Convergence with Physical Viscosity for Nonlinear Elasticity

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1 Introduction

We use the methods of compensated compactness and Young measures to show that solutions of the equations for 1-D elasticity with a physical state law and physical viscosity converge strongly to global weak solutions of the inviscid equations. More precisely, we consider the Cauchy problem for the following degenerate system of parabolic equations

$$u_t^\epsilon - \sigma(v^\epsilon)_x = \epsilon u_{xx}^\epsilon \tag{1}$$

$$v_t^\epsilon - u_x^\epsilon = 0 \tag{2}$$

with initial data $u_0, v_0 \in H^1$. The associated inviscid equations (3), (4) are assumed to be hyperbolic.

$$u_t - \sigma(v)_x = 0 \tag{3}$$

$$v_t - u_x = 0 \tag{4}$$

We show a subsequence of solutions of (1), (2) converges strongly to a global weak solution of (3), (4) (and, assuming uniqueness, the entire sequence converges) under the hypotheses of strict hyperbolicity ($\sigma' \geq c > 0$), loss of genuine nonlinearity at one point ($\sigma''(\lambda_0) = 0$, $\sigma''(\lambda) \neq 0$ for $\lambda \neq \lambda_0$), and where σ is thrice differentiable and has polynomial-like growth near infinity. We will consider later the exact growth hypotheses in more detail.

Previously, DiPerna [5] [6] [7] proved existence of global weak L^∞ solutions of (3), (4) by using artificial viscosity (5), (6), energy estimates and uniform L^∞

bounds. Using the methods of compensated compactness (developed by Murat and Tartar [12], [13], [18]) and Lax's generalized entropies and L^∞-Young measures, DiPerna showed that a subsequence of the viscosity approximations converges strongly to global weak solutions of the inviscid equations (3), (4).

$$u_t^\epsilon - \sigma(v^\epsilon)_x = \epsilon u_{xx}^\epsilon \tag{5}$$

$$v_t^\epsilon - u_x^\epsilon = \epsilon v_{xx}^\epsilon \tag{6}$$

The first equation (5) is balance of momentum, and the second equation represents conservation of mass with artificial viscosity; i.e., unlike the first equation, the diffusion term here is nonphysical. As introduced, the compensated compactness and Young measure method requires uniform L^∞ bounds on the approximating solutions (in addition to energy estimates). The only general method to obtain uniform L^∞ bounds for systems of parabolic equations uses a version of the maximum principle to obtain invariant regions and is given by Chueh, Conley and Smoller in [3] (however, see Dafermos [4] for an alternate method). In the above case, one must use artificial viscosity exactly as in (5), (6) to obtain L^∞ bounds. These bounds are not available for (3), (4).

Subsequently, the method has been generalized to make use of only uniform L^p-like bounds which are available from energy estimates. Shearer [15] proves global existence of weak solutions in L^p for (3), (4) in the case of strict hyperbolicity, genuine nonlinearity ($\sigma'' \neq 0$), the same growth constraints on σ used here, and using only the energy estimates (which are available in cases where L^∞ bounds are not). Lin [11] also considers a similar system and uses a combination of energy estimates and the method of invariant regions to obtain weak global solutions of (3), (4). However, his use of invariant regions forces him to use artificial viscosity exactly as in (5), (6).

We end this section with the precise statements of the hypotheses and the main theorem. In section 2 we derive energy estimates for the viscosity solutions; in sections 3 and 4 we discuss Young measures, compensated compactness and prove Tartar's equation holds for two classes of entropy-entropy flux pairs. In section 5. we use these entropies in Tartar's equation to show that the support of an arbitrary Young measure is a point.

To prove the main theorem. we adapt the methods of Serre [14] and Shearer [15] to the present case. Define $\Sigma(v) = \int_0^v \sigma(s)ds$ (we can assume $\sigma(0) = 0$). We use the following hypotheses.

