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# Hyperbolic conservation laws with discontinuous fluxes and hydrodynamic limit for particle systems

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ABSTRACT

We study the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F(x, \rho) = 0. \tag{0.1}$$

The main feature of such a conservation law is the discontinuity of the flux function in the space variable $x$. Kruzkov's approach for the $L^1$-contraction does not apply since it requires the Lipschitz continuity of the flux function in $x$; an additional jump wave may occur in the solution besides the classical waves; and entropy solutions even for the Riemann problem are not unique under the classical entropy conditions. On the other hand, it is known that, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed-parameter $\lambda(x)$ in the hydrodynamic limit formally lead to scalar hyperbolic conservation laws with discontinuous fluxes of the form

$$\partial_t \rho + \partial_x \big(\lambda(x) h(\rho)\big) = 0. \tag{0.2}$$

The natural question arises which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This paper is a first step and provides an answer to this question for a family of discontinuous flux functions. In particular, we identify the entropy condition for (0.1) and proceed to show the well-posedness by combining our existence result with a uniqueness result of Audusse and Perthame (2005) for the family of flux functions; we establish a compactness framework for the

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hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (0.1), which is based on the notion and reduction of measure-valued entropy solutions; and we finally establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter governed by an $L^\infty$ entropy solution to (0.2).

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## 1. Introduction

We are concerned with the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F\big(x, \rho(t,x)\big) = 0 \tag{1.1}$$

and with initial data

$$\rho|_{t=0} = \rho_0(x), \tag{1.2}$$

where $F(x,\rho)$ is continuous at all points of $(\mathbb{R}\setminus\mathcal{N})\times\mathbb{R}$ except on a set $\mathcal{N}\subset\mathbb{R}$ of measure zero.

The main feature of (1.1) is the discontinuity of the flux function in the space variable $x$. This feature causes new important difficulties in conservation laws. Kruzkov's approach in [18] for the $L^1$-contraction does not apply; an additional jump wave may occur in the solution besides the classical waves; and entropy solutions even for the Riemann problem of (1.1) are not unique under the classical entropy conditions.

Several different entropy conditions have been suggested in the literature (see [1,2,4,5,10,15,17,21] and the references therein). One type of entropy conditions involves a rule how the solution should behave at the jump wave induced by the discontinuity in the flux, that is, the solution is required to satisfy an additional condition on its traces at the discontinuous points of the flux function, for which the existence of traces of the solution is needed. An alternative entropy condition in [2,4] is an adapted entropy condition that uses steady-state solutions to replace the constant parameter in the Kruzkov entropy inequality. This is quite an attractive notion since it does not require the traces of the entropy solution, which allows the solution only in $L^\infty$. In this paper, we establish the well-posedness in $L^\infty$ for conservation laws with a certain class of flux functions (cf. conditions (H1)–(H2) and (H3) or (H3$'$) in Section 2 below) by providing an existence proof to supplement the uniqueness result in [2].

The entropy condition based on the traces of solutions at the jump waves has lead to the existence and uniqueness of the solutions for a wider class of flux functions than those satisfying (H1)–(H2) and (H3) or (H3$'$) in Section 2. The Cauchy problem (even the Riemann problem) may lead to different solutions depending on which choice of the conditions on the traces of solutions is made (for example, see [2]). If one restricts oneself to the flux functions satisfying (H1)–(H2) and (H3$'$) in Section 2 (in which $F(x,\cdot)$ in (1.1) is monotone) and to the entropy solutions in the class of functions of bounded variation, the two notions of entropy conditions addressed above will lead to the same solution. This is not the case for the flux functions satisfying (H1)–(H2) and (H3) in which $F(x,\cdot)$ is non-monotone.

On the other hand, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed-parameter $\lambda(x)$ in the hydrodynamic limit formally lead to scalar hyperbolic conservation laws with discontinuous flux of the form

$$\partial_t \rho + \partial_x\big(\lambda(x)h(\rho)\big) = 0 \tag{1.3}$$

and with initial data (1.2), where $\lambda(x)$ is continuous except on a set of measure zero and $h(\rho)$ is Lipschitz continuous. Equation (1.3) is equivalent to the following $2\times 2$ hyperbolic system of conservation laws:

$$\begin{cases} \partial_t\rho + \partial_x\big(\lambda h(\rho)\big) = 0, \\ \partial_t\lambda = 0. \end{cases} \tag{1.4}$$

In particular, when $h(\rho)$ is not strictly monotone, system (1.4) is nonstrictly hyperbolic, one of the main difficulties in conservation laws (cf. [7,9]). The natural question is which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This paper is a first step and provides an answer to this question for a family of discontinuous flux functions via an interacting particle system, namely, the attractive zero range process (ZRP). This ZRP leads to a conservation law of the form (1.3) with $\lambda(x) > 0$ and $h(\rho)$ being monotone in $\rho$. Furthermore, its hydrodynamic limit naturally gives rise to an entropy condition of the type described in [2,4].

Motivated by the hydrodynamic limit of the ZRP, in this paper we adopt the notion of entropy solutions in the sense of Audusse and Perthame [2] for a class of conservation laws with discontinuous flux functions, including the non-monotone case, and establish the existence of such an entropy solution via the method of compensated compactness in Section 3. This completes the well-posedness in $L^\infty$ by combining a uniqueness result established in [2] for this class of conservation laws under their notion of entropy solutions.

In order to establish the hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (1.1) rigorously, we establish a compactness framework for (1.1)–(1.2) in Section 2. This mathematical framework is based on the notion and reduction of measure-valued entropy solutions developed in Section 2, which is also applied for another proof of the existence of entropy solutions for the non-monotone case in Section 3.

In Section 4, we establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter $\lambda(x)$ governed by the unique entropy solution of the Cauchy problem (1.2)–(1.3).

## 2. Notion and reduction of measure-valued entropy solutions

In this section, we first develop the notion of measure-valued entropy solutions and establish their reduction to entropy solutions in $L^\infty$ (provided that they exist) of the Cauchy problem (1.1)–(1.2) satisfying that

(H1) $F(x,\rho)$ is continuous at all points of $(\mathbb{R}\setminus\mathcal{N})\times\mathbb{R}$ with $\mathcal{N}$ a closed set of measure zero;
(H2) $\exists$ continuous functions $f, g$ such that, for any $x\in\mathbb{R}$ and large $\rho$, $f(\rho)\leqslant|F(x,\rho)|\leqslant g(\rho)$ with $f(\rho)\geqslant 0$ and $f(\pm\infty)=\infty$;
(H3) there exist a function $\rho_m(x)$ from $\mathbb{R}$ to $\mathbb{R}$ and a constant $M_0$ such that, for $x\in\mathbb{R}\setminus\mathcal{N}$, $F(x,\rho)$ is a locally Lipschitz, one-to-one function from $(-\infty,\rho_m]$ and $[\rho_m,\infty)$ to $[M_0,\infty)$ (or $(-\infty,M_0]$) with $F(x,\rho_m(x))=M_0$ and with common Lipschitz constant $L_I$ for all $x\in\mathbb{R}\setminus\mathcal{N}$ and all $\rho\in I$ that is any bounded interval in $\mathbb{R}$;

or

(H3′) for $x\in\mathbb{R}\setminus\mathcal{N}$, $F(x,\cdot)$ is a locally Lipschitz, one-to-one function from $\mathbb{R}$ to $\mathbb{R}$ with common Lipschitz constant $L_I$ for all $x\in\mathbb{R}\setminus\mathcal{N}$ and all $\rho\in I$ that is any bounded interval in $\mathbb{R}$.

One example of the flux functions satisfying (H1)–(H2) and (H3) or (H3′) is

$$F(x,\rho)=\lambda(x)h(\rho), \tag{2.1}$$

where $\lambda(x)$ is continuous in $x \in \mathbb{R}$ with $0 < \lambda_1 \leqslant \lambda(x) \leqslant \lambda_2 < \infty$ for some constants $\lambda_1$ and $\lambda_2$, except on a closed set $\mathcal{N}$ of measure zero, and $h(\rho)$ is locally Lipschitz and is either monotone or convex (or concave) with $h(\rho_m) = 0$ for some $\rho_m$ in which case $M_0 = 0$.

It is easy to check that, if the flux function $F(x,\rho)$ satisfies (H1)–(H3), then, for any constant $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$), there are two steady-state solutions $m_\alpha^+$ from $\mathbb{R}$ to $[\rho_m(x), \infty)$ and $m_\alpha^-$ from $\mathbb{R}$ to $(-\infty, \rho_m(x)]$ of (1.1) such that

$$F\big(x, m_\alpha^\pm(x)\big) = \alpha \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.2}$$

In the case (H1)–(H2) and (H3′), $m_\alpha^+(x) = m_\alpha^-(x)$ which is even simpler.

### 2.1. Notion of measure-valued entropy solutions

First, the notion of entropy solutions in $L^\infty$ introduced in Audusse and Perthame [2] and Baiti and Jenssen [4] can be further formulated into the following.

**Definition 2.1** *(Notion of entropy solutions in $L^\infty$).* We say that an $L^\infty$ function $\rho : \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is an entropy solution of (1.1)–(1.2) provided that, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m_\alpha^\pm(x)$ of (1.1),

$$\begin{aligned} &\int \Big(\big|\rho(t,x) - m_\alpha^\pm(x)\big|\partial_t J + \operatorname{sign}\big(\rho(t,x) - m_\alpha^\pm(x)\big)\big(F\big(x, \rho(t,x)\big) - \alpha\big)\partial_x J\Big)\,dt\,dx \\ &\quad + \int \big|\rho_0(x) - m_\alpha^\pm(x)\big| J(0,x)\,dx \geqslant 0 \end{aligned} \tag{2.3}$$

for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.

It is easy to see that any entropy solution is a weak solution of (1.1)–(1.2) by choosing $\alpha$ such that $m_\alpha^+(x) \geqslant \|\rho\|_{L^\infty}$ and $m_\alpha^-(x) \leqslant -\|\rho\|_{L^\infty}$, respectively, for a.e. $x \in \mathbb{R}$.

From the uniqueness argument in Audusse and Perthame [2] (also see [8]), one can deduce that, for any $L > 0$,

$$\lim_{t \to 0} \int_{|x| \leqslant L} \big|\rho(t,x) - \rho_0(x)\big|\,dx = 0. \tag{2.4}$$

Following the notion of entropy solutions, we introduce the corresponding notion of measure-valued entropy solutions. We denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

**Definition 2.2** *(Notion of measure-valued entropy solutions).* We say that a measurable map

$$\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$$

is a measure-valued entropy solution of (1.1)–(1.2) provided that $\langle \pi_{0,x}; k\rangle = \rho_0(x)$ for a.e. $x \in \mathbb{R}$ and, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m_\alpha^\pm(x)$ of (1.1),

$$\begin{aligned} &\int \Big(\big\langle \pi_{t,x}; \big|k - m_\alpha^\pm(x)\big|\big\rangle \partial_t J + \big\langle \pi_{t,x}; \operatorname{sign}\big(k - m_\alpha^\pm(x)\big)\big(F(x,k) - \alpha\big)\big\rangle \partial_x J\Big)\,dx\,dt \\ &\quad + \int \big|\rho_0(x) - m_\alpha^\pm(x)\big| J(0,x)\,dx \geqslant 0 \end{aligned} \tag{2.5}$$

for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.

If a measure-valued entropy solution $\pi_{t,x}(k)$ is a Dirac mass with the associated profile $\rho(t,x)$, i.e. $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$, then $\rho(t,x)$ is an entropy solution of (1.1)–(1.2), which is unique as shown in [2].

Note that, when the flux function $F(x,\rho)$ is locally Lipschitz in $\rho$ and globally Lipschitz in $x$, one can use the Kruzkov entropy inequality, instead of (2.5), to formulate the following notion of measure-valued solutions:

$$\partial_t\big\langle\pi_{t,x};\,|k-c|\big\rangle + \partial_x\big\langle\pi_{t,x};\,\mathrm{sign}(k-c)\big(F(x,k)-F(x,c)\big)\big\rangle + \big\langle\pi_{t,x};\,\mathrm{sign}(k-c)\partial_x F(x,c)\big\rangle \leqslant 0 \tag{2.6}$$

in the sense of distributions and to establish their reduction as in DiPerna [14]. One of the new features in our formulation (2.5) in Definition 2.2 is that the constant $c$ in (2.6) is replaced by the steady-state solutions $m_\alpha^\pm(x)$ such that the additional third term in (2.6) vanishes, as in [2,4], and thereby allows the discontinuity of the flux functions on a closed set of measure zero for measure-valued entropy solutions.

### 2.2. Reduction of measure-valued entropy solutions

In this section we first establish the reduction of measure-valued entropy solutions of (1.1)–(1.2) and prove that any measure-valued entropy solution $\pi_{t,x}(k)$ in the sense of Definition 2.2 is the Dirac solution such that the associated profile $\rho(t,x)$ is an entropy solution in the sense of Definition 2.1. That is, our goal is to establish that, when $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$,

$$\pi_{t,x}(k) = \delta_{\rho(t,x)}(k), \tag{2.7}$$

where $\rho : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is the unique entropy solution determined by (2.3). The reduction proof is achieved by two theorems. We start with the following theorem which yields the $L^1$-contraction between the measure-valued entropy solution $\pi_{t,x}$ and the unique entropy solution $\rho(t,x)$ of (1.1)–(1.2).

**Theorem 2.1** *($L^1$-contraction). Assume that there exists a measure-valued entropy solution $\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$ of* (1.1) *in the sense of Definition* 2.2 *with $\pi_{t,x}$ having a fixed compact support for a.e. $(t,x)$. Assume that there exists a function $\rho : \mathbb{R}_+^2 \mapsto \mathbb{R}$ with initial data $\rho_0 \in L^\infty(\mathbb{R})$ and $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ for a.e. $x \in \mathbb{R}$ satisfying the following inequality*:

$$\int \Big(\big\langle\pi_{t,x};\,\big|k-\rho(t,x)\big|\big\rangle\partial_t J + \big\langle\pi_{t,x};\,\mathrm{sign}\big(k-\rho(t,x)\big)\big(F(x,k)-F\big(x,\rho(t,x)\big)\big)\big\rangle\partial_x J\Big)\,dx\,dt \geqslant 0 \tag{2.8}$$

*for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$. Then the function $\int\langle\pi_{t,x};\,|k-\rho(t,x)|\rangle\,dx$ is non-increasing in $t>0$, which implies $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ when $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ for a.e. $x \in \mathbb{R}$. Furthermore, $\rho$ is the unique entropy solution of* (1.1)–(1.2) *in the sense of Definition* 2.1.

