Structure preservation issues for mean-field games and entropic conservation laws

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A typical Mean Field Game for Social Sciences

$$\partial_t\rho + \nabla\cdot(\rho\nabla\theta) = \nu\Delta\rho, \quad \partial_t\theta + \frac{1}{2}|\nabla\theta|^2 + \nu\Delta\theta = f(\rho)$$

$\rho(t,x) \geq 0$, $\theta(t,x)$ being respectively prescribed at $t = 0$ and $t = T$. Here $T > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}$ are given.

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This MFG à la Lasry-Lions is "well-posed" with respect to data ρ_0 and θ_T provided $f' \geq 0$ and $\nu \geq 0$.

Euler, but...with an imaginary speed of sound!

For $\nu = 0$, our MFG reads, in terms of $q = \rho \nabla \theta$,

$$\partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left(\frac{q \otimes q}{\rho}\right) = -\nabla(p(\rho)),$$

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COROLLARY: SOCIAL SCIENCES $\neq$ PHYSICS!!!

The numerical analysis of MFG has been done by Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta, A. Porretta.

A key point of the analysis is the careful preservation at the discrete level of the dual backward-forward structure of the MFG.

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A key point of the analysis is the careful preservation at the discrete level of the dual backward-forward structure of the MFG.

In particular, the linearized operators

$$\rho \to \partial_t \rho - \nu \Delta \rho, \quad \theta \to \partial_t \theta + \nu \Delta \theta$$

must be discretized in a consistent way.

Remark 1. *An important property of $\mathcal{T}$ is that the operator $m \mapsto \left(-\nu(\Delta_h m)_{i,j} - \mathcal{T}_{i,j}(u,m)\right)_{i,j}$ is the adjoint of the linearized version of the operator $u \mapsto \left(-\nu(\Delta_h u)_{i,j} + g(x_{i,j}, [\nabla_h u]_{i,j})\right)_{i,j}$.*
This property implies that the structure of (1.1)-(1.2) is preserved in the discrete version (2.7)-(2.9). In particular, it implies the uniqueness result stated in Theorem 2.2 below.

Summary. The fully discrete scheme for system (1.1),(1.2),(1.3) is therefore the following: for all $0 \le i,j < N_h$ and $0 \le k < N_T$

$$\begin{cases} \frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} - \nu(\Delta_h u^{k+1})_{i,j} + g(x_{i,j}, [\nabla_h u^{k+1}]_{i,j}) & = F(m_{i,j}^k), \\ \frac{m_{i,j}^{k+1} - m_{i,j}^{k}}{\Delta t} + \nu(\Delta_h m^{k})_{i,j} + \mathcal{T}_{i,j}(u^{k+1}, m^k) & = 0, \end{cases} \tag{2.11}$$

with the initial and terminal conditions

$$u_{i,j}^0 = u_0(x_{i,j}), \qquad m_{i,j}^{N_T} = \frac{1}{h^2} \int_{|x - x_{i,j}|_\infty \le h/2} m_T(x)\,dx, \qquad 0 \le i,j < N_h. \tag{2.12}$$

The following theorem was proved in [5] (using essentially Brouwer's fixed point theorem and estimates on the solutions of the discrete Bellman equation):

from Yves Achdou, Alessio Porretta, 2015 hal-01137705: Convergence of a finite difference scheme to weak solutions of the system of partial differential equation arising in mean field games

Well-posedness: a variational proof

Provided $f' \geq 0$, this MFG is nothing but the optimality system for the CONCAVE MAXIMIZATION PROBLEM

$$\sup_{\theta(T,\cdot)=\theta_T} \quad -\int_0^T \int_D G(\partial_t \theta + \nu \Delta \theta, \nabla \theta) - \int_D \rho_0 \, \theta(0,\cdot)$$

where D is the spatial domain (say $D = \mathbb{T}^d$) and:

$$G(r,w) = \sup_{\rho \geq 0, q \in \mathbb{R}^d} r\rho + w \cdot q - \frac{|q|^2}{2\rho} - \int_0^\rho f(s)ds.$$

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This (roughly) explains why the MFG is well-posed.

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In our opinion, this opens the way to challenging structure preservation problems at the numerical level.

Entropic system of conservation laws

$$\partial_t U + \nabla \cdot (F(U)) = 0, \;\; U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m, \;\; x \in \mathbb{T}^d,$$

involve a strictly convex "entropy" $\mathcal{E} : \mathcal{W} \to \mathbb{R}$ (where $\mathcal{W}$ is convex) and an "entropy flux" $\mathcal{Z} \in \mathcal{W} \to \mathbb{R}^d$, such that each smooth solution U satisfies the extra conservation law $\partial_t(\mathcal{E}(U)) + \nabla \cdot (\mathcal{Z}(U)) = 0.$