$H1$ Strict Hyperbolicity: $\sigma'(v) \geq \sigma_0 > 0$ with σ_0 =constant

$H2$ Genuine Nonlinearity Except at a Point:

$\sigma''(\lambda_0) = 0$ and $\sigma''(\lambda) \neq 0$ for $\lambda \neq \lambda_0$

Growth Constraints:

$H3$ $\dfrac{\sigma''}{(\sigma')^{5/4}}, \dfrac{\sigma'''}{(\sigma')^{7/4}} \in L^2$; $\dfrac{\sigma''}{(\sigma')^{3/2}}, \dfrac{\sigma'''}{(\sigma')^2} \in L^\infty$

$H4$ $\dfrac{\sigma(v)}{\Sigma(v)} \to 0$ as $|v| \to \infty$ and there are constants c, q with $q > 1/2$ such that $(\sigma'(v))^q \leq c(1 + \Sigma(v))$

Hypotheses H3. H4 are somewhat messy but are required for technical reasons. Roughly, they require that higher derivatives of σ must grow more slowly than lower derivatives. For example, a flux function σ with exponential growth is not allowed. On the other hand, if σ behaves like a polynomial, then the extra derivatives in the numerators in hypotheses H3, H4 imply that the terms are appropriately bounded or decay to zero sufficiently fast for the hypotheses to hold. For example, it is easy to check that if σ', $|\sigma''|$, $|\sigma'''|$ grow like $|v|^\alpha$, $|v|^{\alpha-1}$, $|v|^{\alpha-2}$ then H3, H4 hold for any $\alpha \geq 0$. H3 and H4 also hold if $\sigma''' \in L^\infty$ and σ', $|\sigma''|$ grow like $|v|^\alpha$, $|v|^{\alpha-1}$ with $\alpha > 2/7$ or if σ'', $\sigma''' \in L^\infty$ and σ' grows like $|v|^\alpha$, $\alpha > 2/5$. Other, more general behavior for σ is also possible.

Theorem 1 Global Existence and Convergence *We assume hypotheses H1-H4. Let u^ϵ, v^ϵ be viscosity solutions of (1), (2) with initial data $u_0^\epsilon, v_0^\epsilon \in H^1(\mathrm{R})$, satisfying $\int \frac{1}{2}(u_0^\epsilon)^2 + \Sigma(v_0^\epsilon)dx \leq C$ = a constant and with $u_0^\epsilon \rightharpoonup \overline{u}_0$ and*

$v_0^\epsilon \rightharpoonup \overline{v}_0$ such that $\epsilon \int (v_{0x}^\epsilon)^2 dx \to 0$ as $\epsilon \to 0$. Then there is a subsequence $u^{\epsilon_k}, v^{\epsilon_k}$ which converges strongly in $L^p_{loc}(\mathbb{R}^2)$, $p < 2$ to global weak solutions $\overline{u}, \overline{v}$ of (3), (4) with initial data $\overline{u}_0, \overline{v}_0$. If $\frac{|\lambda|^q}{\Sigma(\lambda)} \to 0$ as $|\lambda| \to \infty$, then v^{ϵ_k} converges strongly in L^q_{loc}. If the viscosity approximations converge to a unique solution of (3), (4), then the entire sequence u^ϵ, v^ϵ converges strongly to $\overline{u}, \overline{v}$.

Remarks We can, of course, choose for the viscosity solutions of (1), (2) initial data in H^1 independent of ϵ. However, to obtain the most general existence theorem, we assume that the initial data converges weakly to $\overline{u}_0, \overline{v}_0$ in the sense of distributions. From Shearer [15], it follows that $\overline{u}_0, \overline{v}_0$ also has finite energy; i.e., $\int \frac{1}{2}(\overline{u}_0)^2 + \Sigma(\overline{v}_0) dx \leq C$, and so we have global existence of solutions with arbitrary finite energy initial data. Also from [15] we have $\overline{u}, \overline{v} \in L^\infty(\mathbb{R}_+; L^\eta) \cap Lip(\mathbb{R}_+; W^{1,\infty}(\mathbb{R})^*_{loc})$ where L^η is the space of functions $(u, v) \in L^1_{loc}(\mathbb{R})$ with finite energy (i.e., $\eta(u, v) = \frac{1}{2}u^2 + \Sigma(v)$ and $\int_{\mathbb{R}} \eta(u, v) dx < \infty$) and $W^{1,\infty}(\mathbb{R})^*$ denotes the dual of $W^{1,\infty}(\mathbb{R})$.