**Proof.** In expression (2.8), we choose the test function as the product test function $J_j(t)H(x)$, with $J_j(t)$ converging to the indicator function $\mathbb{1}_{[t_1,t_2]}(t)$ as $j\to\infty$ for $t_2 > t_1 \geqslant 0$. Then (2.8) is equivalent to

$$\begin{aligned}&\int H(x)\big\langle\pi_{t_1,x}(k);\,\big|k-\rho(t_1,x)\big|\big\rangle\,dx - \int H(x)\big\langle\pi_{t_2,x}(k);\,\big|k-\rho(t_2,x)\big|\big\rangle\,dx\\ &\quad+\int_{t_1}^{t_2}\!\!\int H'(x)\big\langle\pi_{t,x}(k);\,\mathrm{sign}\big(k-\rho(t,x)\big)\big(F(x,k)-F\big(x,\rho(t,x)\big)\big)\big\rangle\,dx\,dt \geqslant 0.\end{aligned} \tag{2.9}$$

In (2.9), we choose

$$H(x) = e^{-\gamma\sqrt{1+|x|^2}}\chi\left(\frac{x}{N}\right), \quad \gamma, N > 0,$$

for $\chi \in C_0^\infty(-2,2)$ with $\chi(x)=1$ when $x \in [-1,1]$ and $\chi(x) \geqslant 0$. Letting $N \to \infty$ first and $\gamma \to 0$ then yields that, for any $t_2 > t_1 \geqslant 0$,

$$\int \langle \pi_{t_2,x}; |k-\rho(t_2,x)| \rangle \, dx - \int \langle \pi_{t_1,x}; |k-\rho(t_1,x)| \rangle \, dx \leqslant 0.$$

In particular, when $t_2 = t > 0$, $t_1 \to 0$, then $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ implies

$$\int \langle \pi_{t,x}; |k-\rho(t,x)| \rangle \, dx \leqslant 0$$

so that $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ for any $t > 0$.

Plugging this into inequality (2.5), we obtain inequality (2.3). Thus, $\rho(t,x)$ is an entropy solution which is unique by [2]. □

It thus remains to prove inequality (2.8).

**Theorem 2.2.** *Assume that $\rho : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is the unique entropy solution of* (1.1)–(1.2) *with initial data $\rho_0 \in L^\infty(\mathbb{R})$. Assume that there exists a measure-valued entropy solution $\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$ of* (1.1) *in the sense of Definition* 2.2 *with $\pi_{t,x}$ having a fixed compact support for a.e. $(t,x)$ and $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ for a.e. $x \in \mathbb{R}$. Then inequality* (2.8) *holds for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$.*

**Proof.** The proof is divided into nine steps.

*Step 1.* We first notice the following:

- Under assumption (H3′), $F(x,\rho)$ is continuous in $x$ a.e. Then we can define a function $\tilde{\rho}(s,y,x)$ for a.e. $(s,y,x) \in \mathbb{R}_+ \times \mathbb{R}^2$ such that, for fixed $(s,y)$,

$$F\big(x, \tilde{\rho}(s,y,x)\big) := F\big(x, m_{F(y,\rho(s,y))}(x)\big) = F\big(y, \rho(s,y)\big), \tag{2.10}$$

  where the last equality follows from (2.2). Thus, we define

$$\tilde{\rho}(s,y,x) = m_{\beta(s,y)}(x) \quad \text{with } \beta(s,y) := F\big(y, \rho(s,y)\big).$$

  In the same way, we can define a function $\tilde{m}(x,c,y)$ for any constant $c \in \mathbb{R}$ and for a.e. $(x,y) \in \mathbb{R}^2$ such that, for fixed $x$,

$$F\big(y, \tilde{m}(x,c,y)\big) := F\big(y, m_{F(x,c)}(y)\big) = F(x,c). \tag{2.11}$$

  Thus, we define

$$\tilde{m}(x,c,y) = m_{\gamma(x,c)}(y) \quad \text{with } \gamma(x,c) := F(x,c).$$

- For the case (H3), we define $\tilde{\rho}(s,y,x)$ such that the sign of the difference between $\tilde{\rho}(s,y,x)$ and $\rho_m(y)$ is the same as the sign of the difference between the corresponding solution and $\rho_m(y)$, that is,

$$\operatorname{sign}\big(\rho(s,y) - \rho_m(y)\big) = \operatorname{sign}\big(\tilde{\rho}(s,y,x) - \rho_m(y)\big). \tag{2.12}$$

It can be achieved by defining

$$\tilde{\rho}(s,y,x) := m^{+}_{\beta(s,y)}(x)\,\mathrm{sign}_{+}\big(\rho(s,y)-\rho_m(y)\big) + m^{-}_{\beta(s,y)}(x)\,\mathrm{sign}_{-}\big(\rho(s,y)-\rho_m(y)\big), \tag{2.13}$$

since $\rho_m(y)$ is the minimum (or maximum) point of the flux function with $F(y,\rho_m(y)) = M_0$. Similarly, we define

$$\tilde{m}(x,c,y) := m^{+}_{\gamma(x,c)}(y)\,\mathrm{sign}_{+}\big(c-\rho_m(x)\big) + m^{-}_{\gamma(x,c)}(y)\,\mathrm{sign}_{-}\big(c-\rho_m(x)\big). \tag{2.14}$$

Then we have as in (2.10) and (2.11),

$$F\big(x,\tilde{\rho}(s,y,x)\big) = F\big(y,\rho(s,y)\big) = \beta(s,y)$$

and

$$F\big(y,\tilde{m}(x,c,y)\big) = F(x,c) = \gamma(x,c).$$

With these notations, we can rewrite inequality (2.5) as follows:

$$\partial_t\big\langle\pi_{t,x};\,\big|k-\tilde{\rho}(s,y,x)\big|\big\rangle + \partial_x\big\langle\pi_{t,x};\,\mathrm{sign}\big(k-\tilde{\rho}(s,y,x)\big)\big(F(x,k)-F\big(y,\rho(s,y)\big)\big)\big\rangle \leqslant 0 \tag{2.15}$$

in the sense of distributions, and inequality (2.3) can be rewritten as

$$\partial_s\big|\rho(s,y)-\tilde{m}(x,k,y)\big| + \partial_y\big(\mathrm{sign}\big(\rho(s,y)-\tilde{m}(x,k,y)\big)\big(F\big(y,\rho(s,y)\big)-F(x,k)\big)\big) \leqslant 0$$

for any $k \in \mathbb{R}$, which implies

$$\partial_s\big\langle\pi_{t,x};\,\big|\rho(s,y)-\tilde{m}(x,k,y)\big|\big\rangle + \partial_y\big\langle\pi_{t,x};\,\mathrm{sign}\big(\rho(s,y)-\tilde{m}(x,k,y)\big)\big(F\big(y,\rho(s,y)\big)-F(x,k)\big)\big\rangle \leqslant 0 \tag{2.16}$$

in the sense of distributions.

*Step 2.* We next perform an integration by parts against a test function of the form

$$J_{\tau,\omega}(t,x,s,y) = J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y) \geqslant 0. \tag{2.17}$$

Here $J \in C_0^\infty(\mathbb{R}^2_+)$ and the two families of functions $\bar{H}_\tau,\ H_\omega \in C_0^\infty(\mathbb{R})$ are defined as

$$\bar{H}_\tau(z) = \frac{1}{\tau}\bar{H}\left(\frac{z}{\tau}\right) \quad \text{and} \quad H_\omega(z) = \frac{1}{\omega}H\left(\frac{z}{\omega}\right) \quad \text{for } \tau,\omega>0,$$

for a positive, compactly supported function $H \in C_0^\infty(\mathbb{R})$ and a positive function $\bar{H} \in C_0^\infty(\mathbb{R})$ with compact support in $(-1,1)$ such that $\int_{\mathbb{R}} H(z)\,dz = \int_{\mathbb{R}} \bar{H}(z)\,dz = 1$.

We first choose the test function in (2.15) as defined above for fixed $(s,y)$ and then integrate the resulting inequality with respect to $(s,y)$ to obtain

$$\begin{aligned}
&\int \big\langle\pi_{t,x};\,\big|k-\tilde{\rho}(s,y,x)\big|\big\rangle\partial_t J_{\tau,\omega}(t,x,s,y)\,dt\,dx\,ds\,dy \\
&\quad+\int \big\langle\pi_{t,x};\,\mathrm{sign}\big(k-\tilde{\rho}(s,y,x)\big)\big(F(x,k)-\beta(s,y)\big)\big\rangle\partial_x J_{\tau,\omega}(t,x,s,y)\,dt\,dx\,ds\,dy \\
&\quad+\int \big|\rho_0(x)-\tilde{\rho}(s,y,x)\big|J_{\tau,\omega}(0,x,s,y)\,dx\,ds\,dy \geqslant 0.
\end{aligned} \tag{2.18}$$

Furthermore, after integration, it follows from (2.16) that

$$\begin{aligned}
&\int \langle \pi_{t,x}; \left|\rho(s,y) - \tilde{m}(x,k,y)\right|\rangle \partial_s J_{\tau,\omega}(t,x,s,y)\,dt\,dx\,ds\,dy \\
&\quad + \int \langle \pi_{t,x}; \operatorname{sign}\bigl(\rho(s,y) - \tilde{m}(x,k,y)\bigr)\bigl(F\bigl(y,\rho(s,y)\bigr) - \gamma(x,k)\bigr)\rangle \partial_y J_{\tau,\omega}(t,x,s,y)\,dt\,dx\,ds\,dy \\
&\quad + \int \langle \pi_{t,x}; \left|\rho_0(y) - m(x,k,y)\right|\rangle J_{\tau,\omega}(t,x,0,y)\,dt\,dx\,dy \geqslant 0. \qquad (2.19)
\end{aligned}$$

We next add (2.18) and (2.19) together to obtain the following inequality:

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \geqslant 0, \qquad (2.20)$$

where

$$\begin{aligned}
T_1 :={}& \frac{1}{2}\int \langle \pi_{t,x}; \left|k - \tilde{\rho}(s,y,x)\right|\rangle \partial_t J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy \\
&+ \frac{1}{2}\int \langle \pi_{t,x}; \operatorname{sign}\bigl(k - \tilde{\rho}(s,y,x)\bigr)\bigl(F(x,k) - \beta(s,y)\bigr)\rangle \\
&\qquad \times \partial_x J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy, \\
T_2 :={}& \frac{1}{2}\int \langle \pi_{t,x}; \left|\rho(s,y) - \tilde{m}(x,k,y)\right|\rangle \partial_s J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy \\
&+ \frac{1}{2}\int \langle \pi_{t,x}; \operatorname{sign}\bigl(\rho(s,y) - \tilde{m}(x,k,y)\bigr)\bigl(F\bigl(y,\rho(s,y)\bigr) - \gamma(x,k)\bigr)\rangle \\
&\qquad \times \partial_y J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy, \\
T_3 :={}& \int \langle \pi_{t,x}; \left|k - \tilde{\rho}(s,y,x)\right| - \left|\rho(s,y) - \tilde{m}(x,k,y)\right|\rangle J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}'_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy, \\
T_4 :={}& \int \langle \pi_{t,x}; \bigl(F(x,k) - F\bigl(y,\rho(s,y)\bigr)\bigr)\bigl(\operatorname{sign}\bigl(k - \tilde{\rho}(s,y,x)\bigr) + \operatorname{sign}\bigl(\rho(s,y) - \tilde{m}(x,k,y)\bigr)\bigr)\rangle \\
&\quad \times J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H'_\omega(x-y)\,dt\,dx\,ds\,dy, \\
T_5 :={}& \int \langle \pi_{t,x}; \left|\rho_0(y) - \tilde{m}(x,k,y)\right|\rangle J\left(\frac{t}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t)H_\omega(x-y)\,dt\,dx\,dy, \\
T_6 :={}& \int \left|\rho_0(x) - \tilde{\rho}(s,y,x)\right| J\left(\frac{s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(-s)H_\omega(x-y)\,dx\,ds\,dy.
\end{aligned}$$

*Step 3.* We first show that $T_4 = 0$. This requires to show that

$$\operatorname{sign}\bigl(k - \tilde{\rho}(s,y,x)\bigr) = \operatorname{sign}\bigl(\tilde{m}(x,k,y) - \rho(s,y)\bigr). \qquad (2.21)$$

With this result, the integrand of $T_4$ cancels for a.e. $(t,x,s,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, which yields that $T_4 = 0$ for every $\omega, \tau > 0$.

To prove (2.21), we apply (2.13) and (2.14). For a.e. $(t,x,s,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, we obtain

$$F(x,k) - F\bigl(x, \tilde{\rho}(s,y,x)\bigr) = F\bigl(y, \tilde{m}(x,k,y)\bigr) - F\bigl(y, \rho(s,y)\bigr).$$

Under (H3′), the result follows immediately, since $F$ is monotone in the second variable.

Under (H3), we find from (2.12) that

$$\operatorname{sign}\bigl(k-\rho_m(x)\bigr)-\operatorname{sign}\bigl(\tilde m(x,k,y)-\rho_m(x)\bigr)$$
$$=0=\operatorname{sign}\bigl(\rho(s,y)-\rho_m(y)\bigr)-\operatorname{sign}\bigl(\tilde\rho(s,y,x)-\rho_m(y)\bigr). \tag{2.22}$$

We have two cases:

If $\operatorname{sign}(k-\rho_m(x))=\operatorname{sign}(\rho(s,y)-\rho_m(y))$, the problem is reduced to the monotone case since $F(x,\cdot)$ is monotone on each interval $[-\infty,\rho_m(x)]$ and $[\rho_m(x),\infty]$;

If $\operatorname{sign}(k-\rho_m(x))\neq\operatorname{sign}(\rho(s,y)-\rho_m(y))$, the result follows immediately from (2.22).

In Steps 4–6, we will show that, in the limit as $\omega\to 0$ first and $\tau\to 0$ second, inequality (2.8) follows from $T_1+T_2+T_3+T_5+T_6\geqslant 0$.

*Step 4.* We first show that

$$\tilde\rho(s,y,x)\xrightarrow{x\to y}\tilde\rho(s,y,y)=\rho(s,y)\quad\text{for a.e. }(s,y)\in\mathbb{R}_+^2, \tag{2.23}$$

and

$$\tilde m(x,k,y)\xrightarrow{y\to x}\tilde m(x,k,x)=k\quad\text{for a.e. }x\in\mathbb{R}. \tag{2.24}$$

For the case (H3′), since the flux function is continuous outside a negligible set $\mathcal{N}$, then, for $y\in\mathbb{R}\setminus\mathcal{N}$,

$$F\bigl(x,\tilde\rho(s,y,y)\bigr)\xrightarrow{x\to y}F\bigl(y,\tilde\rho(s,y,y)\bigr).$$

On the other hand, we have $F(y,\tilde\rho(s,y,y))=F(x,\tilde\rho(s,y,x))$. Therefore, we have

$$F\bigl(x,\tilde\rho(s,y,x)\bigr)-F\bigl(x,\tilde\rho(s,y,y)\bigr)\xrightarrow{x\to y}0,$$

and (2.23) is a consequence of the fact that $F(x,\cdot)$ is a one-to-one function.

