \approx \omega_p^2$. In this case the exponential terms become very small and can also be neglected. This leads to the simplified equation

$$1 - \frac{n_0e^2}{2k^2v_{th}^2m\eta^2} - \frac{3n_0e^2}{4k^2v_{th}^2m\eta^4} = 0.$$

Inserting the definition of η from (4.22) and the definition of the plasma frequency from (1.2) gives

$$1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2v_{th}^2}{\omega^2}\right) = 0, \tag{4.30}$$

which shall be solved for ω. For $\frac{k^2v_{th}^2}{\omega^2} \ll 1$ we can solve equation (4.30) iteratively. In zeroth order, i.e. for $\frac{k^2v_{th}^2}{\omega^2} = 0$, we get $\omega^2 = \omega_p^2$. For the first-order approximation this roughly approximated solution can be inserted in the small correction term in (4.30). We get

$$1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2v_{th}^2}{\omega_p^2}\right) \approx 0,$$

which can be equivalently rewritten as

$$\omega^2 \approx \omega_p^2 + 3k^2v_{th}^2. \tag{4.31}$$

This solution corresponds to the well-known Bohm-Gross dispersion relation (3.25) for longitudinal electron oscillations in a dilute plasma.

4.3.3 Solution for the imaginary part γ of the plasma wave frequency

Correspondingly, we consider the imaginary part of (4.29)

$$\begin{aligned}
&-\frac{\nu}{\sqrt{2}kv_{th}\eta}-\frac{\nu}{2\sqrt{2}kv_{th}\eta^3}+\frac{\nu\sqrt{2\pi}\xi\eta\exp(-\eta^2)}{kv_{th}}-\frac{\nu\omega}{2k^2v_{th}^2\eta^2}-\frac{3\nu\omega}{4k^2v_{th}^2\eta^4}\\
&-\frac{\sqrt{\pi}\nu\omega\xi\exp(-\eta^2)}{k^2v_{th}^2}+\frac{\sqrt{\pi}\nu\omega\xi 2\eta^2\exp(-\eta^2)}{k^2v_{th}^2}-\frac{\sqrt{\pi}\nu\gamma\eta\exp(-\eta^2)}{k^2v_{th}^2}\\
&-\frac{\nu\gamma\xi}{k^2v_{th}^2\eta^3}+\frac{\sqrt{\pi}n_0e^2\eta\exp(-\eta^2)}{k^2v_{th}^2m}+\frac{n_0e^2\xi}{k^2v_{th}^2m\eta^3}=0.
\end{aligned}$$

As for the real part, we insert the definition of ξ from (4.22) back into the equation and we obtain

$$\begin{aligned}
&-\frac{\nu}{\sqrt{2}kv_{th}\eta}-\frac{\nu}{2\sqrt{2}kv_{th}\eta^3}+\frac{\nu\sqrt{2\pi}(\nu+\gamma)\eta\exp(-\eta^2)}{\sqrt{2}k^2v_{th}^2}-\frac{\nu\omega}{2k^2v_{th}^2\eta^2}-\frac{3\nu\omega}{4k^2v_{th}^2\eta^4}\\
&-\frac{\sqrt{\pi}\nu\omega(\nu+\gamma)\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3}+\frac{\sqrt{\pi}\nu\omega(\nu+\gamma)2\eta^2\exp(-\eta^2)}{\sqrt{2}k^3v_{th}^3}-\frac{\sqrt{\pi}\nu\gamma\eta\exp(-\eta^2)}{k^2v_{th}^2}\\
&-\frac{\nu\gamma(\nu+\gamma)}{\sqrt{2}k^3v_{th}^3\eta^3}+\frac{\sqrt{\pi}n_0e^2\eta\exp(-\eta^2)}{k^2v_{th}^2m}+\frac{n_0e^2(\nu+\gamma)}{\sqrt{2}k^3v_{th}^3m\eta^3}=0.
\end{aligned}$$

Neglecting terms in $\nu\gamma$ and quadratic terms in ν results in

$$-\frac{\nu}{\sqrt{2}kv_{th}\eta}-\frac{\nu}{2\sqrt{2}kv_{th}\eta^3}-\frac{\nu\omega}{2k^2v_{th}^2\eta^2}-\frac{3\nu\omega}{4k^2v_{th}^2\eta^4}+\frac{\sqrt{\pi}n_0e^2\eta\exp(-\eta^2)}{k^2v_{th}^2m}+\frac{n_0e^2(\nu+\gamma)}{\sqrt{2}k^3v_{th}^3m\eta^3}=0.$$

With the definition of η from (4.22) and the definition of the plasma frequency from (1.2) this equation can be rewritten as

$$-\frac{2\nu}{\omega}-\frac{\nu k^2v_{th}^2}{\omega^3}-\frac{3\nu k^2v_{th}^2}{\omega^3}+\frac{\sqrt{\pi}\omega_p^2\omega\exp\left(-\frac{\omega^2}{2k^2v_{th}^2}\right)}{\sqrt{2}k^3v_{th}^3}+\frac{2\nu\omega_p^2}{\omega^3}+\frac{2\gamma\omega_p^2}{\omega^3}=0.$$

Solving for the possible damping coefficient γ gives

$$\gamma=-\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega^4}{(kv_{th})^3}\exp\left(-\frac{\omega^2}{2k^2v_{th}^2}\right)+\frac{\nu\omega^2}{\omega_p^2}-\nu+\frac{2\nu k^2v_{th}^2}{\omega_p^2}.$$

With the solution for the real part of the plasma wave frequency (4.31) we can express $2k^2v_{th}^2$ in the last summand as $\omega^2-\omega_p^2-k^2v_{th}^2$. We get

$$\gamma=-\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega^4}{(kv_{th})^3}\exp\left(-\frac{\omega^2}{2k^2v_{th}^2}\right)+\nu\left(\frac{2\omega^2}{\omega_p^2}-2-\frac{k^2v_{th}^2}{\omega_p^2}\right).$$

