Finding an Approximate Riemann Solver via Relaxation: Concept and Advantages

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Abstract We first explain the general concept of relaxation models using the Jin-Xin model for scalar conservation laws. Then we consider the Suliciu model specifically for the homogeneous Euler equations. In a third step, using a two-speed relaxation model for the Euler equations with gravity as an example, we show how the construction of the relaxation system can endow the resulting method with useful properties. Finally, these properties are verified in numerical tests.

1 Introduction

Finite volume methods are a popular way to solve systems of partial differential equations in fluid dynamics. In 1959 Godunov introduced the revolutionary idea to solve Riemann problems in order to compute solutions for non-linear hyperbolic conservation laws [10]. In his approach, Godunov computes the exact solution of the Riemann problems which makes his scheme rather cumbersome and computationally inefficient. Later, Roe pointed out that the Riemann problem need not be solved exactly, but that an approximate solution is sufficient in many cases. The result of his work was the Roe solver [14]. Since the Riemann solver is the core part in Godunov type methods, the efficiency of the scheme can be greatly increased by the use of approximate Riemann solvers. While with Roe's solver it was possible to compute quite accurate solutions in an efficient way, the solver still lacked some important properties. A major flaw of Roe's solver is that it does not necessarily satisfy a discrete entropy inequality. Building on Roe's ideas, other approximate

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Riemann solvers were developed in the following years to overcome this problem. One class of such solvers are the Harten-Lax-van Leer (HLL) solvers. The first solver of this kind is presented in [11]. The key idea is to assume that the solution to the Riemann problem consists of three constant states separated by two waves. By using the integral form of the conservation law over the Riemann fan, it is possible to determine the intermediate state in the solution. This approach yields accurate results for systems with only two equations, such as the shallow water equations, but performs poorly for larger systems like the full Euler equations because it focuses only on the outer waves and ignores intermediate waves. Newer solvers like HLLC for the Euler equations also consider intermediate waves and therefore lead to more accurate solutions [18].
In the 1990s, the concept of relaxation schemes emerged [12, 8, 2]. The basic idea is to construct a new enlarged relaxation system, including a relaxation term on the right-hand side, that is an approximation of the original system. The numerical scheme then solves the relaxation system in two steps:

1. First solve the left-hand side of the relaxation system, which consists of a linear transport and is therefore numerically easy to solve.
2. Then project the solution of the first step back onto the equilibrium variables, i.e. use only the variables of the original system to solve the next time step.

The resulting numerical method is thus simple and yet leads to rather accurate results. Furthermore, the approximate Riemann solver naturally satisfies a discrete form of the entropy inequality, which results in an increased robustness of the method. Since there is a certain degree of freedom in how to construct the relaxation system, it is possible to equip the approximate Riemann solver with additional desirable properties.

2 Concept of relaxation

In order to familiarize ourselves with the concept of relaxation, let us first consider the simple case of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1}$$

To solve this equation, Jin and Xin [12] introduced the relaxation system

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2 \partial_x u^\varepsilon &= \frac{1}{\varepsilon}\left(f(u^\varepsilon) - v^\varepsilon\right), \end{aligned} \tag{2}$$

with a constant relaxation speed a and relaxation parameter ε. This system is a diffusive approximation of the original scalar conservation law in (1). This can be illustrated by a Chapman-Enskog expansion [6]. For this procedure we consider a formal expansion of v in terms of ε

$$v^\varepsilon = v_0^\varepsilon + \varepsilon v_1^\varepsilon + O(\varepsilon^2) \tag{3}$$

and insert this expansion into the system (2)

$$\partial_t u^\varepsilon + \partial_x \left(v_0^\varepsilon + \varepsilon v_1^\varepsilon\right) = 0, \tag{4}$$

$$\partial_t \left(v_0^\varepsilon + \varepsilon v_1^\varepsilon\right) + a^2 \partial_x u^\varepsilon = \frac{1}{\varepsilon}\left(f(u^\varepsilon) - v_0^\varepsilon - \varepsilon v_1^\varepsilon\right). \tag{5}$$