Similarly, for $x\in\mathbb{R}\setminus\mathcal{N}$, we have

$$F(y,k)\xrightarrow{y\to x}F(x,k),$$

while $F(x,k)=F(y,\tilde m(x,k,y))$. Therefore, we have

$$F\bigl(y,\tilde m(x,k,y)\bigr)-F(y,k)\xrightarrow{y\to x}0,$$

and (2.24) is a consequence of the fact that $F(y,\cdot)$ is a one-to-one function.

For the case (H3), it is clear from the definition of $\tilde\rho(s,y,x)$ and $\tilde m(x,k,y)$ in (2.13) and (2.14), respectively.

*Step 5.* We show that, when $\omega\to 0$ first and $\tau\to 0$ second, $T_1$ converges to

$$\frac{1}{2}\int\Bigl(\bigl\langle\pi_{t,x};\,\bigl|k-\rho(t,x)\bigr|\bigr\rangle\partial_t J(t,x)+\bigl\langle\pi_{t,x};\operatorname{sign}\bigl(k-\rho(t,x)\bigr)\bigl(F(x,k)-F\bigl(x,\rho(t,x)\bigr)\bigr)\bigr\rangle\partial_x J(t,x)\Bigr)\,dt\,dx. \tag{2.25}$$

Observe that

$$\begin{aligned}
&\left|\int \langle \pi_{t,x}; |k-\tilde{\rho}(s,y,x)|\rangle \partial_t J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy\right.\\
&\qquad\left.-\int \langle \pi_{t,x}; |k-\tilde{\rho}(s,y,y)|\rangle \partial_t J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy\right|\\
&\quad\leqslant \int\left(\int \left|\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)\right| H_\omega(x-y)\left|\partial_t J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\right| dx\right)\bar{H}_\tau(t-s)\,dt\,ds\,dy\\
&\quad\to 0 \quad \text{when } \omega\to 0,
\end{aligned} \tag{2.26}$$

by the Dominated Convergence theorem and the fact that

$$\int \left|\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)\right| H_\omega(x-y)\left|\partial_t J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\right| dx \to 0,$$

when $\omega\to 0$ for a.e. $(s,y)\in\mathbb{R}^2_+$ since $\tilde{\rho}(s,y,x)\xrightarrow{x\to y}\tilde{\rho}(s,y,y)=\rho(s,y)$ by Step 4. Furthermore,

$$\begin{aligned}
&\int \langle \pi_{t,x}; |k-\rho(s,y)|\rangle \left|\partial_t J\left(\frac{t+s}{2},\frac{x+y}{2}\right)-\partial_t J(t,x)\right| \bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy\\
&\quad=\mathcal{O}(\omega)+\mathcal{O}(\tau)\to 0,
\end{aligned} \tag{2.27}$$

when $\omega\to 0$ first and $\tau\to 0$ second. Then, to find the limit of the first part of $T_1$, it suffices to compute the limit of

$$\int \langle \pi_{t,x}; |k-\rho(s,y)|\rangle \partial_t J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy. \tag{2.28}$$

Thus, it suffices to show that $\rho(s,y)$ can be replaced by $\rho(t,x)$ in (2.28), i.e., when $\omega\to 0$ first and $\tau\to 0$ second,

$$\begin{aligned}
&\int \left|\rho(t,x)-\rho(s,y)\right|\partial_t J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy\\
&\quad=\int \left|\rho(t,x)-\rho(t+\tau r,x+\omega z)\right|\partial_t J(t,x)\bar{H}(-r)H(-z)\,dt\,dx\,dr\,dz\to 0.
\end{aligned} \tag{2.29}$$

This is guaranteed by the fact that

$$\lim_{\tau\to 0}\lim_{\omega\to 0}\int \left|\rho(t,x)-\rho(t+\tau r,x+\omega z)\right| dt\,dx=0,$$

and the Dominated Convergence theorem since all the functions involved are bounded. This implies that, in (2.28), we can indeed replace $\rho(s,y)$ by $\rho(t,x)$.

On the other hand, hypothesis (H2) on $F(x,\rho)$ implies

$$\begin{aligned}
&\left|\operatorname{sign}\big(k-\tilde{\rho}(s,y,x)\big)\big(F(x,k)-\beta(s,y)\big)-\operatorname{sign}\big(k-\tilde{\rho}(s,y,y)\big)\big(F(x,k)-F\big(x,\tilde{\rho}(s,y,y)\big)\big)\right|\\
&\quad=\left|\operatorname{sign}\big(k-\tilde{\rho}(s,y,x)\big)\big(F(x,k)-F\big(x,\tilde{\rho}(s,y,x)\big)\big)-\operatorname{sign}\big(k-\rho(s,y)\big)\big(F(x,k)-F\big(x,\rho(s,y)\big)\big)\right|\\
&\quad\leqslant C\left|\tilde{\rho}(s,y,x)-\rho(s,y)\right|.
\end{aligned}$$

Integrating the last expression with respect to $x$ against the function $H_\omega(x-y)$ yields its convergence to 0 by the same argument as above when $\omega \to 0$. Since $J \in C_0^\infty(\mathbb{R}_+^2)$, as above, the limit of the second part of $T_1$ is the same as the limit of

$$\int \big\langle \pi_{t,x}; \operatorname{sign}\big(k-\rho(s,y)\big)\big(F(x,k)-F\big(x,\rho(s,y)\big)\big)\big\rangle \partial_x J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy,$$

and it suffices to prove that, when $\omega \to 0$ first and $\tau \to 0$ second,

$$\int \big\langle \pi_{t,x}; \big|\operatorname{sign}\big(k-\rho(s,y)\big)\big(F(x,k)-F\big(x,\rho(s,y)\big)\big) - \operatorname{sign}\big(k-\rho(t,x)\big)\big(F(x,k)-F\big(x,\rho(t,x)\big)\big)\big|\big\rangle$$
$$\times\, \partial_x J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy \to 0.$$

Using the Lipschitz property and fact (2.29), we achieve the result for the second part of $T_1$.

*Step 6.* $T_2$ converges to (2.25) as well. This follows by the same argument as used already in Step 5 and observing that

$$\begin{aligned}
&\left| \int \big\langle \pi_{t,x}; \big|\rho(s,y)-\tilde{m}(x,k,y)\big|\big\rangle \partial_s J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)dt\,dx\,ds\,dy \right.\\
&\quad \left. - \int \big\langle \pi_{t,x}; \big|\rho(s,y)-k\big|\big\rangle \partial_s J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy \right|\\
&\quad \leqslant \int \big\langle \pi_{t,x}; \big|\tilde{m}(x,k,y)-k\big|\big\rangle \left|\partial_s J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\right|\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy.
\end{aligned}$$

Again the right-hand side of the last expression converges to zero when $\omega \to 0$. Using the same argument as in Step 5, we achieve the result for $T_2$.

*Step 7.* $T_3$ converges to 0 when $\omega \to 0$. Since

$$\begin{aligned}
&\lim_{\omega\to 0}\left| \int \big\langle \pi_{t,x}; \big|k-\tilde{\rho}(s,y,x)\big| - \big|\rho(s,y)-\tilde{m}(x,k,y)\big|\big\rangle H_\omega(x-y)\,dx\,dy \right|\\
&\quad \leqslant \lim_{\omega\to 0}\int \big\langle \pi_{t,x}; \big|k-\tilde{m}(x,k,y)\big|\big\rangle H_\omega(x-y)\,dx\,dy + \lim_{\omega\to 0}\int \big\langle \pi_{t,x}; \big|\rho(s,y)-\tilde{\rho}(s,y,x)\big|\big\rangle H_\omega(x-y)\,dx\,dy\\
&\quad = 0,
\end{aligned}$$

the result follows as in Steps 5 and 6.

*Step 8.* $T_6$ converges to zero when $\tau \to 0$ after $\omega \to 0$. Note that

$$\begin{aligned}
&\int \big|\big|\rho_0(x)-\tilde{\rho}(s,y,x)\big| - \big|\rho_0(x)-\tilde{\rho}(s,y,y)\big|\big| J\left(0, \frac{x+y}{2}\right)\bar{H}_\tau(-s)H_\omega(x-y)\,dx\,ds\,dy\\
&\quad \leqslant \int \big|\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)\big| J\left(0, \frac{x+y}{2}\right)\bar{H}_\tau(-s)H_\omega(x-y)\,dx\,ds\,dy.
\end{aligned}$$

Again with (2.29), the right-hand side converges to zero when $\omega \to 0$. We therefore next compute the limit when $\omega \to 0$ first and $\tau \to 0$ second of

$$\int \big|\rho_0(x)-\rho(s,y)\big| J\left(\frac{s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(-s)H_\omega(x-y)\,dx\,ds\,dy.$$

As before,

$$\lim_{\omega\to 0}\int \left|\rho(s,x)-\rho(s,y)\right| J\left(\frac{s}{2},\frac{x+y}{2}\right) H_\tau(-s) H_\omega(x-y)\,dx\,ds\,dy = 0.$$

Therefore, the next goal is to compute the limit when $\tau \to 0$ of

$$\int \left|\rho_0(x)-\rho(s,x)\right| J\left(\frac{s}{2},x\right)\bar{H}_\tau(-s)\,dx\,ds\,dy = \int \left|\rho_0(x)-\rho(\tau r,x)\right| J\left(\frac{\tau r}{2},x\right)\bar{H}(-r)\,dx\,dr. \tag{2.30}$$

Since all the functions are bounded and $\operatorname{supp}\bar{H} \subset (-1,1)$, by the Dominated Convergence theorem, this converges to 0 when $\tau \to 0$, and thereby (2.30) converges to 0.

*Step 9.* $T_5$ converges to zero by the analogous argument as in Step 8 and using the fact that $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$.

With Steps 3–9 and by (2.20), we complete the proof. □

## 3. Existence of entropy solutions

In this section, we establish the existence of entropy solutions (1.1)–(1.2) in the sense of Definition 2.1, as required for the reduction of measure-valued entropy solutions. More precisely, for each fixed $\varepsilon > 0$, $\rho^\varepsilon$ denotes the unique Kruzkov solution of (1.1)–(1.2) in the sense (3.3), where the flux function depends smoothly on the space variable $x$; then it is shown that the sequence $\rho^\varepsilon$ converges to an entropy solution of (1.1)–(1.2).

### 3.1. Existence of entropy solutions when $F$ is smooth

Define $F^\varepsilon(x,\rho)$ the standard mollification of $F(x,\rho)$ in $x\in\mathbb{R}$:

$$F^\varepsilon(x,\rho) := \left(F(\cdot,\rho)*\theta^\varepsilon\right)(x) \to F(x,\rho) \quad \text{a.e. as } \varepsilon\to 0, \tag{3.1}$$

with $\theta^\varepsilon(x) := \theta(\frac{x}{\varepsilon})$, $\theta(x)\geqslant 0$, $\operatorname{supp}\theta(x)\subset[-1,1]$, and $\int_{-1}^{1}\theta(x)\,dx = 1$. For fixed $\varepsilon > 0$, consider the following Cauchy problem:

$$\begin{cases} \partial_t\rho + \partial_x F^\varepsilon(x,\rho) = 0, \\ \rho|_{t=0} = \rho_0(x) \geqslant 0. \end{cases} \tag{3.2}$$

Kruzkov's result in [18] indicates that there exists a unique solution $\rho^\varepsilon$ of (3.2) satisfying the Kruzkov entropy inequality:

$$\begin{aligned} &\partial_t\left|\rho^\varepsilon(t,x)-c\right| + \partial_x\left(\operatorname{sign}\left(\rho^\varepsilon(t,x)-c\right)\left(F^\varepsilon\left(x,\rho^\varepsilon(t,x)\right)-F^\varepsilon(x,c)\right)\right) \\ &\quad + \operatorname{sign}\left(\rho^\varepsilon(t,x)-c\right)\partial_x F^\varepsilon(x,c) \leqslant 0 \end{aligned} \tag{3.3}$$

in the sense of distributions. Notice that, since $F^\varepsilon$ is now smooth in the first variable, we can define steady-state solutions $m_\alpha^{\varepsilon,\pm}(x)$ for each $x\in\mathbb{R}$. In particular, the steady-state solutions $m_\alpha^{\varepsilon,\pm}(x)$ also satisfy the Kruzkov entropy inequality (3.3):

$$\begin{aligned} &\partial_t\left|m_\alpha^{\varepsilon,\pm}(x)-c\right| + \partial_x\left(\operatorname{sign}\left(m_\alpha^{\varepsilon,\pm}(x)-c\right)\left(F^\varepsilon\left(y,m_\alpha^{\varepsilon,\pm}(x)\right)-F^\varepsilon(x,c)\right)\right) \\ &\quad + \operatorname{sign}\left(m_\alpha^{\varepsilon,\pm}(x)-c\right)\partial_x F^\varepsilon(x,c) \leqslant 0 \end{aligned} \tag{3.4}$$

in the distributional sense. This can be also seen as follows: Since the level set $\{x \in \mathbb{R}:\ m_\alpha^{\varepsilon,\pm}(x) = c\}$ is discrete for a.e. $c, \alpha$, and this level set coincides with the set $\{x \in \mathbb{R}:\ F^\varepsilon(x,c) = \alpha\}$, it follows from the Sard theorem that the set of critical values of the function $S(m_\alpha^{\varepsilon,\pm}(x)) := \operatorname{sign}(m_\alpha^{\varepsilon,\pm}(x) - c)(F^\varepsilon(x, m_\alpha^{\varepsilon,\pm}(x)) - F^\varepsilon(x,c))$ has measure zero, which implies (3.4).

We now prove that the entropy solution $\rho^\varepsilon$ also satisfies (2.3).