Approximating ω^2 in the zeroth order by ω_p^2 gives us

$$\gamma\approx-\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_p^4}{(kv_{th})^3}\exp\left(-\frac{\omega^2}{2k^2v_{th}^2}\right)-\nu\frac{k^2v_{th}^2}{\omega_p^2}.$$

The last summand can be rewritten using the definition of the thermal velocity (3.26), the plasma frequency (1.2) and the Debye length (1.1). We finally obtain

$$\begin{aligned}\gamma &\approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_p^4}{(kv_{th})^3} \exp\left(-\frac{\omega^2}{2k^2 v_{th}^2}\right) - \nu k^2 \lambda_D^2 \\ &= -\gamma_L - \gamma_{BGK}.\end{aligned}$$

This is a strictly negative expression. Hence, the expected damping effect of the electric field for the Vlasov-Poisson-BGK system is proven. In addition, we were able to show that the damping coefficient γ is made up of a Landau damping part γ_L that we already know from the solution (3.27) of the Vlasov-Poisson system and a collisional damping γ_{BGK} that arises from the BGK relaxation and depends on the small collision frequency ν.

4.4 Numerical treatment of the dispersion relation

The damping effect of the electric field for the Vlasov-Poisson-BGK system shall now be verified numerically. To this end, we solve the dispersion relation in a numerical way and study the position and behaviour of the zeros of the dispersion relation for different values of the wave vector k and the collision frequency ν. Here, we mainly follow and adapt the ideas from [27].

4.4.1 The plasma dispersion function

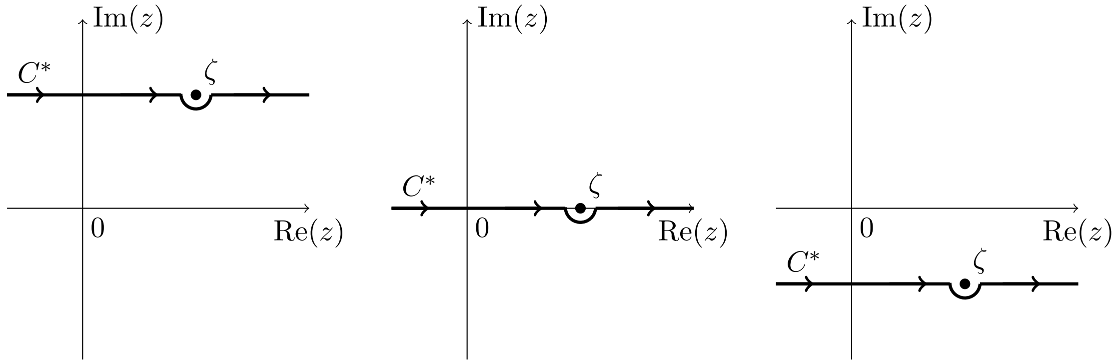
If the equilibrium distribution function is a Maxwellian it is possible to express the dispersion relation in terms of the plasma dispersion function. It was introduced and extensively studied by Burton D. Fried and Samuel D. Conte in [14] and is defined as follows.

Definition 4.4 (Plasma dispersion function, [27])**.** Let $\zeta \in \mathbb{C}$ and C be an integration contour as displayed in Figure 3.1. Then,

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{-z^2}}{z-\zeta} \, \mathrm{d}z \tag{4.32}$$

is called the *plasma dispersion function.*

It can be shown that the integral appearing in the definition of the plasma dispersion function does not depend on the explicit integration contour C [27]. Instead, let us integrate in the form shown in Figure 4.2.



(a) $\mathrm{Im}(\zeta) > 0$ **(b)** $\mathrm{Im}(\zeta) = 0$ **(c)** $\mathrm{Im}(\zeta) < 0$

Figure 4.2: Integration contour C^* used in the plasma dispersion.

We denote this integration contour by C^*. It passes around the pole from below and thus exhibits the same behaviour as the initial integration contour C.

The plasma dispersion function can be rewritten in several ways. One of it makes use of the imaginary error function introduced in Definition 2.21.

Lemma 4.5 ([27])**.** The plasma dispersion function defined in (4.32) can also be expressed as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \left[P \int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u} \,\mathrm{d}u + \pi i e^{-\zeta^2} \right] \tag{4.33}$$

$$= \sqrt{\pi} e^{-\zeta^2} \left[i - \operatorname{erfi}(\zeta) \right]. \tag{4.34}$$

Proof. We prove the two representations one after another.

1. For expression (4.33) let us start by considering the integral in the definition (4.32) of the plasma dispersion function. The integration contour shall be chosen as in Figure 4.2 and split up into the integral C_1^* along the straight line and the integral C_2^* along the semicircle, i.e.

$$\int_{C^*} \frac{e^{-z^2}}{z-\zeta} \,\mathrm{d}z = \int_{C_1^*} \frac{e^{-z^2}}{z-\zeta} \,\mathrm{d}z + \int_{C_2^*} \frac{e^{-z^2}}{z-\zeta} \,\mathrm{d}z.$$