Proof: We collect the results from the rest of the paper to give the proof. In section 2 we obtain uniform L^p-like bounds from the first energy estimate (7). From lemma 2 in section 3, there is a family of L^η-Young measures $\nu = \nu_{x,t}$ associated with a subsequence of the viscosity solutions $u^{\epsilon_k}, v^{\epsilon_k}$. The latter converge weakly to candidate solutions $\overline{u}, \overline{v}$. To prove the theorem, we need to show that the weak limits $\overline{u}, \overline{v}$ are actually strong limits and that they satisfy the weak formulation of the inviscid equation:

$$
\begin{aligned}
-\int_{\mathbb{R}} \phi(x, 0) u_0(x) dx - \int_0^T \int_{\mathbb{R}} \phi_t u - \phi_x \sigma(v) dx dt &= 0 \\
-\int_{\mathbb{R}} \phi(x, 0) v_0(x) dx - \int_0^T \int_{\mathbb{R}} \phi_t v - \phi_x u dx dt &= 0
\end{aligned}
$$

for all smooth, compactly supported test functions $\phi \in C_c^\infty(\mathbb{R}^2)$. To do this, we start by writing the degenerate parabolic equation in the weak form and take the limit along the subsequence ϵ_k.

$$
-\int_{\mathbb{R}} \phi(x, 0) u_0^\epsilon(x) dx - \int_0^T \int_{\mathbb{R}} \phi_t u^\epsilon - \phi_x \sigma(v^\epsilon) dx dt = \int_0^T \int_{\mathbb{R}} \epsilon \phi_{xx} u^\epsilon dx dt
$$

$$- \int_{\mathbb{R}} \phi(x,0) v_0'(x)dx - \int_0^T \int_{\mathbb{R}} \phi_t v^\epsilon - \phi_x u^\epsilon \, dx dt \;\; = \;\; 0$$

The linear terms clearly converge to the correct weak limits. The difficulty is with the nonlinear composition $\sigma(v^\epsilon)$. In section 5 we show that the support of the L^q Young measures associated with the subsequence $u^{\epsilon_k}, v^{\epsilon_k}$ is a point for a.e. x, t. So from lemma 2 concerning Young measures we have strong convergence in $L^1_{loc}(\mathbb{R}^2)$ of $g(u^{\epsilon_k}, v^{\epsilon_k})$ to $g(\overline{u}, \overline{v})$ when $\frac{g(u,v)}{\eta(u,v)} \to 0$ as $|u| + |v| \to \infty$. Since $\Sigma(v) \geq \sigma_0 v^2/2$ we can choose $g(u,v) = |u|^p + |v|^p$ for any $p < 2$ and so we have strong convergence in $L^p_{loc}(\mathbb{R}^2)$. Similarly, $v^{\epsilon_k} \to \overline{v}$ in $L^q_{loc}(\mathbb{R}^2)$ if $|v|^q/\Sigma(v) \to 0$ as $|v| \to \infty$. Finally, by hypothesis H4, $\sigma/\Sigma \to 0$ as $|v| \to \infty$, so we also have $\sigma(v^{\epsilon_k}) \to \sigma(\overline{v})$ and consequently $\overline{u}, \overline{v}$ are global weak solutions to the inviscid equations.