From collecting all terms of order $O(1/\varepsilon)$, we can determine

$$v_0^\varepsilon = f(u^\varepsilon). \tag{6}$$

For the terms with order $O(1)$, on the other hand, we gain the system

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v_0^\varepsilon &= 0,\\ \partial_t v_0^\varepsilon + a^2 \partial_x u^\varepsilon &= -v_1^\varepsilon. \end{aligned} \tag{7}$$

We can reformulate the second equation using both (6) and the chain rule

$$v = f(u) - \varepsilon \left(a^2 - f'(u)^2\right) \partial_x u. \tag{8}$$

This expression can be plugged into the first equation of (2) and we derive

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\left(a^2 - f'(u)^2\right) \partial_x u\right). \tag{9}$$

Clearly this equation is diffusive as long as the stability criterion

$$-a \le f'(u) \le a \tag{10}$$

is satisfied. This criterion is called subcharacteristic condition [12]. The Chapman-Enskog expansion shows that the relaxation system is a suitable approximation of the original conservation law. Therefore it is sufficient to determine the solution of the relaxation system. We do that by applying the following splitting approach. In a first step we solve the left-hand side of (2)

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0,\\ \partial_t v^\varepsilon + a^2 \partial_x u^\varepsilon &= 0. \end{aligned} \tag{11}$$

All eigenvalues of this system are linearly degenerate so that it is easy to find the solution to the associated Riemann problem. In the second step, the projection step, we solve the system in the limit $\varepsilon \to 0$, i.e.

$$\begin{aligned} \partial_t u^\varepsilon &= 0,\\ \partial_t v^\varepsilon &= \frac{1}{\varepsilon}\left(f(u^\varepsilon) - v^\varepsilon\right). \end{aligned} \tag{12}$$

In practice, we simply take the solution of the first step for u^ε and use this as the initial value when calculating the solution at the next time step. This projection step was first introduced in [4].

3 The Suliciu relaxation model

While the Jin-Xin relaxation system consists of twice as many equations as the original conservation law (this also applies to systems of conservation laws), it is also possible to construct relaxation systems with fewer additional equations. For the compressible Euler equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0, \\ \partial_t E + \partial_x((E + p)u) &= 0, \end{aligned} \tag{13}$$

with density ρ, velocity u, total energy E and pressure p, one such system is the so-called Suliciu relaxation model [16, 7, 2]. The main idea of this approach is to derive an evolution equation for the pressure from the continuity equation in (13)

$$\partial_t(\rho p) + \partial_x(\rho u p) + \rho^2 p'(\rho)\partial_x u = 0. \tag{14}$$

In this equation one replaces the pressure p by a relaxation variable π and the sound speed $\rho\sqrt{p'(\rho)}$ by a positive constant relaxation speed a

$$\partial_t(\rho\pi) + \partial_x(\rho u\pi + a^2 u) = \rho\frac{p - \pi}{\varepsilon}. \tag{15}$$

Finally, this new relaxation equation is added to the original Euler equations and the pressure p is replaced in all equations by π so that the resulting Suliciu relaxation system has the form

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi) &= 0, \\ \partial_t E + \partial_x((E + \pi)u) &= 0, \\ \partial_t(\rho\pi) + \partial_x(\rho u\pi + a^2 u) &= \rho\frac{p - \pi}{\varepsilon}, \\ \partial_t(\rho a) + \partial_x(\rho a u) &= 0. \end{aligned} \tag{16}$$

A Chapman-Enskog expansion with similar steps as for the Jin-Xin relaxation leads to the following subcharacteristic stability condition

$$a \geq \rho c, \tag{17}$$

where c represents the sound speed. The eigenvalues of system (16) are given by

$$\lambda^- = u - \frac{a}{\rho}, \quad \lambda^u = u, \quad \lambda^+ = u + \frac{a}{\rho}. \tag{18}$$