These two parts can be parametrized by $C_1^* : t \to t + i\,\mathrm{Im}(\zeta)$ for $|t - \mathrm{Re}(\zeta)| \geq \delta$ with small $\delta > 0$ and $C_2^* : \theta \to \zeta - \delta e^{i\theta}$ for $\theta \in [0, \pi]$. We obtain for the integral along the straight line

$$\begin{aligned} \int_{C_1^*} \frac{e^{-z^2}}{z-\zeta} \,\mathrm{d}z &= \int_{-\infty}^{\mathrm{Re}(\zeta)-\delta} \frac{e^{-(t+i\,\mathrm{Im}(\zeta))^2}}{t + i\,\mathrm{Im}(\zeta) - \zeta} \,\mathrm{d}t + \int_{\mathrm{Re}(\zeta)+\delta}^{\infty} \frac{e^{-(t+i\,\mathrm{Im}(\zeta))^2}}{t + i\,\mathrm{Im}(\zeta) - \zeta} \,\mathrm{d}t \\ &= \int_{-\infty}^{\mathrm{Re}(\zeta)-\delta} \frac{e^{-(t+i\,\mathrm{Im}(\zeta))^2}}{t - \mathrm{Re}(\zeta)} \,\mathrm{d}t + \int_{\mathrm{Re}(\zeta)+\delta}^{\infty} \frac{e^{-(t+i\,\mathrm{Im}(\zeta))^2}}{t - \mathrm{Re}(\zeta)} \,\mathrm{d}t \\ &= \int_{-\infty}^{-\delta} \frac{e^{-(u+\zeta)^2}}{u} \,\mathrm{d}u + \int_{\delta}^{\infty} \frac{e^{-(u+\zeta)^2}}{u} \,\mathrm{d}u, \end{aligned}$$

where we made the substitution $u = t - \mathrm{Re}(\zeta)$ and used that ζ can be written in the form $\zeta = \mathrm{Re}(\zeta) + i\,\mathrm{Im}(\zeta)$. Letting $\delta \to 0$ we get

$$\int_{C_1^*} \frac{e^{-z^2}}{z-\zeta}\,\mathrm{d}z \to P\int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u}\,\mathrm{d}u.$$

For the integral along the semicircle we have

$$\int_{C_2^*} \frac{e^{-z^2}}{z-\zeta}\,\mathrm{d}z = \int_0^{\pi} \frac{e^{-(\zeta-\delta e^{i\theta})^2}}{-\delta e^{i\theta}}(-i\delta e^{i\theta})\,\mathrm{d}\theta = i\int_0^{\pi} e^{-(\zeta-\delta e^{i\theta})^2}\,\mathrm{d}\theta.$$

For $\delta \to 0$ we obtain

$$\int_{C_2^*} \frac{e^{-z^2}}{z-\zeta}\,\mathrm{d}z \to \pi i e^{-\zeta^2}.$$

Hence, putting these results together we get expression (4.33) of the plasma dispersion function

$$Z(\zeta) = \frac{1}{\sqrt{\pi}}\left[P\int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u}\,\mathrm{d}u + \pi i e^{-\zeta^2}\right].$$

2. The second expression (4.34) can be derived from the first one. Let us start by considering the integral appearing in (4.33). We have

$$\begin{aligned}
P\int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u}\,\mathrm{d}u &= e^{-\zeta^2}\,P\int_{-\infty}^{\infty} \frac{e^{-u^2}e^{-2\zeta u}}{u}\,\mathrm{d}u \\
&= e^{-\zeta^2}\left(P\int_{-\infty}^{0} \frac{e^{-u^2}e^{-2\zeta u}}{u}\,\mathrm{d}u + P\int_0^{\infty} \frac{e^{-u^2}e^{-2\zeta u}}{u}\,\mathrm{d}u\right) \\
&= e^{-\zeta^2}\,P\int_0^{\infty} e^{-u^2}\,\frac{(e^{-2\zeta u} - e^{2\zeta u})}{u}\,\mathrm{d}u \\
&= 2e^{-\zeta^2}\,P\int_0^{\infty} e^{-u^2}\,\frac{\sinh(-2\zeta u)}{u}\,\mathrm{d}u \\
&= -2e^{-\zeta^2}\,P\int_0^{\infty} e^{-u^2}\,\frac{\sinh(2\zeta u)}{u}\,\mathrm{d}u.
\end{aligned}$$

Here we used the binomial formula, the definition of the hyperbolic sine and the fact that sinh is an odd function. Until now we considered the principal value integral because of the point $u = 0$ appearing in the denominator. Since $\sinh(2\zeta u) \sim 2\zeta u$ in the neighbourhood of $u = 0$, the singularity at this point vanishes and we can identify the Riemann and the principal value integral. We denote it by

$$y(\zeta) = \int_0^{\infty} e^{-u^2}\,\frac{\sinh(2\zeta u)}{u}\,\mathrm{d}u.$$

Then we obtain

$$y'(\zeta) = 2\int_0^\infty e^{-u^2}\cosh(2\zeta u)\,\mathrm{d}u$$

and

$$y''(\zeta) = 4\int_0^\infty ue^{-u^2}\sinh(2\zeta u)\,\mathrm{d}u.$$

Integration by parts gives us for the last integral

$$4\int_0^\infty ue^{-u^2}\sinh(2\zeta u)\,\mathrm{d}u = 4\zeta\int_0^\infty e^{-u^2}\cosh(2\zeta u)\,\mathrm{d}u = 2\zeta y'(\zeta),$$

hence $y''(\zeta) = 2\zeta y'(\zeta)$. This differential equation is solved by $y'(\zeta) = y'(0)e^{-\zeta^2}$. With $y'(0) = 2\int_0^\infty e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$ we get $y'(\zeta) = \sqrt{\pi}e^{-\zeta^2}$. Then, with $y(0) = 0$ we obtain from the fundamental theorem of calculus and with the definition of the imaginary error function (2.21)

$$y(\zeta) = \sqrt{\pi}\int_0^\zeta e^{t^2}\,\mathrm{d}t = \frac{\pi}{2}\operatorname{erfi}(\zeta).$$

With this result we can rewrite (4.33) as

$$\begin{aligned} Z(\zeta) &= \frac{1}{\sqrt{\pi}}\left[-2e^{-\zeta^2}\frac{\pi}{2}\operatorname{erfi}(\zeta) + \pi i e^{-\zeta^2}\right] \\ &= \sqrt{\pi}e^{-\zeta^2}\left[i - \operatorname{erfi}(\zeta)\right]. \end{aligned}$$

□

The plasma dispersion function and its relation to the imaginary error function shall now be used for the dispersion relation (4.19).