**Proposition 3.1.** *Let $\rho^\varepsilon(t,x)$ be a solution of the Cauchy problem* (3.2) *satisfying the Kruzkov entropy inequality* (3.3)*. Then $\rho^\varepsilon(t,x)$ also satisfies the entropy inequality* (2.3) *with steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon,\pm}(x)$.*

**Proof.** We divide the proof into five steps.

*Step 1.* In (3.3), we choose the constant $c = m_\alpha^{\varepsilon,\pm}(y)$ for any $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) for fixed $(s,y)$, and integrate against the test function (2.17) first in $(t,x)$ and then in $(s,y)$ to obtain the following inequality:

$$\begin{aligned}
&\int \big|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big| \partial_t J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad + \int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(x, \rho^\varepsilon(t,x)\big) - F^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}(y)\big)\big)\partial_x J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad - \int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big)\partial_x F^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}(y)\big) J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad + \int \big|\rho^\varepsilon(0,x) - m_\alpha^{\varepsilon,\pm}(y)\big| J_{\tau,\omega}(t,0,s,y)\, dx\, ds\, dy \geqslant 0.
\end{aligned} \tag{3.5}$$

On the other hand, the Kruzkov entropy inequality (3.3) is satisfied for any steady-state solution $m_\alpha^{\varepsilon,\pm}$, for any $c \in \mathbb{R}$ and $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$). For fixed $(t,x)$, the steady-state solutions $m_\alpha^{\varepsilon,\pm}(y)$ as functions in $y$ satisfy (3.4) with $(s,y)$ replacing $(t,x)$ and the constant $c = \rho(t,x)$. We integrate against the test function $J_{\tau,\omega}$ first in $(s,y)$ and then in $(t,x)$ to obtain the following inequality:

$$\begin{aligned}
&\int \big|m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\big| \partial_s J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad + \int \operatorname{sign}\big(m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\big)\big(F^\varepsilon\big(y, m_\alpha^{\varepsilon,\pm}(y)\big) - F^\varepsilon\big(y, \rho^\varepsilon(t,x)\big)\big)\partial_y J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad - \int \operatorname{sign}\big(m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\big)\partial_y F^\varepsilon\big(y, \rho^\varepsilon(t,x)\big) J_{\tau,\omega}(t,x,s,y)\, dt\, dx\, ds\, dy \\
&\quad + \int \big|m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\big| J_{\tau,\omega}(t,x,0,y)\, dt\, dx\, dy \geqslant 0.
\end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6) together, we then have

$$I_1 + I_2 + I_3 \geqslant 0,$$

where

$$\begin{aligned}
I_1 := {} & \frac{1}{2}\int \big|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big|(\partial_t J + \partial_s J)\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\, dt\, dx\, ds\, dy \\
& + \frac{1}{2}\int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(x, \rho^\varepsilon(t,x)\big) - F^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}(y)\big)\big) \\
& \qquad \times (\partial_x + \partial_y) J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y) dt\, dx\, ds\, dy,
\end{aligned}$$

$$I_2 := \int \big|m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\big| J_{\tau,\omega}(t,x,0,y)\,dt\,dx\,dy + \int \big|\rho^\varepsilon(0,x) - m_\alpha^{\varepsilon,\pm}(y)\big| J_{\tau,\omega}(0,x,s,y)\,dx\,ds\,dy,$$

$$\begin{aligned} I_3 := &\int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big) - F^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big) + F^\varepsilon\big(y,m_\alpha^{\varepsilon,\pm}(y)\big) - F^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)\big) \\ &\quad \times J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H'_\omega(x-y)\,dt\,dx\,ds\,dy \\ &- \int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\big)\big(\partial_x F^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big) - \partial_y F^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)\big) \\ &\quad \times J\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy. \end{aligned}$$

In order to prove Proposition 3.1, we need to show that $I_1 + I_2$ converges to the left-hand side of the entropy inequality (2.3) and $I_3 \to 0$, when $\tau \to 0$ after $\omega \to 0$.

*Step 2.* We start with the following two useful identities:

$$\lim_{\omega\to 0} \int H_\omega(x-y)\big|m_\alpha^{\varepsilon,\pm}(y) - m_\alpha^{\varepsilon,\pm}(x)\big|\,dx\,dy = 0; \tag{3.7}$$

and, for any continuous function $G$ of $m_\alpha^{\varepsilon,\pm}$,

$$\lim_{\omega\to 0} \int H'_\omega(x-y)(x-y)\big|G\big(m_\alpha^{\varepsilon,\pm}(y)\big) - G\big(m_\alpha^{\varepsilon,\pm}(x)\big)\big|\,dx\,dy = 0. \tag{3.8}$$

We first show that the steady-state solutions are continuous on $\mathbb{R}$ for each $\alpha \in [M_0,\infty)$ (or $\alpha \in (-\infty, M_0]$).

We start with $\alpha \neq M_0$: Since the flux function is continuous in the first variable,

$$F^\varepsilon\big(y, m_\alpha^{\varepsilon,+}(x)\big) \xrightarrow{y\to x} F^\varepsilon\big(x, m_\alpha^{\varepsilon,+}(x)\big).$$

On the other hand, $F^\varepsilon(y, m_\alpha^{\varepsilon,+}(y)) = F^\varepsilon(x, m_\alpha^{\varepsilon,+}(x))$. Therefore, we have

$$F^\varepsilon\big(y, m_\alpha^{\varepsilon,+}(y)\big) - F^\varepsilon\big(y, m_\alpha^{\varepsilon,+}(x)\big) \xrightarrow{y\to x} 0,$$

and, as a consequence of the fact that $F^\varepsilon(y,\cdot)$ is a one-to-one function on $[\rho_m(y),\infty)$,

$$m_\alpha^{\varepsilon,+}(y) \xrightarrow{y\to x} m_\alpha^{\varepsilon,+}(x) \quad \text{for any } x \in \mathbb{R}.$$

Similarly, we can show for each $\alpha \neq M_0$ that

$$m_\alpha^{\varepsilon,-}(y) \xrightarrow{y\to x} m_\alpha^{\varepsilon,-}(x) \quad \text{for any } x \in \mathbb{R}.$$

If $\alpha = M_0$, then $m_{M_0}^{\varepsilon,\pm}(x) = \rho_m^\varepsilon(x)$,

$$F^\varepsilon\big(y, \rho_m^\varepsilon(x)\big) \xrightarrow{y\to x} F^\varepsilon\big(x, \rho_m^\varepsilon(x)\big) = M_0.$$

On the other hand, we have $F^\varepsilon(y, \rho_m^\varepsilon(y)) = M_0$. Therefore

$$F^\varepsilon\big(y, \rho_m^\varepsilon(y)\big) - F^\varepsilon\big(y, \rho_m^\varepsilon(x)\big) \xrightarrow{y\to x} 0,$$

and, as a consequence of the fact that $F^\varepsilon$ is a continuous function in the second variable, we obtain

$$\rho_m^\varepsilon(y) \xrightarrow{y\to x} \rho_m^\varepsilon(x) \quad \text{for any } x\in\mathbb{R}.$$

With this, as in the proof of Theorem 2.2, we obtain (3.7).

Notice that, for any continuous function $G$ of $m_\alpha^{\varepsilon,\pm}$, we have

$$\begin{aligned}
&\int H'_\omega(x-y)(x-y)\big|G\big(m_\alpha^{\varepsilon,\pm}(y)\big)-G\big(m_\alpha^{\varepsilon,\pm}(x)\big)\big|\,dx\,dy\\
&\quad=\int H'_\omega(-\omega z)(-\omega z)\big|G\big(m_\alpha^{\varepsilon,\pm}(x+\omega z)\big)-G\big(m_\alpha^{\varepsilon,\pm}(x)\big)\big|\omega\,dx\,dz\\
&\quad=\int zH'(-z)\big|G\big(m_\alpha^{\varepsilon,\pm}(x+\omega z)\big)-G\big(m_\alpha^{\varepsilon,\pm}(x)\big)\big|\,dx\,dz\\
&\quad\to 0 \quad \text{when } \omega\to 0,
\end{aligned}$$

since $m_\alpha^{\varepsilon,\pm}$ is continuous and is in $L^\infty$. Thus, (3.8) follows.

*Step 3.* With (3.7) and

$$\big|\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big|-\big|\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(x)\big|\leqslant\big|m_\alpha^{\varepsilon,\pm}(x)-m_\alpha^{\varepsilon,\pm}(y)\big|,$$

as in the proof of Theorem 2.2, we obtain that, when $\omega\to 0$ first and $\tau\to 0$ second, $I_1$ converges to

$$\int\big(\big|\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(x)\big|\partial_t J(t,x)+\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(x)\big)\big(F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-\alpha\big)\partial_x J(t,x)\big)\,dt\,dx.$$

In the same way, we can replace $m_\alpha^{\varepsilon,\pm}(y)$ by $m_\alpha^{\varepsilon,\pm}(x)$ and $\rho^\varepsilon(t,x)$ by $\rho^\varepsilon(0,x)$ in $I_2$, when $\omega\to 0$ first and $\tau\to 0$ second. Then both terms of $I_2$ converge to

$$\frac{1}{2}\int\big|\rho^\varepsilon(0,x)-m_\alpha^{\varepsilon,\pm}(y)\big|J(0,x)\,dx.$$

*Step 4.* It remains to show that $\lim_{\tau\to 0}\lim_{\omega\to 0} I_3=0$. To avoid confusion, from now on, we denote the derivative of $F^\varepsilon(x,\cdot)$ with respect to the first variable by $F_x^\varepsilon(x,\cdot)$.

Notice that

$$\begin{aligned}
I_3=&\int\Big(-\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)-F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)(y-x)\big)\\
&\quad+\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(y,m_\alpha^{\varepsilon,\pm}(y)\big)-F^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)-F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)(y-x)\big)\Big)\\
&\quad\times J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar H_\tau(t-s)H'_\omega(x-y)\,dt\,dx\,ds\,dy\\
&+\int\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)F_y^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar H_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy\\
&-\int\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)(y-x)\\
&\qquad\times J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar H_\tau(t-s)H'_\omega(x-y)\,dt\,dx\,ds\,dy\\
&-\int\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar H_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy
\end{aligned}$$

$$+\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)(y-x)$$
$$\times J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy$$
$$=\int\Big(-\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)-F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)(y-x)\big)$$
$$+\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F^\varepsilon\big(y,m_\alpha^{\varepsilon,\pm}(y)\big)-F^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)-F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)(y-x)\big)\Big)$$
$$\times J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)H'_\omega(x-y)\,dt\,dx\,ds\,dy$$
$$+\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)\big)\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)$$
$$\times J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)\,dt\,dx\,ds\,dy$$
$$+\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_y^\varepsilon\big(y,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)\big)$$
$$\times J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)H_\omega(x-y)\,dt\,dx\,ds\,dy.$$

The last term of this expression is of order $\mathcal{O}(\omega)$, since $F^\varepsilon$ is at least $C^1$ in the first variable, all the functions in the integrand are bounded, and the support of $H$ is also bounded. Therefore, this term converges to 0 when $\omega\to 0$.

*Step 5.* It remains to show that the first and the second terms in the last expression vanish in the limit. The first term is equal to

$$\frac{1}{2}\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_{xx}^\varepsilon\big(\xi,m_\alpha^{\varepsilon,\pm}(y)\big)-F_{xx}^\varepsilon\big(\xi,\rho^\varepsilon(t,x)\big)+\mathcal{O}\big(|y-x|\big)\big)H'_\omega(x-y)(x-y)^2$$
$$\times J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)\,dt\,dx\,ds\,dy.$$

Since $\rho^\varepsilon\in L^\infty$ and $F^\varepsilon$ is smooth in the first variable, by (H2), the first term of the last expression is bounded above by

$$C\int\frac{1}{\omega^2}H'\bigg(\frac{x-y}{\omega}\bigg)|y-x|^2 J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)\bar H_\tau(t-s)\,dt\,dx\,ds\,dy$$
$$=C\omega\int H'(z)z^2 J\bigg(\frac{t+s}{2},\frac{2x+\omega z}{2}\bigg)\bar H_\tau(t-s)\,dt\,dx\,ds\,dz=\mathcal{O}(\omega)\to 0\quad\text{when }\omega\to 0.$$

The second term is equal to

$$\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)\big)$$
$$\times\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)\bigg(\bigg(J\bigg(\frac{t+s}{2},\frac{x+y}{2}\bigg)-J(t,x)\bigg)+J(t,x)\bigg)H_\tau(t-s)\,dt\,dx\,ds\,dy$$
$$=\mathcal{O}(\omega)+\mathcal{O}(\tau)+\int J(t,x)\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)$$
$$\times\big(\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big)-F_x^\varepsilon\big(x,m_\alpha^{\varepsilon,\pm}(y)\big)\big)$$

$$- \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(x)\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big) - F_x^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}(x)\big)\big)\big)\,dt\,dx\,dy$$

$$+ \int \operatorname{sign}\big(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(x)\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon(t,x)\big) - F_x^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}(x)\big)\big)$$

$$\times \left(\int \big(H_\omega(x-y) + H'_\omega(x-y)(x-y)\big)\,dy\right) J(t,x)\,dt\,dx.$$

Notice that

$$\operatorname{sign}\big(\rho^\varepsilon - m_\alpha^{\varepsilon,\pm}\big)\big(F_x^\varepsilon\big(x,\rho^\varepsilon\big) - F_x^\varepsilon\big(x, m_\alpha^{\varepsilon,\pm}\big)\big)$$

is a continuous function of $m_\alpha^{\varepsilon,\pm}$. Thus, the third term of the last expression goes to zero if $\omega \to 0$ by (3.7) and (3.8).

In the remaining last term, the integral with respect to $y$ is equal to 0 because

$$H_\omega(x-y) + H'_\omega(x-y)(x-y) = -\partial_y\big((x-y)H_\omega(x-y)\big).$$

This concludes that $I_3$ vanishes in the limit when $\omega \to 0$ first and $\tau \to 0$ second. $\square$

Thus we conclude the existence of an entropy solution $\rho_\varepsilon(t,x)$ in the sense of Definition 2.1 for each $F^\varepsilon$ with fixed $\varepsilon > 0$.

**Remark 3.1.** Notice that the sequence of approximate entropy solutions converges to a measure-valued entropy solution when $\varepsilon \to 0$: First, since $\rho_0 \in L^\infty$, we find that, for $\alpha$ big enough,

$$m_\alpha^{\varepsilon,-}(x) \leqslant \rho_0(x) \leqslant m_\alpha^{\varepsilon,+}(x) \quad \text{for all } x \in \mathbb{R}.$$

From [2], it then follows that

$$m_\alpha^{\varepsilon,-}(x) \leqslant \rho^\varepsilon(t,x) \leqslant m_\alpha^{\varepsilon,+}(x),$$

which implies the uniform boundedness of $\rho^\varepsilon(t,x)$ in $\varepsilon$ since $m_\alpha^{\varepsilon,\pm}(x)$ are uniformly bounded in $\varepsilon$. Then there exist a compactly supported family of probability measures $\pi_{t,x}$ on $\mathbb{R}$ (i.e. Young measures; see Tartar [23]) and a subsequence (still denoted by) $\rho^\varepsilon(t,x)$ such that, for any continuous function $f(\rho)$,

$$f\big(\rho^\varepsilon(t,x)\big) \overset{*}{\rightharpoonup} \big\langle \pi_{t,x}, f(k)\big\rangle \quad \text{when } \varepsilon \to 0. \tag{3.9}$$

On the other hand, by Section 3.1, the sequence $\rho^\varepsilon(t,x)$ satisfies the entropy inequality (2.3) for $F^\varepsilon(x,\rho)$ and the steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon,\pm}$. In particular, we use (3.9) and the definition of the sequence $F^\varepsilon(x,\rho)$ in (3.1) to conclude that, when $\varepsilon \to 0$, the compactly supported family of probability measures $\pi_{t,x}$ satisfies that, for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$,

$$\begin{aligned} &\int \Big(\big\langle \pi_{t,x}; \big|k - m_\alpha^\pm(x)\big|\big\rangle \partial_t J + \big\langle \pi_{t,x}; \operatorname{sign}\big(k - m_\alpha^\pm\big)\big(F(x,k) - \alpha\big)\big\rangle \partial_x J\Big)\,dx\,dt \\ &\quad + \int \big|\rho_0(x) - m_\alpha^\pm(x)\big|\, J(0,x)\,dx \geqslant 0. \end{aligned} \tag{3.10}$$

Thus, $\pi_{t,x}$ is a measure-valued entropy solution of (1.1)–(1.2) with compact support for a.e. $(t,x) \in \mathbb{R}_+^2$ in the sense of Definition 2.2.