It can easily be checked that all eigenvalues are linearly degenerate. This is of great advantage because it allows us to solve the Riemann problem exactly. As shown in Fig. 1, the solution has four constant states separated by three waves. The intermediate states can then be computed with the help of the Riemann invariants, which are constant across the corresponding wave. For a large enough relaxation speed a, the resulting approximate Riemann solver preserves positivity of density and internal energy and satisfies a discrete entropy inequality [2].

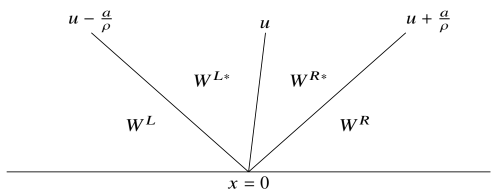

Fig. 1 Structure of the solution to the Riemann problem associated with the Suliciu relaxation model (16).

4 A two-speed relaxation system

In this section we want to present a relaxation system that approximates the full Euler equations with a gravitational source term given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= -\rho \partial_x \Phi, \\ \partial_t E + \partial_x((E + p)u) &= -\rho u \partial_x \Phi, \end{aligned} \tag{19}$$

where Φ represents the gravitational potential. Although the Suliciu scheme in combination with a discretization of the source term is a suitable approximation to (19), it does not provide accuracte solutions to certain problems. For example, if the Mach number M is very small, a method based on the classic Suliciu system leads to very inaccurate solutions, since the diffusion on the velocity in the approximate Riemann solver increases with decreasing Mach number. A second weak point becomes apparent in problems close to hydrostatic equilibrium

$$\begin{cases} u = 0, \\ \partial_x p = -\rho \partial_x \Phi. \end{cases} \tag{20}$$

The scheme resulting from the Suliciu system does not automatically satisfy a discrete equivalent of (20). Therefore these steady states are not preserved exactly and small perturbations around the equilibrium can only be resolved if the resolution is increased sufficiently. This increased computational cost can be circumvented by implementing a well-balancing mechanism that guarantees that an equivalent of (20) is satisfied exactly.

A relaxation system that addresses these problems is introduced in [1], where it is defined by

$$
\begin{aligned}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u v + \pi) &= -\rho \partial_x Z, \\
\partial_t E + \partial_x ((E + \pi) v) &= -\rho v \partial_x Z, \\
\partial_t (\rho v) + \partial_x \left(\rho v^2\right) + \frac{a}{b} \partial_x \pi &= \rho \frac{u - v}{\varepsilon} - \frac{a}{b} \rho \partial_x Z, \\
\partial_t (\rho \pi) + \partial_x (\rho \pi v) + ab \partial_x v &= \rho \frac{p - \pi}{\varepsilon}, \\
\partial_t \rho Z + \partial_x \rho Z v &= \rho \frac{\Phi - Z}{\varepsilon}, \\
\partial_t a + v \partial_x a &= 0, \\
\partial_t b + v \partial_x b &= 0,
\end{aligned}
\tag{21}
$$

where v, π, Z are the relaxation variables and $a, b > 0$ two relaxation speeds. A Chapman-Enskog expansion indicates that this relaxation system is an approximation of the original system (19) under the subcharacteristic conditions

$$
\frac{a}{b} \geq 1 \quad \text{and} \quad ab \geq \rho^2 c^2. \tag{22}
$$

Looking at the relaxation system, it quickly becomes clear that this system is based on the Suliciu approach, because the relaxation equation for the pressure is derived in the same way as for Suliciu. However, a significant difference is already evident in this equation. Instead of using only one relaxation speed as before, in this system it is split into two speeds a and b. In addition, a relaxation equation for the velocity u is also added to the system. These two innovations were first introduced in [3] and serve to make the resulting method asymptotic preserving. By choosing the correct scaling of the relaxation speeds, which is given by $a \sim O(1/M^2)$ and $b \sim O(1)$, the diffusion on the velocity is reduced and in turn transferred to the pressure. For more details on this approach, see also [3, 1].