4.4.2 Rewriting the dispersion relation

Using the definition of the thermal velocity (3.26) and expression (4.5) for the equilibrium distribution function f^{equ}, we get as before from (4.19)

$$1 - \frac{1}{\sqrt{2\pi}v_{th}}\int_{\mathbb{R}}\frac{\left(\nu + \frac{\nu v\bar{\omega}}{kv_{th}^2} - \frac{n_0e^2iv}{mkv_{th}^2}\right)\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{ikv - i\bar{\omega} + \nu}\,\mathrm{d}v = 0.$$

Splitting the integral in its single summands, we have the relation

$$\begin{aligned}1 &- \frac{\nu}{\sqrt{2\pi}ikv_{th}}\int_{\mathbb{R}}\frac{\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v - \frac{\nu\bar{\omega}}{\sqrt{2\pi}ik^2v_{th}^3}\int_{\mathbb{R}}\frac{v\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v\\ &+\frac{n_0e^2}{\sqrt{2\pi}k^2v_{th}^3m}\int_{\mathbb{R}}\frac{v\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v = 0.\end{aligned}$$

Here, we recognize two integrals of the type $\int_{\mathbb{R}}\frac{v\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v$. To those we add a zero and obtain for the dispersion relation

$$\begin{aligned}1 &- \frac{\nu}{\sqrt{2\pi}ikv_{th}}\int_{\mathbb{R}}\frac{\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v - \frac{\nu\bar{\omega}}{\sqrt{2\pi}ik^2v_{th}^3}\int_{\mathbb{R}}\exp\left(-\frac{v^2}{2v_{th}^2}\right)\mathrm{d}v\\ &-\frac{\nu\bar{\omega}}{\sqrt{2\pi}ik^2v_{th}^3}\left(\frac{\bar{\omega}}{k}-\frac{\nu}{ik}\right)\int_{\mathbb{R}}\frac{\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v + \frac{n_0e^2}{\sqrt{2\pi}k^2v_{th}^3m}\int_{\mathbb{R}}\exp\left(-\frac{v^2}{2v_{th}^2}\right)\mathrm{d}v \qquad (4.35)\\ &+\frac{n_0e^2}{\sqrt{2\pi}k^2v_{th}^3m}\left(\frac{\bar{\omega}}{k}-\frac{\nu}{ik}\right)\int_{\mathbb{R}}\frac{\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v = 0.\end{aligned}$$

There are two types of integrals in this dispersion relation. They can be evaluated or rewritten as follows:

1.
$$\int_{\mathbb{R}}\exp\left(-\frac{v^2}{2v_{th}^2}\right)\mathrm{d}v = \sqrt{2\pi}v_{th},$$

2.
$$\begin{aligned}\int_{\mathbb{R}}\frac{\exp\left(-\frac{v^2}{2v_{th}^2}\right)}{v-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\,\mathrm{d}v &= \int_{\mathbb{R}}\frac{\exp(-u^2)}{\sqrt{2}v_{th}u-\frac{\bar{\omega}}{k}+\frac{\nu}{ik}}\sqrt{2}v_{th}\,\mathrm{d}u\\ &= \int_{\mathbb{R}}\frac{\exp(-u^2)}{u-\frac{\bar{\omega}}{\sqrt{2}kv_{th}}+\frac{\nu}{\sqrt{2}ikv_{th}}}\,\mathrm{d}u = \sqrt{\pi}Z\left(\frac{\bar{\omega}+i\nu}{\sqrt{2}kv_{th}}\right),\end{aligned}$$

where we made use of the substitution $u=\frac{v}{\sqrt{2}v_{th}}$ and the definition of the plasma dispersion function given in (4.32).

Inserting these expressions into (4.35) gives us

$$
\begin{aligned}
1 &- \frac{\nu}{\sqrt{2\pi} i k v_{th}} \sqrt{\pi} Z\left(\frac{\bar{\omega}+i\nu}{\sqrt{2} k v_{th}}\right) - \frac{\nu\bar{\omega}}{\sqrt{2\pi} i k^2 v_{th}^3} \sqrt{2\pi} v_{th} \\
&- \frac{\nu\bar{\omega}}{\sqrt{2\pi} i k^2 v_{th}^3} \left(\frac{\bar{\omega}}{k} - \frac{\nu}{ik}\right) \sqrt{\pi} Z\left(\frac{\bar{\omega}+i\nu}{\sqrt{2} k v_{th}}\right) + \frac{n_0 e^2}{\sqrt{2\pi} k^2 v_{th}^3 m} \sqrt{2\pi} v_{th} \\
&+ \frac{n_0 e^2}{\sqrt{2\pi} k^2 v_{th}^3 m} \left(\frac{\bar{\omega}}{k} - \frac{\nu}{ik}\right) \sqrt{\pi} Z\left(\frac{\bar{\omega}+i\nu}{\sqrt{2} k v_{th}}\right) = 0.
\end{aligned}
$$

With the definition of the plasma frequency (1.2) we finally obtain

$$
1 + \frac{\omega_p^2 + i\nu\bar{\omega}}{k^2 v_{th}^2} + Z\left(\frac{\bar{\omega}+i\nu}{\sqrt{2} k v_{th}}\right)\left(\frac{\bar{\omega}(\omega_p^2-\nu^2)}{\sqrt{2} k^3 v_{th}^3} + \frac{i\nu}{\sqrt{2} k v_{th}}\left(1 + \frac{\bar{\omega}^2+\omega_p^2}{k^2 v_{th}^2}\right)\right) = 0. \tag{4.36}
$$

This expression of the dispersion relation depends on the plasma dispersion function Z. It can now be used for our numerical implementation.