### 3.2. Existence of entropy solutions when $F$ is discontinuous in $x$

We are now ready to state the main theorem of this section.

**Theorem 3.1.** *Let $F(x,\rho)$ be strictly convex or concave in $\rho$ for a.e. $x \in \mathbb{R}$ and satisfy* (H1)–(H3)*, or let $F(x,\rho)$ satisfy* (H1)–(H2) *and* (H3′)*. Let $\rho_0(x) \in L^\infty$. Then the sequence of entropy solutions $\rho^\varepsilon$ of the Cauchy problem* (3.2) (*in the sense of Definition* 2.1) *converges to the unique entropy solution of the Cauchy problem* (1.1)–(1.2) *in the sense of Definition* 2.1.

**Proof.** We consider the two cases separately.

For the case (H1)–(H2) and (H3′), that is, the flux function $F$ is monotone in $\rho$, we apply the compactness framework established in Section 2 to establish the convergence. For this case, the existence of entropy solutions has been established in [4]. In Remark 3.1, we have shown that the limit of the entropy solutions $\rho^\varepsilon$ is determined by a measure-valued entropy solution $\pi_{t,x}$. Then, by Theorems 2.1–2.2, $\pi_{t,x}$ is the Dirac measure concentrated on the unique entropy solution $\rho(t,x)$ of (1.1)–(1.2) in the sense of Definition 2.1, which implies the whole sequence converges.

For the case (H1)–(H3), since we have not established the existence of an entropy solution, we employ the compensated compactness method to establish the convergence of the entropy solutions of the Cauchy problem (3.2), which also yields the existence of a unique entropy solution of the Cauchy problem (1.1)–(1.2).

From Remark 3.1, we know that $\rho^\varepsilon$ is uniformly bounded in $L^\infty$ which implies that there exists a subsequence $\rho^\varepsilon$ converging weakly to a compactly supported family of probability measures $\nu_{t,x}$ on $\mathbb{R}_+$ such that, for any function $f(\rho,t,x)$ that is continuous in $\rho$ for a.e. $(t,x)$,

$$f\big(\rho^\varepsilon(t,x),t,x\big) \overset{*}{\rightharpoonup} \big\langle \nu_{t,x}, f(k,t,x)\big\rangle \quad \text{when } \varepsilon \to 0. \tag{3.11}$$

In particular,

$$\rho^\varepsilon(t,x) \overset{*}{\rightharpoonup} \langle \nu_{t,x}, k\rangle =: \rho(t,x) \in L^\infty. \tag{3.12}$$

Our goal is to prove the strong convergence of $\rho^\varepsilon(t,x)$ to $\rho(t,x)$ a.e., equivalently, $\nu_{t,x} = \delta_{\rho(t,x)}$, which implies that $\rho(t,x)$ is an entropy solution of (1.1)–(1.2), that is, $\rho(t,x)$ satisfies the entropy inequality in Definition 2.1.

From Section 3.1, we know that the sequence $\rho^\varepsilon$ exists and satisfies

$$E^\varepsilon := \partial_t\big|\rho_\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)\big| + \partial_x\big(\operatorname{sign}\big(\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)\big)\big(F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big) - \gamma(s,y)\big)\big) \leqslant 0$$

in the sense of distributions, where

$$\hat\rho^\varepsilon(s,y,x) := m^{+,\varepsilon}_{\gamma(s,y)}(x)\operatorname{sign}_+\big(\rho(s,y) - \rho_m(y)\big) + m^{-,\varepsilon}_{\gamma(s,y)}(x)\operatorname{sign}_-\big(\rho(s,y) - \rho_m(y)\big).$$

Notice that $\gamma(s,y) := F(y,\rho(s,y))$ is independent of $\varepsilon$. Thus, for fixed $(s,y)$, we have the strong convergence of $m^{\pm,\varepsilon}_{\gamma(s,y)}(x)$ to a steady-state solution $m^{\pm}_{\gamma(s,y)}(x)$ of (1.1)–(1.2) when $\varepsilon \to 0$. In particular,

$$\|\hat\rho^\varepsilon\|_{L^\infty} \leqslant M, \quad M \text{ independent of } \varepsilon;$$

and, for a.e. $(s,y,x) \in \mathbb{R}^2_+ \times \mathbb{R}$,

$$\hat\rho^\varepsilon(s,y,x) \to \hat\rho(s,y,x) := m^{+}_{\gamma(s,y)}(x)\operatorname{sign}_+\big(\rho(s,y) - \rho_m(y)\big) + m^{-}_{\gamma(s,y)}(x)\operatorname{sign}_-\big(\rho(s,y) - \rho_m(y)\big),$$

when $\varepsilon \to 0$. By Schwartz's lemma, $E^\varepsilon$ is a sequence of measures; by Murat's lemma [20], $E^\varepsilon$ is uniformly bounded measure sequence in the measure space, which implies that

$$E^\varepsilon \text{ is compact in } W^{-1,p}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) \text{ for any } p \in (1,2). \tag{3.13}$$

On the other hand, since the vector-field sequence

$$\Big(\big|\rho^\varepsilon(t,x) - m^{\pm,\varepsilon}_{\gamma(s,y)}(x)\big|, \operatorname{sign}\big(\rho^\varepsilon(t,x) - m^{\pm,\varepsilon}_{\gamma(s,y)}(x)\big)\big(F^\varepsilon\big(x,\rho^\varepsilon(t,x)\big) - \gamma(s,y)\big)\Big)$$

is uniformly bounded in $\varepsilon$ for any fixed $(s,y)$, it follows that

$$E^\varepsilon \text{ is bounded in } W^{-1,\infty}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big). \tag{3.14}$$

With (3.13)–(3.14), we obtain by a compactness interpolation theorem in [6,13] that

$$E^\varepsilon \text{ is compact in } H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big). \tag{3.15}$$

On the other hand,

$$\partial_t \rho^\varepsilon + \partial_x F^\varepsilon\big(x,\rho^\varepsilon\big) = 0 \quad \text{which is automatically compact in } H^{-1}_{\mathrm{loc}}\big(\mathbb{R}^2_+\big). \tag{3.16}$$

Moreover, since $\hat\rho^\varepsilon(s,y,x)$ strongly converges a.e., then we find that, when $\varepsilon \to 0$,

$$\begin{aligned}
\eta^\varepsilon_1\big(\rho^\varepsilon,t,x,s,y\big) &:= \big|\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)\big| \\
&\overset{*}{\rightharpoonup} \big\langle \nu_{t,x}(k); \big|k - \hat\rho(s,y,x)\big|\big\rangle \\
&=: \big\langle \nu_{t,x}; \eta_1(k,t,x,s,y)\big\rangle, \\
q^\varepsilon_1\big(\rho^\varepsilon,t,x,s,y\big) &:= \operatorname{sign}\big(\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)\big)\big(F^\varepsilon\big(x,\rho^\varepsilon\big) - \gamma(s,y)\big) \\
&\overset{*}{\rightharpoonup} \big\langle \nu_{t,x}(k); \operatorname{sign}\big(k - \hat\rho(s,y,x)\big)\big(F(x,k) - \gamma(s,y)\big)\big\rangle \\
&=: \big\langle \nu_{t,x}; q_1(k,t,x,s,y)\big\rangle, \\
\eta^\varepsilon_2\big(\rho^\varepsilon(t,x)\big) &:= \rho^\varepsilon(t,x) \\
&\overset{*}{\rightharpoonup} \big\langle \nu_{t,x}(k); k\big\rangle = \rho(t,x) \\
&=: \big\langle \nu_{t,x}; \eta_2(k)\big\rangle, \\
q^\varepsilon_2\big(\rho^\varepsilon(t,x),x\big) &:= F^\varepsilon\big(x,\rho^\varepsilon\big) \\
&\overset{*}{\rightharpoonup} \big\langle \nu_{t,x}(k); F(x,k)\big\rangle \\
&:= \big\langle \nu_{t,x}; q_2(k,x)\big\rangle,
\end{aligned} \tag{3.17}$$

and

$$\begin{vmatrix} \eta_1(\rho^\varepsilon(t,x),s,y,x) & q_1(\rho^\varepsilon(t,x),s,y,x) \\ \eta_2(\rho^\varepsilon(t,x)) & q_2(\rho^\varepsilon(t,x),x) \end{vmatrix} \overset{*}{\rightharpoonup} \left\langle \nu_{t,x}; \begin{vmatrix} \eta_1(k,s,y,x) & q_1(k,s,y,x) \\ \eta_2(k) & q_2(k,x) \end{vmatrix} \right\rangle, \tag{3.18}$$

where

$$\big(\eta_1(k,t,x,s,y), q_1(k,t,x,s,y)\big) = \big(\big|k-\hat{\rho}(s,y,x)\big|, \operatorname{sign}\big(k-\hat{\rho}(s,y,x)\big)\big(F(x,k)-\gamma(s,y)\big)\big),$$
$$\big(\eta_2(k), q_2(k,x)\big) = \big(k, F(x,k)\big).$$

Together (3.15)–(3.16) with (3.17)–(3.18), we apply the Div–Curl lemma (see Tartar [23] and Murat [19]) to obtain

$$\left\langle \nu_{t,x}; \begin{vmatrix} \eta_1(k,s,y,x) & q_1(k,s,y,x) \\ \eta_2(k) & q_2(k,x) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}; \eta_1(k,s,y,x)\rangle & \langle \nu_{t,x}; q_1(k,s,y,x)\rangle \\ \langle \nu_{t,x}; \eta_2(k)\rangle & \langle \nu_{t,x}; q_2(k,x)\rangle \end{vmatrix}$$

for all $(s,y),(t,x) \in \mathbb{R} \setminus \mathcal{M}$ with $\mathcal{M}$ a set of measure zero in $\mathbb{R}^2_+$. Thus, we have

$$\begin{aligned}&\big\langle \nu_{t,x}; \big|k-\hat{\rho}(s,y,x)\big|F(x,k) - k\operatorname{sign}\big(k-\hat{\rho}(s,y,x)\big)\big(F(x,k)-\gamma(s,y)\big)\big\rangle \\ &\quad = \big\langle \nu_{t,x}; \big|k-\hat{\rho}(s,y,x)\big|\big\rangle\big\langle \nu_{t,x}; F(x,k)\big\rangle - \langle \nu_{t,x}, k\rangle\big\langle \nu_{t,x}; \operatorname{sign}\big(k-\hat{\rho}(s,y,x)\big)\big(F(x,k)-\gamma(s,y)\big)\big\rangle.\end{aligned}$$

Equivalently, we have

$$\begin{aligned}&\big\langle \nu_{t,x}; \big|k-\hat{\rho}(s,y,x)\big|\big(F(x,k)-\big\langle \nu_{t,x}; F(x,k)\big\rangle\big)\big\rangle \\ &\quad - \big\langle \nu_{t,x}; \big(k-\rho(t,x)\big)\operatorname{sign}\big(k-\hat{\rho}(s,y,x)\big)\big(F(x,k)-F\big(y,\rho(s,y)\big)\big)\big\rangle = 0.\end{aligned}$$

Since this is true for all $(s,y)$ and $(t,x)$ except on a set $\mathcal{M}$ of measure zero, we then choose $(s,y) = (t,x)$ for $(t,x) \in \mathbb{R} \setminus \mathcal{M}$ to obtain

$$\begin{aligned}&\big\langle \nu_{t,x}; \big|k-\rho(t,x)\big|\big(F(x,k)-\big\langle \nu_{t,x}; F(x,k)\big\rangle\big)\big\rangle \\ &\quad - \big\langle \nu_{t,x}; \big(k-\rho(t,x)\big)\operatorname{sign}\big(k-\rho(t,x)\big)\big(F(x,k)-F\big(x,\rho(t,x)\big)\big)\big\rangle = 0,\end{aligned}$$

that is,

$$\big\langle \nu_{t,x}; \big|k-\rho(t,x)\big|\big\rangle\big(F\big(x,\rho(t,x)\big)-\big\langle \nu_{t,x}; F(x,k)\big\rangle\big) = 0. \tag{3.19}$$

There are two possibilities:
When $\langle \nu_{t,x}; |k-\rho(t,x)|\rangle = 0$, then we have $\nu_{t,x}(k) = \delta_{\rho(t,x)}(k)$.
When $\langle \nu_{t,x}; F(x,k)\rangle - F(x,\rho(t,x)) = 0$, we note that

$$\begin{aligned}&\big\langle \nu_{t,x}; F(x,k)\big\rangle - F\big(x,\rho(t,x)\big) = \big\langle \nu_{t,x}; F(x,k)-F\big(x,\rho(t,x)\big)\big\rangle \\ &= \left\langle \nu_{t,x}; F_\rho\big(x,\rho(t,x)\big)\big(k-\rho(t,x)\big) + \frac{1}{2}\int_0^1 \theta F_{\rho\rho}\big(x,\theta\rho(t,x)+(1-\theta)k\big)\,d\theta\big(k-\rho(t,x)\big)^2 \right\rangle \\ &= F_\rho\big(x,\rho(t,x)\big)\big\langle \nu_{t,x}; k-\rho(t,x)\big\rangle + \frac{1}{2}\left\langle \nu_{t,x}; \int_0^1 \theta F_{\rho\rho}\big(x,\theta\rho(t,x)+(1-\theta)k\big)\,d\theta\big(k-\rho(t,x)\big)^2 \right\rangle \\ &= \frac{1}{2}\left\langle \nu_{t,x}; \int_0^1 \theta F_{\rho\rho}\big(x,\theta\rho(t,x)+(1-\theta)k\big)\,d\theta\big(k-\rho(t,x)\big)^2 \right\rangle.\end{aligned}$$

Since $F(x,\rho)$ is strictly convex or concave in $\rho$, we conclude

$$\nu_{t,x}(k) = \delta_{\rho(t,x)}(k) \quad \text{for } (t,x) \text{ a.e.} \tag{3.20}$$

Therefore, we have

$$\rho^{\varepsilon}(t,x) \to \rho(t,x) \quad \text{a.e. when } \varepsilon \to 0.$$

Since the limit is unique via the uniqueness result in [2], the whole sequence $\rho^{\varepsilon}(t,x)$ strongly converges to $\rho(t,x)$ a.e. It is easy to check that $\rho(t,x)$ is the unique entropy solution of the Cauchy problem (1.1)–(1.2) in the sense of Definition 2.1. $\square$

**Remark 3.2.** In [5], the existence of entropy solutions (1.1)–(1.2) in the sense of Definition 2.1 is proven for the case $\lambda(x)u^2$. They used the vanishing viscosity method combined with a mollification for $\lambda(x)$.