We expect that solutions of the inviscid equations are unique when entropy is appropriately dissipated (i.e., when an entropy inequality such as $\eta(u,v)_t - (u\sigma(v))_x \leq 0$ is weakly satisfied). Furthermore, the viscosity solutions satisfy the above entropy inequality, so we expect the strong limit of the subsequence to converge to the unique entropy dissipating weak solution of the inviscid equations. From the same argument as in the above paragraph, it follows that any subsequence of the viscosity solutions has a further subsequence which converges strongly to a global weak solution of the inviscid equations. If this limit is unique (i.e., independent of the choice of the initial subsequence), then it follows that the entire sequence u^ϵ, v^ϵ converges strongly to the unique limit.□

2 Viscosity Solutions and Energy Estimates

We assume that there exists global classical (smooth) solutions of (1), (2) for any initial data $u_0, v_0 \in H^1(\mathbb{R})$. We also assume that the solutions and their first few derivatives decay to zero as $|x| \to \infty$ for any $t > 0$.

Lemma 1 Energy Estimates *The viscosity solutions of (1), (2) satisfy for any $T > 0$*

$$(i) \quad E(T) \leq E(0) \tag{7}$$

$$(ii) \quad \epsilon \int_0^T \int_{\mathbb{R}} u_x^2 + \sigma'(v) v_x^2 dx dt \leq 2E(0) + \epsilon \int_{\mathbb{R}} (v_{0x})^2 dx \tag{8}$$

where $E(t) = \int_{\mathbb{R}} \frac{1}{2} u(x,t)^2 + \Sigma(v(x,t)) dx$.

Proof: Multiplying (1) by u and (2) by $\sigma(v)$ and adding we get (we have dropped the superscript ϵ)

$$(\tfrac{1}{2}u^2 + \Sigma(v))_t - (u\sigma(v))_x = \epsilon \tfrac{1}{2}(u^2)_{xx} - \epsilon u_x^2.$$

Integrating in x and t, we obtain (i) and the inequality $\epsilon \int_0^T \int_{\mathbb{R}} u_x^2 dx dt \leq E(0)$ (one should really integrate first on the interval $[\delta, T]$, $\delta > 0$, since we may only have appropriate smoothness and decay for $t > 0$, and then let $\delta \to 0$). To show the second inequality (ii) we use an argument of Greenburg, MacCamy, Mizel [9]. See also DiPerna [5].

Multiply (1) by v_x and use $v_{tx} = u_{xx}$ and integrate to get

$$\int_0^T \int_{\mathbb{R}} v_x u_t dx dt - \int_0^T \int_{\mathbb{R}} \sigma'(v) v_x^2 dx dt = \frac{\epsilon}{2} \int_{\mathbb{R}} v_x^2 |_0^T dx. \tag{9}$$

Integrate the first term by parts twice, first in t then in x after the substitution $v_{xt} = u_{xx}$ to get

$$\int_{\mathbb{R}} v_x u |_0^T dx + \int_0^T \int_{\mathbb{R}} u_x^2 dx dt = \frac{\epsilon}{2} \int_{\mathbb{R}} v_x^2 |_0^T dx + \int_0^T \int_{\mathbb{R}} \sigma'(v) v_x^2 dx dt. \tag{10}$$

Use Cauchy-Schwarz on the first term evaluated at $t = T$ to get

$$\int_{\mathbb{R}} v_x u(T) dx \leq \frac{\epsilon}{2} \int_{\mathbb{R}} v_x^2(T) dx + \frac{1}{2\epsilon} \int_{\mathbb{R}} u^2(T) dx \tag{11}$$