4.4.3 Numerical implementation

For the numerical computation of the zeros of the dispersion relation (4.36) we adapt a Python code that was provided by Eric Sonnendrücker and is used in [27] for the Vlasov-Poisson case. It is mainly based on a search routine presented in the article "A Numerical Method for Locating the Zeros of an Analytic Function" [11] by L. Mike Delves and James N. Lyness which proceeds as follows:

1. Determine the number of zeros of a given analytical function f inside a given integration contour φ. This number K can be obtained by the relation

$$
K = \frac{1}{2\pi i} \int_{\varphi} \frac{f'(z)}{f(z)} \,\mathrm{d}z, \tag{4.37}
$$

   which can be deduced from the residue theorem stated in Theorem 2.16. In practice, we shall choose the integration contour φ in form of a rectangular box.

2. If $K < 5$, go to the next step. Else, subdivide the rectangular box into four smaller rectangular boxes and go back to the first step.

3. Construct a polynomial that has the same zeros $z_1, z_2, ..., z_K$ as the function f. Compute the zeros of this polynomial. Here, Delves and Lyness propose to use the relation

$$
s_m = \sum_{i=1}^{K} z_i^m = \frac{1}{2\pi i} \int_{\varphi} z^m \, \frac{f'(z)}{f(z)} \,\mathrm{d}z, \quad m \in \mathbb{N} \tag{4.38}
$$

   together with Newton's identities for determining the coefficients of the polynomial.

4. Check if the obtained zeros are indeed zeros of the function f. If not, refine them by iterative methods or further subdivide the rectangular box and start again.

This scheme can also be identified in the following pseudo code. It defines the class zafpy2 which is later called to determine the zeros of the dispersion relation (4.36). Its complete formulation is mainly due to Eric Sonnendrücker and can be found in the appendix.

Algorithm 1: Definition of the class zafpy2

Initialization;

Count the number of zeros in a given rectangular box:
Define the function count_zeros:
Data: Vertices xmin, xmax, ymin, ymax of the rectangular box
Result: Number of zeros
Count the number of zeros of a given function by formula (4.37);

Locate the zeros in a given rectangular box:
Define the function get_zeros:
Data: Vertices xmin, xmax, ymin, ymax of the rectangular box, error margin *tol* for the validation of the zeros, error margin $tolK$ for K being an integer, maximal number maxiter for refining the zeros iteratively
Result: Location of the zeros
Get nzeros by applying the function count_zeros;
K=int(round(nzeros.real));

Continuation of the code: See next page;

if *abs(K-nzeros.real)* $> tolK$ *or* $K > 5$ **then**
    Apply the function refine;
else
    Determine the zeros as generalized eigenvalues of Hankel matrices. Construct them as follows:
    if $K > 0$ **then**
        construct an empty vector s of length $2K$;
        for *m in range* $(0, 2K)$ **do**
            Compute the values s_m by formula (4.38) and write these values in the vector s;
        end
        Construct the Hankel matrices and determine the generalized eigenvalues. Write them in the vector w;
        Check if the obtained zeros are indeed zeros of the original function:
        for *i in range (len(w))* **do**
            Consider the i-th entry $ww = w[i]$ and insert it in the original function. The obtained value gives the error;
        end
        while *error* $>$ *tol and it* $<$ *maxiter* **do**
            Refine ww and determine the new error;
        end
        Write the refined value in the vector w;
        if *error* $>$ *tol* **then**
            Apply the function refine;
        else
            Add the obtained zero to the list of zeros;
        end
    end
end

Refine the given rectangular box:
Define the function refine:
Data: Vertices xmin, xmax, ymin, ymax of the rectangular box, error margin *tol*, error margin $tolK$, maximal number maxiter
Result: Refined rectangular box subdivided into four smaller rectangular boxes

This algorithm is used for calculating the zeros of the dispersion relation (4.36) for different fixed values of the wave vector k. We call the class zafpy2 in the next code. It is given here in a pseudo code form .The original detailed code can be found the appendix.

Algorithm 2: Determination of the zeros of the dispersion relation (4.36)

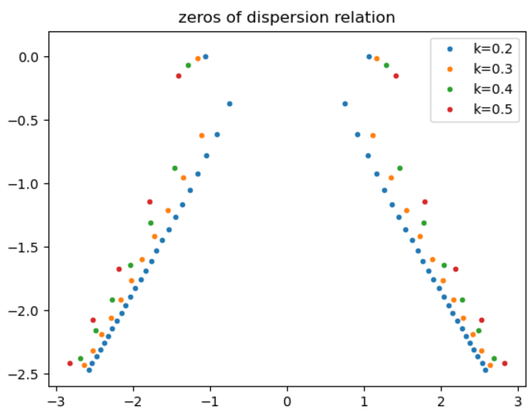
Data: Dispersion relation D, vertices xmin, xmax, ymin, ymax of the rectangular box
Result: Zeros of the dispersion relation
Initialization
Define the dispersion relation D in terms of the plasma dispersion function by (4.36);

Determine the zeros of the dispersion relation and give back the zero with the largest imaginary part:
for k *in arange* $(.2, .6, .1)$ **do**
    zaf=zafpy2(D, $kmode$);
    zeros=zaf.get_zeros(xmin,xmax,ymin,ymax);
    zero_max=zeros[argmax(imag(zeros))];
end

We shall display the numerical results in the next section.