**Remark 3.3.** The conditions on the flux function $F(x,\rho)$ in Theorem 3.1 for the non-monotone case can be relaxed as follows: $F(x,\rho)$ satisfies (H1)–(H3) and is convex or concave with

$$\mathcal{L}^1\big\{\rho\colon\ F_{\rho\rho}(x,\rho)=0\big\}=0 \quad \text{for a.e. } x\in\mathbb{R},$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

## 4. Hydrodynamic limit of a zero range process with discontinuous speed-parameter

In Section 2, we have established a compactness framework for approximate solutions via the reduction of measure-valued entropy solutions of (1.1)–(1.2) in the sense of Definition 2.1. In this section we focus on a microscopic particle system for a zero range process (ZRP) with discontinuous speed-parameter $\lambda(x)$. We apply the compactness framework to show the hydrodynamic limit for the particle system, when the distance between particles tends to zero, to the unique entropy solution of the Cauchy problem

$$\partial_t\rho + \partial_x\big(\lambda(x)h(\rho)\big)=0 \tag{4.1}$$

and with initial data

$$\rho|_{t=0}=\rho_0(x)\geqslant 0, \tag{4.2}$$

where $h(\rho)$ is a monotone function of $\rho$, and $\lambda(x)$ is continuous in $x\in\mathbb{R}$, except on a closed set $\mathcal{N}$ of measure zero, with $0<\lambda_1\leqslant\lambda(x)\leqslant\lambda_2<\infty$ for some constants $\lambda_1$ and $\lambda_2$. Then $m_\alpha^+ = m_\alpha^- := m_\alpha$ for $\alpha\in[0,\infty)$.

Rezakhanlou in [22] first established the hydrodynamic limit of the processus des misanthropes (PdM) with constant speed-parameter. Covert and Rezakhanlou [12] provided a proof of the hydrodynamic limit of a PdM with nonconstant but continuous speed-parameter $\lambda$. Bahadoran [3] proved this for a simple exclusion process. In all these papers, the most important step is to show an entropy inequality at microscopic level, which then implies the (macroscopic) Kruzkov entropy inequality, when the distance between particles tends to zero, and thereby implies the uniqueness of limit points. In this section, we generalize this to the case when the speed-parameter $\lambda(x)$ has jumps for the attractive Zero Range Process (ZRP). In Section 4.1, we analyze some properties of the ZRP. In Section 4.2, we prove the one-dimensional microscopic entropy inequality letting $\varepsilon=\varepsilon(N)=N^{-\sigma}$, $\sigma\in(0,1)$, for a ZRP with discontinuous speed-parameter when $N\to\infty$, where $\varepsilon$ is as in Section 3 and $N$ is the inverse of the distance between particles. In Section 4.3, we show the existence of measure-valued solutions via the microscopic entropy inequality and how inequality (2.3) follows.

*4.1. Some properties of the microscopic interacting particle system*

We consider a system of particles with conserved total mass and evolving on a one-dimensional lattice $\mathbb{Z}$ according to a Markovian law. With the Euler scaling factor $N$, the microscopic particle density is expected to converge to a deterministic limit when $N \to \infty$, which is characterized by a solution of a conservation law. Under the Euler scaling, $\frac{1}{N}$ represents the distance between sites. Obviously we have two space scales: The discrete lattice $\mathbb{Z}$ as embedded in $\mathbb{R}$ with "vertices" $\frac{u}{N}$ and $u \in \mathbb{Z}$. In this way, the distances between particles tend to zero if $N$ increases to infinity. Sites of the microscopic scale $\mathbb{Z}$ are denoted by the letters $u, v$ and correspond to the points $\frac{u}{N}$, $\frac{v}{N}$ in the macroscopic scale $\mathbb{R}$. Points of the macroscopic space scale $\mathbb{R}$ are denoted by the letters $x$, $y$ and correspond to the sites $[xN]$, $[yN]$ in the microscopic space scale, where $[z]$ is the integer part of $z$. We denote by $\eta_t(u)$ the number of particles at time $t > 0$ at site $u$. Then the vector $\eta_t = (\eta_t(u)\colon\ u \in \mathbb{Z})$ is called a configuration at time $t$ with configuration space $\mathbb{N}^{\mathbb{Z}}$.

In general, the ZRP can be described as follows: Infinitely many indistinguishable particles are distributed on a 1-dimensional lattice. Any site of the lattice may be occupied by a finite number of particles. Associated to a given site $u$ there is an exponential clock with rate $\lambda_\varepsilon(\frac{u}{N})g(\eta(u))$ depending on the macroscopic spatial coordinates. Each time the clock rings on the site $u$, one of the particles jumps to the site $v$ chosen with probability $p(u, v)$. The elementary transition probabilities $p\colon \mathbb{Z} \mapsto [0, 1]$ are supposed to be

(i) translation invariant: $p(x, y) = p(0, y - x) =: p(y - x)$;
(ii) normalized: $\sum_y p(x, y) = 1$, $p(x, x) = 0$;
(iii) assumed to be of finite range: $p(x, y) = 0$ for $|y - x|$ sufficiently large;
(iv) irreducible: $p(0, 1) > 0$.

Without loss of generality, we assume that $\sum_z p(z)z = \gamma = 1$; otherwise, for $\gamma \neq 1$, we replace the function $h(\rho)$ by $h(\rho)/\gamma$ in the following argument. The rate $g : \mathbb{N} \to \mathbb{R}_+$ is a positive, nondecreasing function with $g(0) = 0$, $g(\infty) = \infty$, and

$$\frac{g(k)}{k^2} \to 0 \quad \text{when } k \to \infty. \tag{4.3}$$

With this description, the Markov process $\eta_t$ is generated by

$$NL_\varepsilon^N f(\eta) = N \sum_{u,v} \lambda_\varepsilon\left(\frac{u}{N}\right) g\bigl(\eta(u)\bigr) p(v - u)\bigl(f\bigl(\eta^{u,v}\bigr) - f(\eta)\bigr). \tag{4.4}$$

Here $N$ comes from the Euler scaling factor speeding the generator, thus $\eta_t$ denotes a configuration on which this speeded generator $NL_\varepsilon^N$ has acted for time $Nt$, and $\eta^{u,v}$ represents the configuration $\eta$ where one particle jumped from $u$ to $v$:

$$\eta^{u,v}(w) = \begin{cases} \eta(w) & \text{if } w \neq u, v, \\ \eta(u) - 1 & \text{if } w = u, \\ \eta(v) + 1 & \text{if } w = v. \end{cases}$$

For any $\varepsilon = \varepsilon(N) > 0$ and for any constant $\alpha \geqslant 0$, we define a product measure given by

$$\tilde{\nu}_\alpha^N(\eta) := \prod_u \frac{1}{Z(\alpha/\lambda_\varepsilon(\frac{u}{N}))} \frac{\alpha^{\eta(u)}}{(\lambda_\varepsilon(\frac{u}{N}))^{\eta(u)} g(\eta(u))!} := \prod_u \tilde{\nu}_\alpha^N\bigl(\eta(u)\bigr), \tag{4.5}$$

where $Z$ is a partition function equal to

$$Z\left(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(\lambda_\varepsilon(\frac{u}{N}))^n g(n)!}. \tag{4.6}$$

Then the expected value of the occupation variable $\eta(u)$ is equal to

$$E_{\tilde{\nu}_\alpha^N}\big[\eta(u)\big]=\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\frac{Z'(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})})}{Z(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})})}:=R\bigg(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\bigg).$$

Now let $h$ be the inverse function of $R$ to obtain

$$h\bigg(R\bigg(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\bigg)\bigg)=\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\quad\Rightarrow\quad\lambda_\varepsilon\bigg(\frac{u}{N}\bigg)h\big(E_{\tilde{\nu}_\alpha^N}\big[\eta(u)\big]\big)=\alpha\quad\Leftrightarrow\quad E_{\tilde{\nu}_\alpha^N}\big[\eta(u)\big]=m_\alpha\bigg(\frac{u}{N}\bigg),$$

where $m_\alpha$ is a steady-state solution to

$$\partial_t\rho+\partial_x\big(\lambda_\varepsilon(x)h(\rho)\big)=0. \tag{4.7}$$

Furthermore, it follows that

$$E_{\tilde{\nu}_\alpha^N}\big[g\big(\eta(u)\big)\big]=h\bigg(m_\alpha\bigg(\frac{u}{N}\bigg)\bigg).$$

From now on, we set

$$\mu_{m_\alpha}^N(\eta)=\prod_u\nu_{m_\alpha(\frac{u}{N})}\big(\eta(u)\big):=\prod_u\tilde{\nu}^N_{\lambda_\varepsilon(\frac{u}{N})h(m_\alpha(\frac{u}{n}))}\big(\eta(u)\big). \tag{4.8}$$

The important attribute of the ZRP with nonconstant speed-parameter is that the *product* measure $\mu_{m_\alpha}^N(\eta)$ is invariant under the generator $NL_\varepsilon^N$, i.e.,

$$\int L_\varepsilon^N\big(f(\eta)\big)\,d\mu_{m_\alpha}^N(\eta)=0. \tag{4.9}$$

As a reasonable initial distribution, we choose *the product measure $\mu_0^N(\eta)$ associated to a bounded density profile* defined as follows: For a bounded density profile $\rho_0\geqslant 0$, the probability that particles at time $t=0$ are distributed with configuration $\eta$ is equal to

$$\mu_0^N(\eta):=\prod_u\frac{1}{Z(h(\rho_{u,N})/\lambda_\varepsilon(\frac{u}{N}))}\frac{(h(\rho_{u,N}))^{\eta(u)}}{(\lambda_\varepsilon(\frac{u}{N}))^{\eta(u)}g(\eta(u))!}, \tag{4.10}$$

where $\rho_{u,N}\geqslant 0$ is a sequence satisfying $\lim_{N\to\infty}\int|\rho_{[Nx],N}-\rho_0(x)|\,dx=0$ for $[Nx]$ as the integer part of $Nx$. With this definition, we say that a sequence of probability measures $\mu^N$ is associated to a density profile $\rho\geqslant 0$ if

$$\lim_{N\to\infty}\bigg\langle\mu^N(\eta);\bigg|\frac{1}{N}\sum_u J\bigg(\frac{u}{N}\bigg)\eta(u)-\int J(x)\rho(x)\,dx\bigg|\bigg\rangle=0\quad\text{for every test function } J.$$

Furthermore, let $\mu_t^N$ denote the distribution of a configuration at time $t$ initially distributed by $\mu_0^N$:

$$\mu_t^N=S_t^N*\mu_0^N, \tag{4.11}$$

where $S_t^N=e^{tNL_\varepsilon^N}$ is the semigroup corresponding to the generator $NL_\varepsilon^N$. Then the attractiveness for two initial measures $\mu_{\rho_0}^N$ and $\mu_{\omega_0}^N$ with profiles $\rho_t$ and $\omega_t$, respectively, implies that

$$\mu_{\rho_0}^N\leqslant\mu_{\omega_0}^N\quad\Rightarrow\quad\mu_{\rho_t}^N\leqslant\mu_{\omega_t}^N$$

is satisfied by the assumption that $g$ is a nondecreasing function. Moreover, it is easy to prove that $\mu_{\rho_0} \leqslant \mu_{\omega_0}$ if $\rho_0 \leqslant \omega_0$. It then follows by attractiveness and (4.9) that, for any constant $\alpha$ such that $m_\alpha(x) \geqslant \rho_0(x)$, we obtain that the inequality $\mu_0^N \leqslant \mu_{m_\alpha}^N$ implies

$$S_t^N \mu_0^N \leqslant S_t^N \mu_{m_\alpha}^N = \mu_{m_\alpha}^N. \tag{4.12}$$

Since our initial distribution has a bounded density profile, then the density profile remains bounded at later time $t$.

The goal in proving the hydrodynamic limit of a ZRP is that, if we start from a configuration $\eta_0$ distributed with an initial measure $\mu_0^N$ associated to the bounded density profile $\rho_0$, then the distribution $\mu_t^N$ of the configuration $\eta_t$ at later time $t$ is associated to the density profile $\rho(t,\cdot)$, where $\rho$ is the solution of the Cauchy problem (4.1)–(4.2) in the sense of Definition 2.1. In other words, our main theorem in this section is the following.

**Theorem 4.1** *(Hydrodynamic limit of an attractive ZRP with discontinuous speed-parameter). Let $\eta_t$ be an attractive ZRP with* (4.3) *initially distributed by the measure $\mu_0^N$ associated to a bounded density profile $\rho_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ as defined in* (4.10)*. Let $\varepsilon = \varepsilon(N) = N^{-\sigma}$, $\sigma \in (0,1)$. Then, at later time t,*

$$\lim_{N\to\infty} \left\langle \mu_t^N(\eta); \left| \frac{1}{N} \sum_u J\left(\frac{u}{N}\right) \eta_t(u) - \int J(x)\rho(t,x)\,dx \right| \right\rangle = 0 \tag{4.13}$$

*for any test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}$, where $\rho(t,x)$ is the unique solution of the Cauchy problem* (4.1)–(4.2) *in the sense of Definition* 2.1.

To achieve this, we have to establish an entropy inequality in the sense of Definition 2.1 at microscopic level. This will be done in Section 4.2 by using the scaling relation $\varepsilon = \varepsilon(N) = N^{-\sigma}$, $\sigma \in (0,1)$. Associated to each configuration $\eta_t$, we may define the empirical measure viewed as a random measure on $\mathbb{R}$ by

$$\chi_t^N(x) := \frac{1}{N} \sum_u \eta_t(u) \delta_{\frac{u}{N}}(x). \tag{4.14}$$

Then $\langle \chi_t^N(\cdot), J(\cdot)\rangle = \frac{1}{N}\sum_u J(\frac{u}{N})\eta_t(u)$, and we can rewrite (4.13) by

$$\lim_{N\to\infty} \left\langle \mu_t^N(\eta); \left| \left\langle \chi_t^N(\cdot), J(\cdot)\right\rangle - \int J(x)\rho(t,x)\,dx \right| \right\rangle = 0. \tag{4.15}$$

### *4.2. The entropy inequality at microscopic level*

The following proposition is essential towards the hydrodynamic limit.