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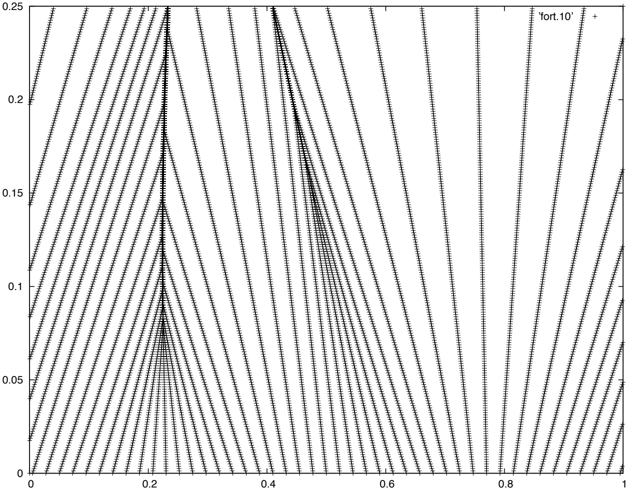
A typical example is the (barotropic) Euler system, where $U = (\rho, q) \in \mathbb{R}_+ \times \mathbb{R}^d$, with entropy $\mathcal{E}(\rho, q) = \frac{|q|^2}{2\rho} + \Phi(\rho)$ and pressure $p(\rho) = \int_0^\rho s\Phi''(s)ds$.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)

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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0,
each test function A acting as a Lagrange multiplier.

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leads to a *concave* maximization problem in A, namely

$$\sup_{A(T,\cdot)=0} \quad \inf_{U} \quad \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0,\cdot) \cdot U_0$$

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$$= \sup_{A(T,\cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0,\cdot) \cdot U_0,$$

where $G(R, W) = \sup_{U \in \mathcal{W} \subset \mathbb{R}^m} R \cdot U + W \cdot F(U) - \mathcal{E}(U),$

for all $(R, W) \in \mathbb{R}^m \times \mathbb{R}^{d \times m}$.

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Observe that G is automatically convex.

Comparison with our initial MFG

$$\sup_{\theta(T,\cdot)=\theta_T} \quad -\int_0^T \int_D G(\partial_t\theta + \nu\Delta\theta, \nabla\theta) - \int_D \rho_0 \, \theta(0,\cdot)$$

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So, our maximization problem to solve the initial value problem can be seen as a generalized variational MFG involving a vector-valued potential $A = A(t,x) \in \mathbb{R}^m$.

Notice that, for the initial MFG

$$\partial_t \rho + \nabla \cdot (\rho \nabla \theta) = \nu \Delta \rho, \quad \partial_t \theta + \frac{1}{2}|\nabla \theta|^2 + \nu \Delta \theta = f(\rho),$$

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while, for our new generalized variational MFG, to solve the IVP for the entropic conservation law with entropy $\mathcal{E}$, $\partial_t U + \nabla \cdot (F(U)) = 0$, we just obtained

$$G(R, W) = \sup_{U \in \mathcal{W} \subset \mathbb{R}^m} R \cdot U + W \cdot F(U) - \mathcal{E}(U).$$

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Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large T.

(*) more precisely if, $\forall\ t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$.

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In that case, we end up with the minimization of

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among all fields $u = u(t,x) \in \mathbb{R}$, $Q = Q(t,x) \in \mathbb{R}^d$, $M = M(t,x) = M^t(t,x) \in \mathbb{R}^{d\times d}$, $\quad M \geq 0$,

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$$u = \partial_t\sigma + \nabla\cdot w,\ \ Q = \partial_t w + \nabla\sigma,\ \ M = \mathbb{I}_d - \nabla w - \nabla w^t$$

where σ and w must vanish at $t = T$.

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Then, we obtain the concave maximization problem

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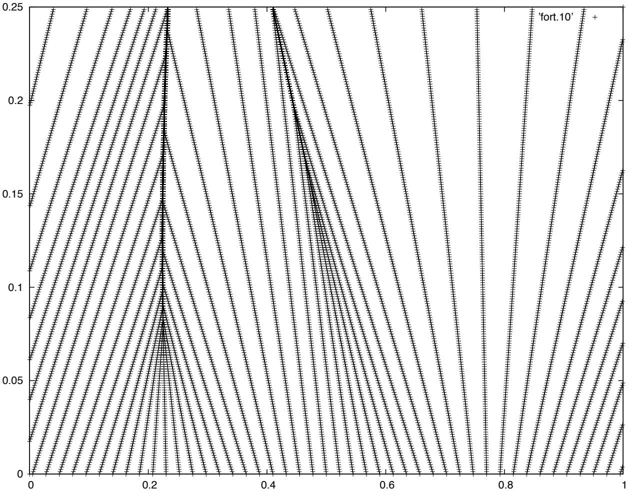
As mentioned, for arbitrarily large T, we may recover, through this problem, the correct "entropy solution" à la Kruzhkov, but only at time T and (surprisingly enough) not for $t < T$, once shocks have formed!