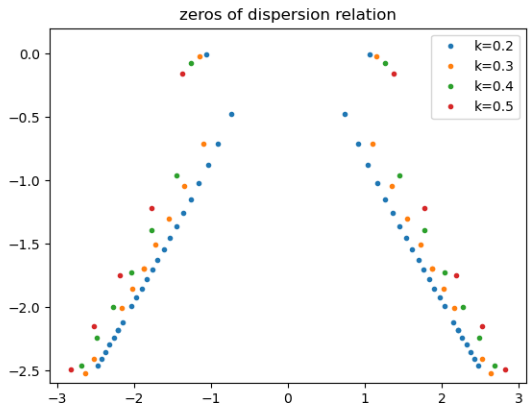
4.4.4 Numerical results

Varying the collision frequency ν, we obtain the following numerical results for the zeros of the dispersion relation (4.36). We consider the different fixed values $k = 0.2$, $k = 0.3$, $k = 0.4$ and $k = 0.5$.



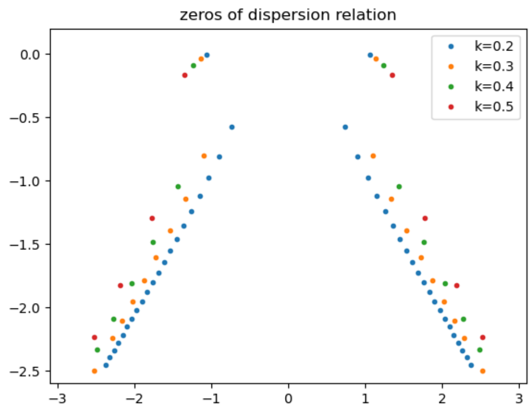
k	Zero of D with largest γ
0.2	$\pm 1.0640 - 5.5107 \cdot 10^{-5} i$
0.3	$\pm 1.1598 - 0.0126 i$
0.4	$\pm 1.2851 - 0.0661 i$
0.5	$\pm 1.4157 - 0.1534 i$

(a) Collision frequency $\nu = 0$.



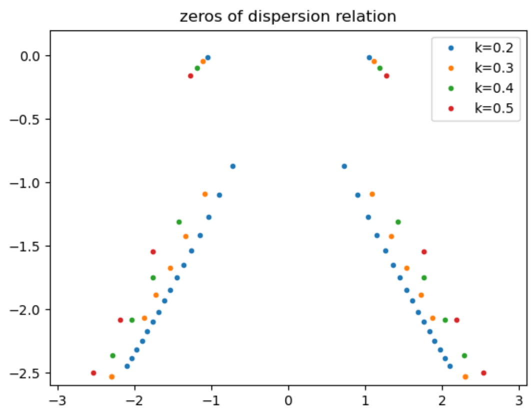
k	Zero of D with largest γ
0.2	$\pm 1.0631 - 0.0055i$
0.3	$\pm 1.1505 - 0.0256i$
0.4	$\pm 1.2621 - 0.0794i$
0.5	$\pm 1.3801 - 0.1620i$

(b) Collision frequency $\nu = 0.1$.



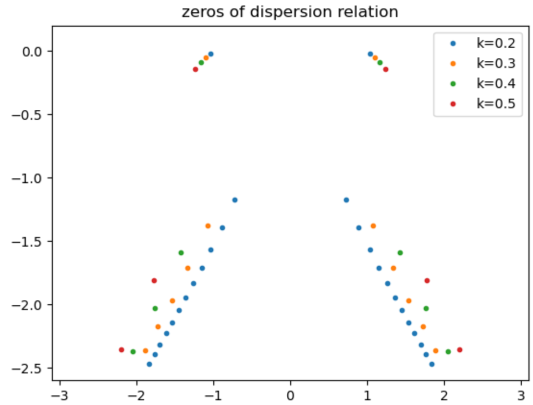
k	Zero of D with largest γ
0.2	$\pm 1.0610 - 0.0101i$
0.3	$\pm 1.1407 - 0.0349i$
0.4	$\pm 1.2412 - 0.0879i$
0.5	$\pm 1.3488 - 0.1657i$

(c) Collision frequency $\nu = 0.2$.



k	Zero of D with largest γ
0.2	$\pm 1.0519 - 0.0184i$
0.3	$\pm 1.1139 - 0.0479i$
0.4	$\pm 1.1917 - 0.0958i$
0.5	$\pm 1.2783 - 0.1608i$

(d) Collision frequency $\nu = 0.5$.



k	Zero of D with largest γ
0.2	$\pm 1.0434 - 0.0208i$
0.3	$\pm 1.0942 - 0.0494i$
0.4	$\pm 1.1590 - 0.0918i$
0.5	$\pm 1.2334 - 0.1471i$

(e) Collision frequency $\nu = 0.8$.

Figure 4.3: Zeros of the dispersion relation for different values of the collision frequency.

First of all, one can see that all imaginary parts of the zeros of the dispersion relation are negative. Hence, we get a damping effect of the electric field. Further, one observes that the zeros with the largest imaginary parts, the so called Landau poles [7], always remain close to the real axis. This is important for maintaining an approximate correspondence between the kinetic Vlasov model and fluid models [5]. In addition, one can see that for increasing collision frequencies the remaining zeros drop more and more below the real axis. The collective damping effect gets larger. Apart from these expected results, one can observe that for larger values of k and increasing collision frequencies ν the imaginary parts γ of the Landau poles become slightly larger. This confirms that both the assumption $k \to 0$ and the assumption of small collision frequencies are crucial for the validity of the analytical result.

Chapter 5

Conclusion and perspectives

In this thesis, we considered the Vlasov-Poisson as well as the Vlasov-Poisson-BGK system. For a basic understanding we first explained several aspects of plasma in general. We gave a definition, presented some applications and introduced the research area of plasma physics. Further, we gave some mathematical and physical prerequisites which are intended to help the reader understand the following considerations.

Then the Vlasov-Poisson system was introduced. We directly followed Landau's approach from [18] and explained the phenomenon of Landau damping in a detailed way. Here we also discussed how to apply the inverse Laplace transform to the algebraic expression $\tilde{E}$ of the electric field and showed how to analytically continue this expression in order to use the residue theorem. We emphasized the importance of the zeros of the dispersion relation of the considered problem. Further, we gave a physical interpretation of Landau damping.