**Proposition 4.1** *(Entropy inequality at microscopic level for $\varepsilon = N^{-\sigma}$ with $\sigma \in (0,1)$ when $N \to \infty$). Let $m_\alpha^\varepsilon$ be the steady-state solutions of* (3.2) *as defined in* (1.3) *with $F^\varepsilon(x,\rho) = \lambda_\varepsilon(x)h(\rho)$. Let $\eta_t$ be the ZRP generated by $NL_\varepsilon^N$ defined by* (4.4) *and initially distributed by the measure $\mu_0^N$ defined by* (4.10)*. Let $\eta^l(u)$ be the average density of particles in large microscopic boxes of size $2l+1$ and centered at $u$:*

$$\eta^l(u) := \frac{1}{2l+1} \sum_{|u-v|\leqslant l} \eta(v).$$

*Then, for every test function $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$,*

$$\lim_{l\to\infty}\lim_{N\to\infty}\mu_t^N\left\{\int_0^t \frac{1}{N}\sum_u\left(\partial_s J\left(s,\frac{u}{N}\right)\left|\eta_s^l(u)-m_\alpha^\varepsilon\left(\frac{u}{N}\right)\right|+\partial_x J\left(s,\frac{u}{N}\right)\left|\lambda_\varepsilon\left(\frac{u}{N}\right)h\left(\eta_s^l(u)\right)-\alpha\right|\right)ds\right.$$
$$\left.+\frac{1}{N}\sum_u J\left(0,\frac{u}{N}\right)\left|\eta_0^l(u)-m_\alpha^\varepsilon\left(\frac{u}{N}\right)\right|\geqslant-\delta\right\}=1. \tag{4.16}$$

Inequality (4.16) is the entropy inequality (2.3) with $\rho$ replaced by the average density of particles in the microscopic boxes of length $2l+1$. To prove the microscopic entropy inequality, we consider the coupled process $(\eta_t,\xi_t)$ generated by $N\bar{L}_\varepsilon^N$, where $\bar{L}_\varepsilon^N$ is defined by

$$\bar{L}_\varepsilon^N f(\eta,\xi)=\sum_{u,v}p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\min\{g(\eta(u)),g(\xi(u))\}\left(f\left(\eta^{u,v},\xi^{u,v}\right)-f(\eta,\xi)\right)$$
$$+\sum_{u,v}p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\{g(\eta(u))-g(\xi(u))\}_+\left(f\left(\eta^{u,v},\xi\right)-f(\eta,\xi)\right)$$
$$+\sum_{u,v}p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\{g(\xi(u))-g(\eta(u))\}_+\left(f\left(\eta,\xi^{u,v}\right)-f(\eta,\xi)\right). \tag{4.17}$$

Furthermore, denote the initial distribution of $(\eta_t,\xi_t)$ by $\bar{\mu}_0^N=\mu_0^N\times\mu_{m_\alpha^\varepsilon}^N$, where $\mu_0^N$ is the initial measure with density profile $\rho_0$ defined by (4.10) and $\mu_{m_\alpha^\varepsilon}^N$ denotes the invariant measure as defined in (4.8).

Then, to prove Proposition 4.1, it suffices to prove the following proposition.

**Proposition 4.2.** *Let $(\eta_t,\xi_t)$ be the coupled process, starting from $\bar{\mu}_0^N$, generated by $N\bar{L}_\varepsilon^N$ as defined by* (4.17)*. Let $\bar{\mu}_t^N=\bar{S}_t^N*\bar{\mu}_0^N$, where $\bar{S}_t^N$ is the semigroup corresponding to the generator $N\bar{L}_\varepsilon^N$. Then, for every test function $J:\mathbb{R}_+^2\mapsto\mathbb{R}_+$ and every $\varepsilon=N^{-\sigma}$ with $\sigma\in(0,1)$,*

$$\lim_{l\to\infty}\lim_{N\to\infty}\bar{\mu}_t^N\left\{\int_0^T \frac{1}{N}\sum_u\left\{\partial_s J\left(s,\frac{u}{N}\right)\left|\eta_s^l(u)-\xi_s^l(u)\right|+\partial_x J\left(s,\frac{u}{N}\right)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|h\left(\eta_s^l(u)\right)-h\left(\xi_s^l(u)\right)\right|\right\}ds\right.$$
$$\left.+\frac{1}{N}\sum_u J\left(0,\frac{u}{N}\right)\left|\eta_0^l(u)-\xi_0^l(u)\right|\geqslant-\delta\right\}=1.$$

Recall that a microscopic entropy inequality leading to the Kruzkov entropy inequality has been proved in [12] for the process of PdM with nonconstant but continuous speed-parameter $\lambda_\varepsilon$. Since there does not exist an invariant product measure for a PdM in general such that $E_{\mu_{m_\alpha^\varepsilon}^N}[\xi(u)]=m_\alpha^\varepsilon(\frac{u}{N})$, to replace the process $\xi$ by $m_\alpha^\varepsilon(\frac{u}{N})$, one has to apply the relative entropy method of Yau [24].

In our case of a space-dependent ZRP, the invariant product measure is available so that we can approximate the steady-state solution $m_\alpha^\varepsilon$ by a process $\xi$ distributed by the invariant measure $\mu_{m_\alpha^\varepsilon}^N$ for any $\alpha\in(0,\infty)$. Then, Proposition 4.1 indeed directly follows from Proposition 4.2.

### 4.3. *Proof of Proposition 4.2*

We split the proof in three steps.

*Step 1: Lower bound for the martingale*. For a test function $J$ with compact support in $\mathbb{R}_+^2$, define by $M_t^J$ the martingale vanishing at time $t=0$:

$$M_t^J = \frac{1}{N}\sum_u J\left(t, \frac{u}{N}\right)\left|\eta_t(u) - \xi_t(u)\right| - \frac{1}{N}\sum_u J\left(0, \frac{u}{N}\right)\left|\eta_0(u) - \xi_0(u)\right|$$
$$- \int_0^t \left(\partial_s + N\bar{L}_\varepsilon^N\right)\left(\frac{1}{N}\sum_u J\left(s, \frac{u}{N}\right)\left|\eta_s(u) - \xi_s(u)\right|\right) ds.$$

Since $J$ has compact support, then, for $t$ large enough,

$$M_t^J = -\frac{1}{N}\sum_u J\left(0, \frac{u}{N}\right)\left|\eta_0(u) - \xi_0(u)\right| - \int_0^t \left(\partial_s + N\bar{L}_\varepsilon^N\right)\left(\frac{1}{N}\sum_u J\left(s, \frac{u}{N}\right)\left|\eta_s(u) - \xi_s(u)\right|\right) ds.$$

We now calculate

$$\begin{aligned}
\bar{L}_\varepsilon^N\left|\eta(u) - \xi(u)\right| &= \sum_{v,w} p(w-v)\lambda_\varepsilon\left(\frac{v}{N}\right)\Big\{\min\big\{g\big(\eta(v)\big), g\big(\xi(v)\big)\big\}\big(\left|\eta^{v,w}(u) - \xi^{v,w}(u)\right| - \left|\eta(u) - \xi(u)\right|\big) \\
&\quad + \big\{g\big(\eta(v)\big) - g\big(\xi(v)\big)\big\}_+\big(\left|\eta^{v,w}(u) - \xi(u)\right| - \left|\eta(u) - \xi(u)\right|\big) \\
&\quad + \big\{g\big(\xi(v)\big) - g\big(\eta(v)\big)\big\}_+\big(\left|\eta(u) - \xi^{v,w}(u)\right| - \left|\eta(u) - \xi(u)\right|\big)\Big\} \\
&= \sum_v \big(1 - G_{u,v}(\eta,\xi)\big)\bigg(-p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|g\big(\eta(u)\big) - g\big(\xi(u)\big)\right| \\
&\quad + p(u-v)\lambda_\varepsilon\left(\frac{v}{N}\right)\left|g\big(\eta(v)\big) - g\big(\xi(v)\big)\right|\bigg) \\
&\quad - \sum_v G_{u,v}(\eta,\xi)\bigg(p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|g\big(\eta(u)\big) - g\big(\xi(u)\big)\right| \\
&\quad + p(u-v)\lambda_\varepsilon\left(\frac{v}{N}\right)\left|g\big(\eta(v)\big) - g\big(\xi(v)\big)\right|\bigg),
\end{aligned} \tag{4.18}$$

where $G_{u,v}$ is the indicator function that equals to 1 if $\eta$ and $\xi$ are not ordered, i.e.,

$$G_{u,v}(\eta,\xi) = \mathbb{1}\big\{\eta(u) < \xi(u);\ \eta(v) > \xi(v)\big\} + \mathbb{1}\big\{\eta(u) > \xi(u);\ \eta(v) < \xi(v)\big\}.$$

Notice that the second sum is nonpositive. Therefore, plugging the last expression in the martingale $M_t^J$ and then interchanging $u$ and $v$ in the last term, we can bound the martingale below by

$$-\frac{1}{N}\sum_u J\left(0, \frac{u}{N}\right)\left|\eta_0(u) - \xi_0(u)\right| - \int_0^t \frac{1}{N}\sum_u \partial_s J\left(s, \frac{u}{N}\right)\left|\eta_s(u) - \xi_s(u)\right| ds$$
$$+ \int_0^t \sum_{u,v}\left(J\left(s, \frac{u}{N}\right) - J\left(s, \frac{v}{N}\right)\right)p(v-u)\big(1 - G_{u,v}(\eta_s,\xi_s)\big)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|g\big(\eta_s(u)\big) - g\big(\xi_s(u)\big)\right| ds.$$

Since the transition probability $p$ is of finite range, i.e. $p(z) = 0$ if $|z| > r$ for some $r$, then

$$\left(J\left(s, \frac{u}{N}\right) - J\left(s, \frac{v}{N}\right)\right)p(v-u) = -\frac{1}{N}(v-u)p(v-u)\partial_x J\left(s, \frac{u}{N}\right) + O\left(\frac{1}{N^2}\right).$$

With $v = u + y$, it then follows that the martingale is bounded below by

$$-\int_0^t \frac{1}{N}\sum_u \biggl\{\partial_s J\left(s, \frac{u}{N}\right)\left|\eta_s(u) - \xi_s(u)\right| + \partial_x J\left(s, \frac{u}{N}\right)\lambda_\varepsilon\left(\frac{u}{N}\right)\tau_u\left(\sum_y yp(y)(1 - G_{0,y})\right)\left|g\bigl(\eta_s(0)\bigr) - g\bigl(\xi_s(0)\bigr)\right|\biggr\}\,ds - \frac{1}{N}\sum_u J\left(0, \frac{u}{N}\right)\left|\eta_0(u) - \xi_0(u)\right| + O\left(\frac{1}{N}\right).$$

*Step 2.* We show

$$\lim_{N\to\infty} E_{\bar{\mu}_t^N}\left[\left(M_t^J\right)^2\right] = 0. \tag{4.19}$$

Recall that

$$N_t^J := \left(M_t^J\right)^2 - \int_0^t \left(N\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right)^2 - 2A^J(s,\eta,\xi)N\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right)\right)ds$$

is a martingale vanishing at time $t = 0$, where $A^J$ is defined by

$$A^J(t,\eta,\xi) = \frac{1}{N}\sum_u J\left(t, \frac{u}{N}\right)\left|\eta_t(u) - \xi_t(u)\right|.$$

Then, by definition, $E_{\bar{\mu}_s^N}[N_s^J] = 0$ for all $0 \leqslant s \leqslant t$. Thus, it suffices to show that the expectation of the integral term of $N_t^J$ converges to zero when $N \to \infty$. In order to prove this, we first find that, by a careful calculation,

$$\begin{aligned} &N\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right)^2 - 2NA^J(s,\eta,\xi)\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right) \\ &\quad = \sum_{v,w} p(w-v)N\lambda_\varepsilon\left(\frac{v}{N}\right)\biggl\{\left|g\bigl(\eta_s(v)\bigr) - g\bigl(\xi_s(v)\bigr)\right|\frac{1}{N^2}\bigl(1 - G_{v,w}(\eta_s,\xi_s)\bigr)\left(J\left(s,\frac{w}{N}\right) - J\left(s,\frac{v}{N}\right)\right)^2 \\ &\qquad + \left|g\bigl(\xi_s(v)\bigr) - g\bigl(\eta_s(v)\bigr)\right|\frac{1}{N^2}G_{v,w}(\eta_s,\xi_s)\left(J\left(s,\frac{v}{N}\right) + J\left(s,\frac{w}{N}\right)\right)^2\biggr\}. \end{aligned}$$

Since $J$ is a smooth function, the first term of this expression is less than $\mathcal{O}(\frac{g(CN)}{N^2})$ for some constant $C$ depending on the total initial mass and therefore converges to zero when $N \to \infty$ by (4.3). For the second term, we know that $(J(s, \frac{v}{N}) + J(s, \frac{w}{N}))^2 \leqslant 4\|J\|_\infty^2$, which implies

$$\begin{aligned} &N\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right)^2 - 2NA^J(s,\eta,\xi)\bar{L}_\varepsilon^N\left(A^J(s,\eta,\xi)\right) \\ &\quad = \mathcal{O}\left(\frac{g(CN)}{N^2}\right) + \frac{4\|J\|_\infty^2}{N}\sum_{v,w} G_{v,w}(\eta_s,\xi_s)p(w-v)\lambda_\varepsilon\left(\frac{v}{N}\right)\left|g\bigl(\xi_s(v)\bigr) - g\bigl(\eta_s(v)\bigr)\right|. \end{aligned}$$

Then, to conclude the proof of (4.19), it suffices to show

$$E_{\bar{\mu}_t^N}\left[\int_0^t \left(\sum_{v,w} G_{v,w}(\eta_s,\xi_s)p(w-v)\lambda_\varepsilon\left(\frac{v}{N}\right)\left|g\big(\xi_s(v)\big)-g\big(\eta_s(v)\big)\right|\right)ds\right]=\mathcal{O}(1). \tag{4.20}$$

For this, we use the martingale $M_t^J$ vanishing at 0 with $J\equiv 1$, that is,

$$M_t := \frac{1}{N}\sum_u \left|\eta_t(u)-\xi_t(u)\right| - \frac{1}{N}\sum_u \left|\eta_0(u)-\xi_0(u)\right| - \int_0^t \frac{1}{N}\sum_u N\bar{L}_\varepsilon^N \left|\eta_s(u)-\xi_s(u)\right| ds.$$

By (4.18), the integral term of the martingale is equal to

$$\int_0^t \frac{2}{N}\sum_{u,v} NG_{u,v}(\eta_s,\xi_s)p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|g\big(\eta_s(u)\big)-g\big(\xi_s(u)\big)\right| ds,$$

by interchanging $u$ and $v$ in some terms. Then we find

$$\begin{aligned}
&E_{\bar{\mu}_t^N}\left[\int_0^t 2\sum_{u,v} G_{u,v}(\eta_s,\xi_s)p(v-u)\lambda_\varepsilon\left(\frac{u}{N}\right)\left|g\big(\eta_s(u)\big)-g\big(\xi_s(u)\big)\right| ds\right]\\
&\quad = E_{\bar{\mu}_t^N}\left[\int_0^t \frac{1}{N}\sum_u \left|\eta_0(u)-\xi_0(u)\right| ds\right] - E_{\bar{\mu}_t^N}\left[\int_0^t \frac{1}{N}\sum_u \left|\eta_t(u)-\xi_t(u)\right| ds\right]\\
&\quad \leqslant E_{\bar{\mu}_t^N}\left[\int_0^t \frac{1}{N}\sum_u \left|\eta_0(u)-\xi_0(u)\right| ds\right].
\end{aligned}$$

Since we assumed that both marginals of $\bar{\mu}_t^N$ are bounded, (4.20) follows, which leads to (4.19).