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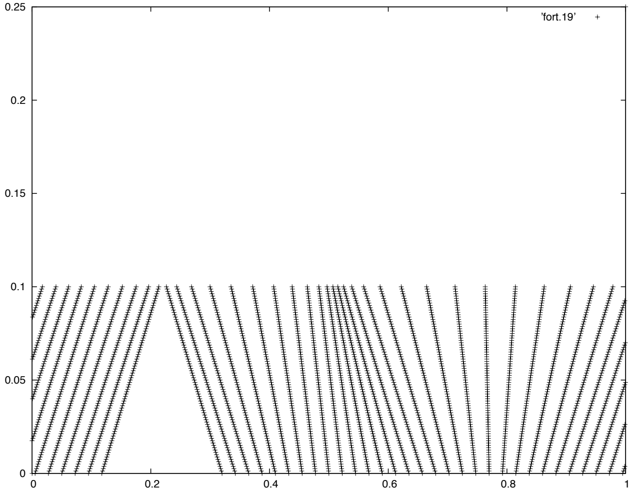

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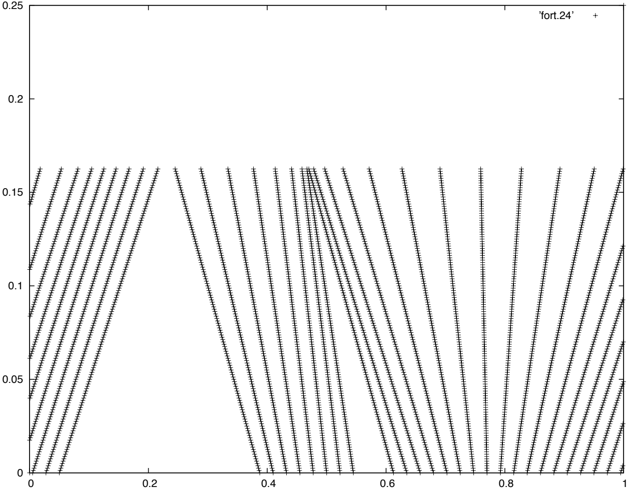
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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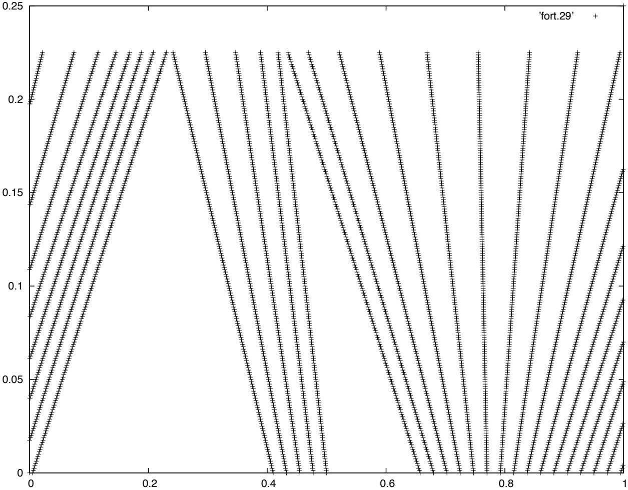
Recovery of the solution at time T=0.16 by convex optimisation.

Observe the formation of a second vacuum zone as the second shock has formed.



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Recovery of the solution at time T=0.225 by convex optimisation.

Observe the extension of the two vacuum zones.

Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)

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Thanks for your attention! For more details, see Y.B. CMP 2018.

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$$\inf\{\int_{[0,T]\times\mathbb{T}^d} \frac{q^2}{4\sigma} - \sigma_0 q,\ \text{s.t.}\ \partial_\theta \sigma + \Delta q = 0,\ \ \sigma(T,\cdot) = 1\}$$

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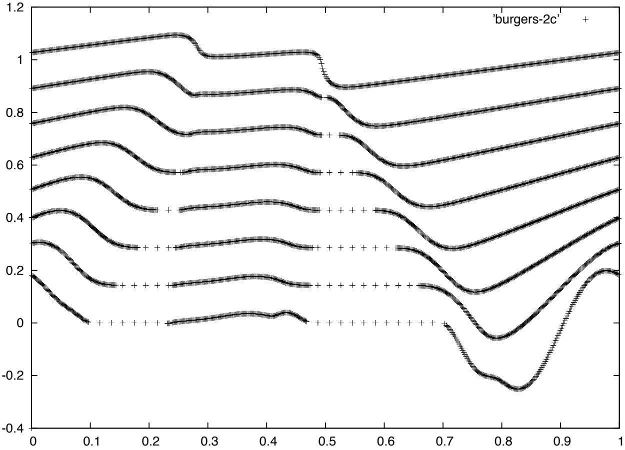
$$\inf\{\int_{[0,T]\times\mathbb{T}^d} \frac{q^2}{4\sigma} - \sigma_0 q, \ \text{s.t.}\ \partial_\theta \sigma + \Delta q = 0, \ \ \sigma(T,\cdot) = 1\}$$

which, in 1D, is a backward-forward version of the Martingale Optimal Transport Problem recently introduced by Huesmann and Trevisan.



Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!