The main chapter was centered around the Vlasov-Poisson-BGK system. We explained the concept of the BGK relaxation and added such an operator to the Vlasov-Poisson equation. We adapted Landau's approach and similarly derived a dispersion relation for the Vlasov-Poisson-BGK system. This relation was first solved analytically. Here, we followed [35] and obtained sensible results. For the real part of the plasma wave frequency we got the well-known Bohm-Gross dispersion relation for electron plasma waves. For the imaginary part γ we could derive a strictly negative expression showing the damping effect of the electric field. Further, we were able to observe that the damping coefficient γ is composed additively of a Landau damping part γ_L and a collisional damping γ_{BGK} due to the BGK relaxation in the form

$$\gamma = -\gamma_L - \gamma_{BGK}.$$

In particular, for the collision frequency $\nu = 0$ we obtained the same damping coefficient as in the Vlasov-Poisson case. Then, we aimed at solving the dispersion relation numerically. We introduced the plasma dispersion function in order to rewrite the dispersion relation using the imaginary error function erfi. For the implementation we mainly used a code provided by Eric Sonnendrücker. Its operating principle is given in [11, 27] and was also explained in this work. The numerical results showed that the imaginary parts of the zeros of the dispersion relation are always negative which gives rise to the damping effect of the electric field. In addition, the dependence of the damping effect on the wave vector k and the collision frequency ν could be observed.

Further research on the topic of damping phenomena for the Vlasov-Poisson-BGK system could be done in a multiple species framework. We propose to consider the two species kinetic BGK model presented by Marlies Pirner in [25]. It explicitly takes the contribution of two particle species into account. This leads to a more detailed description of the considered gas mixture. For a plasma, let us use the index i for all quantities related to the ions. The index e shall stand for all electron quantities. Then, the one-dimensional two species BGK model is given in the form

$$\begin{aligned} &\frac{\partial}{\partial t} f_i + v \frac{\partial}{\partial x} f_i + \frac{eE}{m_i} \frac{\partial}{\partial v} f_i = \nu_{ii} n_i (M_i - f_i) + \nu_{ie} n_e (M_{ie} - f_i), \\ &\frac{\partial}{\partial t} f_e + v \frac{\partial}{\partial x} f_e - \frac{eE}{m_e} \frac{\partial}{\partial v} f_e = \nu_{ee} n_e (M_e - f_e) + \nu_{ei} n_i (M_{ei} - f_e). \end{aligned} \tag{5.1}$$

Here, the Maxwellian distributions are defined as

$$\begin{aligned} M_k(x,v,t) &= \frac{n_k(x,t)}{\left(2\pi T_k(x,t)/m_k\right)^{1/2}} \exp\left(-\frac{|v-u_k(x,t)|^2}{2T_k(x,t)/m_k}\right), \quad k = i, e, \\ M_{kj}(x,v,t) &= \frac{n_{kj}(x,t)}{\left(2\pi T_{kj}(x,t)/m_k\right)^{1/2}} \exp\left(-\frac{|v-u_{kj}(x,t)|^2}{2T_{kj}(x,t)/m_k}\right), \quad k, j = i, e, \ k \neq j, \end{aligned}$$

with $n_k(x,t)$ being the macroscopic density, $u_k(x,t)$ the macroscopic velocity and $T_k(x,t)$ the macroscopic temperature of the particles of species k. The further macroscopic quantities are chosen such that conservation of the number of particles, of the total momentum and of the total energy holds. The equations from (5.1) are coupled to Maxwell's equations through the electric field. It holds

$$\frac{\partial E}{\partial x} = e \int_{\mathbb{R}} (f_i(x,v,t) - f_e(x,v,t))\, \mathrm{d}v. \tag{5.2}$$

We propose to consider the coupled system consisting of (5.1) and (5.2) and show a damping effect of the electric field for an initially slightly disturbed plasma. For the two species BGK model (5.1) without a BGK operator on the right-hand side this is already done in [25]. For the entire system with BGK operator we presume that Landau's approach is applicable in a similar way as in the one species Vlasov-Poisson-BGK case.