Another major difference is the inclusion of the gravitational potential in the approximate Riemann solver by relaxing the potential and introducing a separate transport equation for it. This idea was first introduced in [9]. By adding this new equation, the Riemann problem becomes under-determined, resulting in an additional degree of freedom when deriving the approximate Riemann solver. This is used to enforce a discrete equivalent of the second equation in (20) into the Riemann solver, which guarantees that the solver, and hence the entire scheme, preserves steady states at rest.

Furthermore, as in the Suliciu relaxation system, the approximate Riemann solver preserves the positivity of the density and internal energy under the condition of suitably chosen relaxation speeds and satisfies a discrete form of the entropy inequality. Based on this inequality, it can also be shown that no checkerboard modes can arise for the velocity and the pressure. We summarize the properties in the following theorem.

Theorem 1 *In compliance with the CFL condition, the approximate Riemann solver that is associated with the relaxation system (21) with suitably chosen relaxation speeds a and b*

1. *satisfies a discrete version of the entropy inequality,*
2. *prevents the occurrence of checkerboard modes in the velocity and pressure,*
3. *preserves the positivity of density and internal energy,*
4. *leads to an asymptotic preserving scheme in the sense that the scheme is first order uniformly in M and that it is consistent at first order with the incompressible limit,*
5. *is well-balanced in the sense that it preserves discrete steady states at rest.*

Proof For reasons of clarity, we will not provide a proof at this point. The detailed proofs of the properties can be found in [1]. □

Remark 1 In the theorem, we do not specify the conditions imposed on the relaxation speeds in order for the properties to be satisfied. These can be found in [1] as well as a specific definition for a and b that satisfies these conditions.

5 Numerical results

In this section we want to underline the importance of the additional properties that are achieved by the construction of the two-speed relaxation system. Therefore, we compare the numerical results of this scheme with the classic Suliciu relaxation scheme. We will solve the Euler equations with gravity given in (19) closed by an ideal gas law for the pressure given by $p = \rho e(\gamma - 1)$. Since gravity is part of the system, we add a suitable discretization for the source term to the Suliciu solver [13]. The source term in the two-speed relaxation scheme must match the approximate Riemann solver for well-balancing to work. The specific discretization can be found in [1]. For the Suliciu scheme, we use a linear reconstruction with a minmod limiter, whereas for the two-speed scheme a special hydrostatic reconstruction is used. For more details see [1]. The two schemes are therefore second order accurate in space. For time integration both schemes are combined with a standard third order Strong Stability Preserving (SSP) Runge Kutta method [15].

5.1 A low Mach number test

In this first test case, we want to investigate the advantage of two relaxation speeds in a low Mach number environment. As a problem, we choose a stationary vortex in

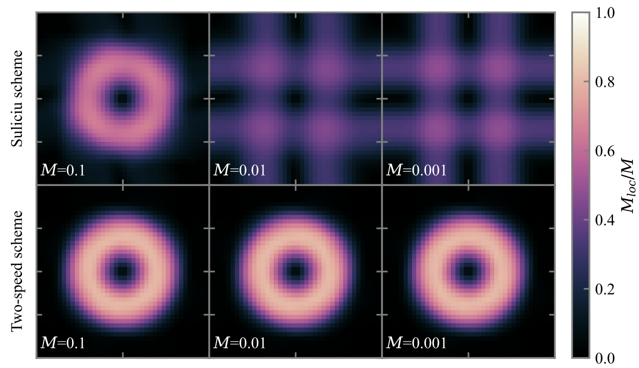

Fig. 2 Numerical solutions for different maximum Mach numbers M after one full turnover. The local Mach number relative to the respective M is color coded

a gravitational field [17]. This vortex is derived in a similar way as the well-known Gresho vortex for the homogeneous Euler equations [13]. In contrast to the Gresho vortex, the constant density in this case is replaced by a density distribution with respect to an isothermal equilibrium. The centrifugal forces are then balanced by a temperature rise in the outer parts of the vortex. The result is a stationary vortex whose maximum local Mach number can be controlled by a parameter M in the initial values. For more information see [17].