With the result of Step 1 and (4.19) and using the Chebichev inequality, we obtain

$$\begin{aligned}
&\bar{\mu}_t^N\left\{\frac{1}{N}\sum_u J\left(0,\frac{u}{N}\right)\left|\eta_0(u)-\xi_0(u)\right| + \int_0^t \frac{1}{N}\sum_u\left\{\partial_s J\left(s,\frac{u}{N}\right)\left|\eta_s(u)-\xi_s(u)\right|\right.\right.\\
&\qquad\left.\left. + \partial_x J\left(s,\frac{u}{N}\right)\lambda_\varepsilon\left(\frac{u}{N}\right)\tau_u\left(\sum_y yp(y)(1-G_{0,y})(\eta,\xi)\right)\left|g\big(\eta_s(0)\big)-g\big(\xi_s(0)\big)\right|\right\} ds + O\left(\frac{1}{N}\right) < -\delta\right\}\\
&\quad \leqslant \bar{\mu}_t^N\left\{M_t^J > \delta\right\} \leqslant \bar{\mu}_t^N\left\{\left|M_t^J\right| > \delta\right\} \leqslant \frac{1}{\delta^2}E_{\bar{\mu}_t^N}\left[\left(M_t^J\right)^2\right],
\end{aligned} \tag{4.21}$$

which converges to 0 when $N\to\infty$, for all $\delta>0$.

*Step 3.* We next use the following summation by parts formula: *For any bounded function $a$ of $\eta(\cdot)$ with $a(0)=0$ and for any smooth test function $J:\mathbb{R}\mapsto\mathbb{R}$, we obtain that, for any $L>0$,*

$$\frac{1}{N}\sum_{|u|\leqslant LN} J\left(\frac{u}{N}\right)a\big(\eta(u)\big) = \frac{1}{N}\frac{1}{(2l+1)}\sum_{|u|\leqslant LN} J\left(\frac{u}{N}\right)\sum_{|u-v|\leqslant l} a\big(\eta(v)\big) + \mathcal{O}\left(\frac{l\,\|J\|_{\mathrm{Lip}}}{N}\right). \tag{4.22}$$

Since we restrict $\varepsilon = N^{-\sigma}, \sigma \in (0,1)$, then $\|\lambda_\varepsilon\|_{\mathrm{Lip}} \leqslant C/\varepsilon = CN^\sigma$ and $\mathcal{O}(\frac{l\|\lambda_\varepsilon\|_{\mathrm{Lip}}}{N}) = \mathcal{O}(\frac{l}{N^{1-\sigma}}) \to 0$ when $N \to \infty$ so that we can use this summation by parts formula (4.22) to replace inequality (4.21) by

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{N\to\infty} \bar{\mu}_t^N \Bigg\{ \frac{1}{N}\sum_u J\left(0, \frac{u}{N}\right)\frac{1}{2l+1}\sum_{|z-u|\leqslant l}\big|\eta_0(z) - \xi_0(z)\big| \\
&\qquad + \int_0^t \frac{1}{N}\sum_u \partial_s J\left(s, \frac{u}{N}\right)\frac{1}{2l+1}\sum_{|z-u|\leqslant l}\big|\eta_s(z)-\xi_s(z)\big|\,ds \\
&\qquad + \int_0^t \frac{1}{N}\sum_u \partial_x J\left(s, \frac{u}{N}\right)\lambda_\varepsilon\left(\frac{u}{N}\right)\frac{1}{2l+1} \\
&\qquad\quad \times \sum_{|z-u|\leqslant l}\tau_z\left(\sum_y yp(y)(1-G_{0,y})(\eta_s,\xi_s)\right)\big|g\big(\eta_s(0)\big) - g\big(\xi_s(0)\big)\big|\,ds < -\delta \Bigg\} \\
&= 0.
\end{aligned} \tag{4.23}$$

Notice that, in (4.23), since $J$ is a positive function, by the triangle inequality, we can remove the sum inside the absolute value in the first line. Following the same argument as in [12,22] (also [11]), since we first set $\varepsilon = \frac{1}{N^\sigma}$, independent of $\lambda_\varepsilon(x)$, we can obtain the following *one block estimates*:

$$\lim_{l\to\infty}\lim_{N\to\infty} E_{\bar{\mu}_t^N}\left\{\int_0^t \frac{1}{N}\sum_u \left|\frac{1}{2l+1}\sum_{|u-z|\leqslant l}\big|\eta_s(z)-\xi_s(z)\big| - \big|\eta_s^l(u) - \xi_s^l(u)\big|\right| ds\right\} = 0 \tag{4.24}$$

and

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{N\to\infty} E_{\bar{\mu}_t^N}\Bigg\{\int_0^t \frac{1}{N}\sum_u \tau_u\bigg|\frac{1}{2l+1}\sum_{|z|\leqslant l}\tau_z\left(\sum_y yp(y)(1-G_{0,y})(\eta_s,\xi_s)\right)\big|g\big(\eta_s(0)\big) - g\big(\xi_s(0)\big)\big| \\
&\qquad\qquad - \big|h\big(\eta_s^l(0)\big) - h\big(\xi_s^l(0)\big)\big|\bigg|\,ds\Bigg\} = 0.
\end{aligned} \tag{4.25}$$

Combining (4.23) with (4.24)–(4.25), we complete the proof of Proposition 4.2.

### *4.4. Existence of measure-valued entropy solutions*

In this section, we prove that Theorem 4.1 implies the existence of a measure-valued entropy solution associated to the configuration $\eta_t$. We recall the empirical measure $\chi_t^N(x)$ associated to a configuration $\eta_t$ in (4.14). We define the Young measures associated to $\eta_t$ as follows:

$$\pi_t^{N,l}(x,k) := \frac{1}{N}\sum_u \delta_{\frac{u}{N}}(x)\delta_{\eta_t^l(u)}(k), \tag{4.26}$$

which implies $\langle \pi_t^{N,l}; J\rangle = \frac{1}{N}\sum_u J(\frac{u}{N}, \eta_t^l(u))$ for any $J \in C_0(\mathbb{R}\times\mathbb{R}_+)$. If $E$ is the configuration space, then these two measures are finite positive measures on $E$ and, for any $J \in C_0(\mathbb{R})$, they are related by the formula

$$\big\langle \pi_t^{N,l}; kJ(x)\big\rangle \approx \big\langle \chi_t^N(\cdot); J(\cdot)\big\rangle. \tag{4.27}$$

Notice that, since there are jumps, the probability measure $\mu_t^N$ defined by (4.11) must be defined on the Skorohod space $D[(0,\infty), E]$, which is the space of right continuous functions with left limits taking values in $E$. Then, using the one-to-one correspondence between the configuration $\eta_t$ and the empirical measure $\chi_t^N(\cdot)$, the law of $\chi^N$ with respect to $\mu_t^N$ will give us a probability measure $Q^N$ on the Skorohod space $D[(0,\infty), \mathcal{M}_+(\mathbb{R})]$, for the space $\mathcal{M}_+(\mathbb{R})$ of finite positive measures on $\mathbb{R}$ endowed with the weak topology.

In the same way, we can associate a probability measure $\tilde{Q}^{N,l}$ on the space $D[(0,\infty), \mathcal{M}_+(\mathbb{R}_+^2)]$. With these definitions, we can state the main theorem of this section as follows.

**Theorem 4.2** *(Law of large numbers for the Young measures). Let $(\mu^N)_{N\geqslant 1}$ be a sequence of probability measures, as defined by* (4.10)*, associated to a bounded density profile $\rho_0 : \mathbb{R} \mapsto \mathbb{R}_+$. Then the sequence $\tilde{Q}^{N,l}$ converges, when $N \to \infty$ first and $l \to \infty$ second, to the probability measure $\tilde{Q}$ concentrated on the measure-valued entropy solution $\pi_{t,x}$ in the sense of Definition* 2.2.

**Proof.** In order to be allowed to take the limit of $Q^N$ and $\tilde{Q}^{N,l}$, we must know that the sequences are tight (weakly relatively compact). If $\tilde{Q}^{N,l}$ is weakly relatively compact, we can take $\tilde{Q}^l$ as a limit point if $N \to \infty$ for each $l$. Denote by $\tilde{Q}$ a limit point of $\tilde{Q}^{N,l}$ if $N \to \infty$ first and $l \to \infty$ second. Therefore, the proof of Theorem 4.2 consists in two main steps: The first is to show that $\tilde{Q}^{N,l}$ is weakly relatively compact and the second is to show the uniqueness of limit points. The key point in the proof is that these can be achieved independent of the choice of mollification $\lambda_\varepsilon$ of the discontinuous speed-parameter $\lambda$ with our choice of the notion of measure-valued entropy solutions.

These can be achieved by following exactly the standard argument in [12,16,22] since it requires only the uniform boundedness of $\lambda_\varepsilon$ in the proof. That is, we can conclude the following: Let $\mu_t^N$ be a measure defined by (4.11). Then

(i) the sequence $Q^N$ defined above is tight in $D[(0,\infty), \mathcal{M}_+(\mathbb{R})]$ and all its limit points $Q$ are concentrated on weakly continuous paths $\chi(t,\cdot)$;

(ii) similarly, the sequence $\tilde{Q}^{N,l}$ is tight in $D[(0,\infty), \mathcal{M}_+(\mathbb{R}\times\mathbb{R}_+)]$ and all its limit points $\tilde{Q}$ are concentrated on weakly continuous paths $\pi(t,\cdot,\cdot)$;

(iii) for every $t \geqslant 0$, $\pi(t,x,k) := \pi_t(x,k)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, $\tilde{Q}$ a.s. That is, $\tilde{Q}$ a.s.

$$\pi_t(x,k) = \pi_{t,x}(k) \otimes dx; \tag{4.28}$$

(iv) for every $t \in [0,T]$, $\pi_{t,x}(k)$ is compactly supported, that is, there exists $k_0 > 0$ such that

$$\pi_{t,x}\big([0,k_0]^c\big) = 0 \qquad \forall (t,x) \in [0,T] \times \mathbb{R};$$

(v) $\pi_{t,x}$ is a measure-valued entropy solution in the sense of Definition 2.2 for any $\alpha \in [M_0,\infty)$, i.e.,

$$\partial_t \langle \pi_{t,x}; |k - m_\alpha(x)| \rangle + \partial_x \langle \pi_{t,x}; |h(k)\lambda(x) - \alpha| \rangle \leqslant 0 \tag{4.29}$$

in the sense of distributions on $\mathbb{R}_+^2$ for any $\alpha \in [M_0,\infty)$ or $\alpha \in (-\infty, M_0]$.

The last result follows from the entropy inequality at microscopic level in Theorem 4.1. Indeed, in terms of the Young measures, the expression (4.16) of Proposition 4.1:

$$\lim_{l\to\infty}\lim_{N\to\infty} \mu_t^N \left\{ \int_0^\infty \frac{1}{N} \sum_u \left\{ \partial_t H\left(t, \frac{u}{N}\right) \left| \eta_t^l(u) - m_\alpha\left(\frac{u}{N}\right) \right| \right.\right.$$
$$\left.\left. + \partial_x H\left(t, \frac{u}{N}\right) \left| \lambda_{\varepsilon(N)}\left(\frac{u}{N}\right) h\big(\eta_t^l(u)\big) - \alpha \right| \right\} dt \geqslant -\delta \right\} = 1$$

can be restated as

$$\lim_{l\to\infty}\lim_{N\to\infty}\tilde{Q}^{N,l}\left\{\int_0^T\Big(\big\langle\pi_t(x,k);\,\big|k-m_\alpha(x)\big|\partial_t H(t,x)\big\rangle + \big\langle\pi_t(x,k);\,\big|\lambda_{\varepsilon(N)}(x)h(k)-\alpha\big|\partial_x H(t,x)\big\rangle\Big)\,dt \geqslant -\delta\right\} = 1$$

for every smooth function $H:(0,T)\times\mathbb{R}\mapsto\mathbb{R}_+$ with compact support, any $\alpha\in[M_0,\infty)$ or $\alpha\in(-\infty,M_0]$, and any $\delta>0$. Since $\tilde{Q}$ is a weak limit point concentrated on absolutely continuous measures and since we already proved that $\pi_{t,x}$ is concentrated on a compact set (and therefore the integrand is a bounded function), we obtain from the last expression that

$$\tilde{Q}\left\{\int_0^T\int\Big(\big\langle\pi_{t,x};\,\big|k-m_\alpha(x)\big|\big\rangle\partial_t H(t,x)+\big\langle\pi_{t,x};\,\big|\lambda(x)h(k)-\alpha\big|\big\rangle\partial_x H(t,x)\Big)\,dx\,dt\geqslant-\delta\right\}=1.$$

Letting $\delta\to 0$, we have that $\tilde{Q}$ a.s. (4.29) holds on $(0,T)\times\mathbb{R}$ in the sense of distributions for every $\alpha\in[0,\infty)$. This proves the uniqueness of $\tilde{Q}$ and thereby concludes the proof of Proposition 4.2. □

Then Theorem 4.1 follows immediately from this result since the measure-valued entropy solution reduces to the Dirac mass concentrated on the unique entropy solution $\rho(t,x)$ of (4.1)–(4.2) as we noticed in Section 3.2.


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