Appendix

Here the original Python code used in Chapter 4 is displayed. The first one defines the class zafpy2.

```
import numpy as np
import sympy as sym
import mpmath as mp  # multiple precision arithmetic
from scipy.linalg import hankel,eigvals
from scipy.special import erfi
from scipy.integrate import quad
mp.dps = 30

mods=['numpy',{'erfi':erfi}]

def Z(x):
    """ Plasma dispersion function """
    return sym.sqrt(sym.pi)*sym.exp(-x**2)*(1j-sym.erfi(x))

class zafpy2:
    def __init__(self,D,kmode,max_zeros=20):
        self.kmode = kmode
        self.zeros = []
        omegabar = sym.symbols('omega')
        self.D = sym.lambdify(omegabar,D(omegabar,kmode),'mpmath')
        self.max_zeros = max_zeros
        self.Dprime_over_D = sym.lambdify(omegabar,
            sym.diff(D(omegabar,kmode),omegabar)/D(omegabar,kmode),'mpmath')
        self.D_over_Dprime = sym.lambdify(omegabar,
            D(omegabar,kmode)/sym.diff(D(omegabar,kmode),omegabar),'mpmath')

    def count_zeros(self, xmin,xmax,ymin,ymax,tol=1.e-3):
        """ Count the number of zeros in the box defined by xmin,xmax,ymin,
                ymax; Returns the number of zeros """
        k=self.kmode
        s1,err1=quad(lambda t: np.float(self.Dprime_over_D(xmax+1j*t).real),
                ymin,ymax,epsrel=tol)
        s2,err2=quad(lambda t: np.float(self.Dprime_over_D(xmin+1j*t).real),
                ymin,ymax,epsrel=tol)
        s3,err3=quad(lambda t: np.float(self.Dprime_over_D(t+1j*ymin).imag),
                xmin,xmax,epsrel=tol)
        s4,err4=quad(lambda t: np.float(self.Dprime_over_D(t+1j*ymax).imag),
                xmin,xmax,epsrel=tol)
        return (s1-s2+s3-s4)/(2*np.pi)

    def get_zeros(self,xmin,xmax,ymin,ymax,deg=3,tol=1e-12,tolK=0.01,
        maxiter=10,verbose=False):
```

```python
        """ Count zeros in the rectangular box """
        if verbose:
            print ('Exploring box:'+str(xmin)+','+str(xmax)+','+str(ymin)+','
                +str(ymax))
        nzeros = self.count_zeros(xmin,xmax,ymin,ymax)
        K=int(round(nzeros.real))
        if abs(K-nzeros.real) > tolK or K>5:
            if verbose:
                print ('refining: error=', abs(K-nzeros.real), ' K=', K)
            self.refine(xmin,xmax,ymin,ymax,deg,tol,tolK,maxiter,verbose)
        else:
            if verbose:
                print ('found '+str(K) + ' zeros,Error='+str(abs(K-nzeros.real)))
            # Compute s_m if K>0
            if K>0:
                s=np.zeros(2*K,'complex')
                for m in range(0,2*K):
                    s1=mp.quad(lambda t: (xmax+1j*t)**m*
                        self.Dprime_over_D(xmax+1j*t),[ymin,ymax],maxdegree=deg)/
                        (2*mp.pi)
                    s2=mp.quad(lambda t: (xmin+1j*t)**m*
                        self.Dprime_over_D(xmin+1j*t),[ymin,ymax],maxdegree=deg)/
                        (2*mp.pi)
                    s3=mp.quad(lambda t: (t+1j*ymin)**m*
                        self.Dprime_over_D(t+1j*ymin),[xmin,xmax],maxdegree=deg)/
                        (2*1j*mp.pi)
                    s4=mp.quad(lambda t: (t+1j*ymax)**m*
                        self.Dprime_over_D(t+1j*ymax),[xmin,xmax],maxdegree=deg)/
                        (2*1j*mp.pi)
                    s[m]= s1-s2+s3-s4

                # Compute zeros as generalised eigenvalues of Hankel matrices
                H = hankel(s[0:K], s[K-1:2*K-1])
                H2 = hankel(s[1:K+1], s[K:2*K])
                w = eigvals(H2,H)
                # Check error on zero and perform Newton refinement if necessary
                error_flag = False
                for i in range(len(w)):
                    ww=w[i]
                    error = abs(self.D(ww))
                    it=0
                    while error > tol and it<maxiter:
                        ww=ww-self.D_over_Dprime(ww)
                        error = abs(self.D(ww))
                        it = it + 1
                    w[i]=ww
                    if verbose:
                        print (str(ww)+': error on zero= ' + str(error) +
                                ' #iter='+str(it))
                    if (error > tol):
                        error_flag = True
                        break
                if error_flag:
                    self.refine(xmin,xmax,ymin,ymax,deg,tol,tolK,maxiter,verbose)
```

```
                else:
                    self.zeros=self.zeros+w.tolist()

        return self.zeros

    def refine(self,xmin,xmax,ymin,ymax,deg,tol,tolK,maxiter,verbose):
        """ Get zeros in refined box """
        h =[float(xmax-xmin),float(ymax-ymin)]
        self.get_zeros(xmin,xmin+0.5*h[0],ymin+0.5*h[1],ymax,deg,tol,tolK,
                maxiter,verbose)
        self.get_zeros(xmin+0.5*h[0],xmax,ymin+0.5*h[1],ymax,deg,tol,tolK,
                maxiter,verbose)
        self.get_zeros(xmin+0.5*h[0],xmax,ymin,ymin+0.5*h[1],deg,tol,tolK,
                maxiter,verbose)
        self.get_zeros(xmin,xmin+0.5*h[0],ymin,ymin+0.5*h[1],deg,tol,tolK,
                maxiter,verbose)
```

The second code determines the zeros of the considered dispersion relation (4.36). Note that the physical constants are set to one.

```
from zafpy2 import *
from pylab import *
import cmath

# Physical constants
vth=1  # Thermal velocity
omegap=1 # Plasma frequency
alpha=1  # Normalization
nu=0.1 # Collision frequency

# Dispersion relation
def D(omegabar,k):
   return 1+ alpha*((omegap**2+1j*omegabar*nu)/(vth*k)**2) +
   alpha*Z((omegabar+nu*1j)/(k*vth*sym.sqrt(2)))*((omegabar*(omegap**2-nu**2))/
    (sym.sqrt(2)*vth**3*k**3)+(nu*1j)/(k*vth*sym.sqrt(2))*
    (1+(omegabar**2+omegap**2)/(vth**2*k**2)))

# Determine the zeros of the dispersion relation
for kmode in arange(.2,.6,.1):
    print ('----------------------------------------')
    print ('mode:', kmode.round(decimals=2) )
    zaf=zafpy2(D,kmode)
    xmin=-3;xmax=3
    # ymin=-5*kmode;ymax=.1
    ymin=-2.5;ymax=.1
    zeros=zaf.get_zeros(xmin,xmax,ymin,ymax)#, verbose=True)

    # Determine the zero with the largest imaginary part
    zero_max=zeros[argmax(imag(zeros))]

    print('-------------------------')
    print('k=',kmode.round(decimals=2))
```

```
        print ('zero with largest imaginary part (omega):', zero_max)
        plot(real(zeros),imag(zeros),'.',label='k='+str(kmode.round(decimals=2)))
axis([xmin-.1,xmax+.1,ymin-.1,ymax+.1])
title('zeros of dispersion relation')
legend()
show()
```

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Würzburg, 13.09.2021

Lena Baumann