We compare the results of the Sulicu and the two-speed scheme on a 40×40 grid for different maximum Mach numbers. The approximated solutions after one turnover are depicted in Fig. 2. Even for a relatively high Mach number of M=0.1, the diffusion in the Suliciu scheme becomes too large so that the vortex is no longer resolved accurately. For smaller Mach numbers, this effect increases even more. The two-speed scheme, on the other hand, is able to resolve the vortex well regardless of the Mach number. This result is based on the fact that the artificial diffusion in the method is independent of the Mach number.

5.2 A well-balancing test

The use of well-balanced methods is particularly important for simulating flows around hydrostatic equilibria. Therefore, we illustrate the property with a test problem taken from [5], in which the initial values are in hydrostatic equilibrium and a small perturbation of this equilibrium is additionally added to the pressure, i.e.

$$p(x, y, 0) = p_e(x, y, 0) + \eta \exp\left(-100\rho_0 g((x-0.3)^2 + (y-0.3)^2)/p_0\right). \qquad (23)$$

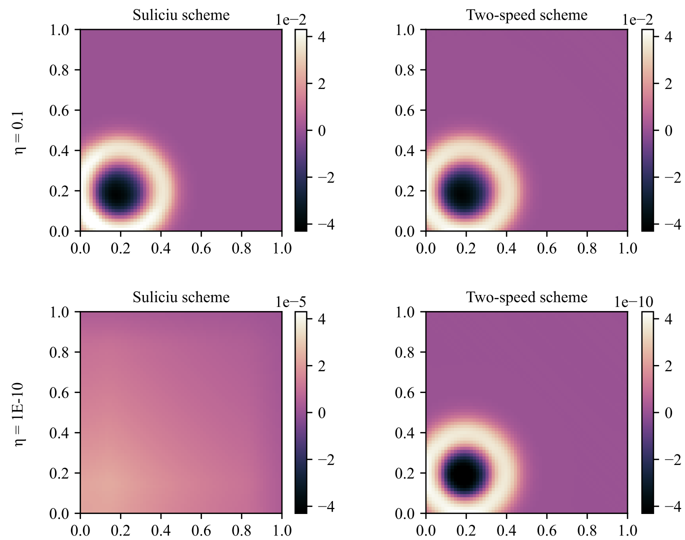

Fig. 3 Pressure perturbation of an isothermal atmosphere at $T = 0.15$. Top row: perturbation with strength $\eta = 0.1$. Bottom row: perturbation with strength $\eta = 1E - 10$

The parameter η can be used to control the strength of the perturbation. A well-balanced method should be able to resolve even a very small perturbation on a coarse grid. We solve this test problem on a 64×64 grid for a large perturbation ($\eta = 0.1$) and a small perturbation ($\eta = 1E - 10$). The results produced by the Suliciu and the two-speed scheme at final time $T = 0.15$ are shown in Fig. 3.

While for the large perturbation both schemes resolve the perturbation equally well, we can observe large differences in accuracy for the smaller perturbation. On the one hand, the Suliciu scheme is not capable of resolving the structure of the perturbation at all. In the solution of the well-balanced two-speed scheme, on the other hand, the perturbation is still well resolved. This very different behaviour of the two methods underlines the importance of well-balancing.

6 Conclusion

Relaxation systems are a useful concept for solving hyperbolic conservation laws since they lead to efficient methods due to the use of an approximate Riemann solver. Furthermore, they naturally satisfy an entropy inequality and preserve the positivity of density and internal energy. A certain freedom in the construction of relaxation systems can be used to incorporate even more useful properties into the approximate Riemann solver. This is exemplified by the two-speed relaxation system in which a low Mach and well-balanced property is integrated.

Acknowledgements

We acknowledge the use of the *Seven-League Hydro Code* (https://slh-code.org) for our numerical experiments. The work of Claudius Birke and Christian Klingenberg is supported by the German Research Foundation (DFG) through the grant KL 566/22-1.

References

1. Birke, C., Chalons, C., Klingenberg, C.: A low Mach two-speed relaxation scheme for the compressible Euler equations with gravity. arXiv:2112.02986 (2021)
2. Bouchut, F.: Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws and Well-Balanced Schemes for Sources. Frontiers in Mathematics, Birkhäuser, Basel (2004)
3. Bouchut, F., Chalons, C., Guisset, S.: An entropy satisfying two-speed relaxation system for the barotropic Euler equations. Application to the numerical approximation of low Mach number flows. Numer. Math. **145**, 35–76 (2020)
4. Brenier, Y.: Averaged multivalued solutions for scalar conservation laws. SIAM J. Numer. Anal. **21**, 1013–1037 (1984)
5. Chandrashekar, P., Klingenberg, C.: A second order well-balanced finite volume scheme for Euler equations with gravity. SIAM J. of Sci. Comput. **37**(3), B382–B402 (2015)
6. Chapman, S., Cowling, T.: Mathematical Theory of Non-Uniform Gases. Cambridge University Press, Cambridge (1970)
7. Coquel, F., Godlewski, E., Perthame, B., In, A., Rascle, P.: Some New Godunov and Relaxation Methods for Two-Phase Flow Problems. In: Toro, E.F. (eds) Godunov Methods, pp. 179–188. Springer, New York (2001)
8. Coquel, F., Perthame, B.: Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics. SIAM J. Num. Anal. **35**, 2223–2249 (1998)
9. Desveaux, V., Zenk, M., Berthon, C., Klingenberg, C.: Well-balanced schemes to capture non-explicit steady states in the Euler equations with gravity. Int. J. Num. Methods Fluids **81**(2), 104–127 (2016)
10. Godunov, S.K.: A Finite Difference Method for the Computation of Discontinuous Solutions of the Equations of Fluid Dynamics. Mat. Sb. **47**, 357–393 (1959)
11. Harten, A., Lax, P.D., van Leer, B.: On Upstream Differencing and Godunov-Type Schemes for Hyperbolic Conservation Laws. SIAM Rev. **25**(1), 35–61 (1983)
12. Jin, S., Xin, Z.: The relaxation schemes for systems of conservation laws in arbitrary space dimensions. Commun. Appl. Math. **48**(3), 235–276 (1995)
13. Miczek, F., Röpke, F.K., Edelmann, P.V.F.: New numerical solver for flows at various Mach numbers. Astron. Astrophys. **576**(A50) (2015)
14. Roe, P.L.: Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes. J. Comput. Phys. **43**, 357–372 (1981)
15. Shu, C.-W., Osher, S.: Efficient implementation of essentially non-oscillatory shock-capturing schemes. J. Comput. Phys. **77**, 439–471 (1988)
16. Suliciu, I.: On modelling phase transitions by means of rate-type constitutive equations. Shock wave structure. Internat. J. Eng. Sci. **28**, 829–841 (1990)
17. Thomann, A., Puppo, G., Klingenberg, C.: An all speed second order well-balanced IMEX relaxation scheme for the Euler equations with gravity. J. Comput. Phys. **420** (2020)
18. Toro, E.F., Spruce, M., Speares, W.: Restoration of the Contact Surface in the HLL-Riemann Solver. Shock Waves **4**, 25–34 (1994)