and do the same for the first term at $t = 0$ and we are done. □

3 Young Measures and Compensated Compactness

In [18], Tartar introduces Young measures and compensated compactness in the context of hyperbolic conservation laws to give a new proof of global existence for the scalar conservation law. Subsequent generalizations (starting with DiPerna's) have led to many new existence results in conservation laws. Tartar used the Young measures to represent limiting oscillations in subsequences of uniformly bounded L^∞ functions, $u^\epsilon : \mathrm{R}^n \to \mathrm{R}^m$, $\|u^\epsilon\|_{L^\infty} \leq M$. Roughly, there is a subsequence u^{ϵ_k} such that each Young measure $\nu_x(\lambda)$ is the density measure for the probability distribution of the tail of the sequence of values $\{u^{\epsilon_k}(x)\}$ as $\epsilon_k \to 0$. These L^∞-Young measures are a weakly measurable family of probability measures, compactly supported inside a single set; namely the cube in R^m centered at the origin with sides of length $2M$. If the sequence $\{u^{\epsilon_k}(x)\}$ converges pointwise to $\overline{u}(x)$ for a.e. x, then the Young measures are Dirac masses, $\nu_x(\lambda) = \delta_{\overline{u}(x)}(\lambda)$ for a.e. x and the convergence is strong in $L^p_{loc}(\mathrm{R}^n)$, $p < \infty$. If the sequence continually bounces around (limiting oscillations are present), then the Young measures are spread out and the convergence is only weak. Stated another way, if ν_x is supported on a single point for a.e. x, then there are no oscillations and the associated subsequence converges strongly in $L^p_{loc}(\mathrm{R}^n)$ for any $p < \infty$.

It is possible to associate Young measures with more general sequences of measurable functions, such as those with uniform L^p bounds, $p < \infty$. Such functions can take on arbitrarily large values, so L^p-Young measures need not be compactly supported and may have mass less than one. However, the fundamental correspondence of Young measures supported on points iff there are no oscillations iff the subsequence converges strongly still holds. I.e., if ν_x is supported at a point for a.e. x, then the subsequence converges strongly in

L^q_{loc} for any $q < p$.

We will state only the results needed for our application. For proofs and a more complete discussion of this and related matters see Ball [1] and Shearer [17], [15]. For other more general Young measures capable of representing concentrations and for related issues, see DiPerna and Majda [8], Ball and Murat [2] and Shearer [16].

Defining $\eta(u,v) = \frac{1}{2}u^2 + \Sigma(v)$, we can rewrite the first energy estimate as

$$\int_{\mathrm{R}} \eta(u^\epsilon(t), v^\epsilon(t))dx \;\; \leq \;\; \int_{\mathrm{R}} \eta(u^\epsilon_0, v^\epsilon_0)dx \leq C. \tag{12}$$

By assumption in theorem 1, the right hand side is bounded by a constant C independent of ϵ. With this uniform L^p-like control we have

Lemma 2 Young Measures and Convergence Properties *If (12) holds, then there is a subsequence $u^{\epsilon_k}, v^{\epsilon_k}$ and a weakly measurable family of nonnegative measures $\{\nu_{x,t}\}_{x,t\in\mathrm{R}^2}$ with mass equal to one for a.e. x,t such that*

(i) For any continuous $g(u,v)$ with $\frac{g(u,v)}{\eta(u,v)} \to 0$ as $|u| + |v| \to \infty$ we can define

$$\overline{g}(x,t) \;\; = \;\; \int_{\mathrm{R}^2} g(\lambda_1, \lambda_2)d\nu_{x,t}(\lambda_1, \lambda_2).$$

Then $\overline{g} \in L^1_{loc}$, and $g(u^{\epsilon_k}, v^{\epsilon_k}) \rightharpoonup \overline{g}$ in the weak topology of $L^1_{loc}(\mathrm{R}^2)$ induced by $C_c(\mathrm{R}^2)$, the space of continuous, compactly supported functions on R^2.

(ii) If $g(u,v)$ is continuous, $\frac{g(u,v)}{\eta(u,v)} \to 0$ as $|u| + |v| \to \infty$ and if the support of $\nu_{x,t}$ is a point for a.e. x,t, then $g(u^{\epsilon_k}, v^{\epsilon_k}) \to g(\overline{u}, \overline{v})$ strongly in $L^1_{loc}(\mathrm{R}^2)$ and $\nu_{x,t}(\lambda_1, \lambda_2) = \delta_{\overline{u}(x,t)}(\lambda_1) \otimes \delta_{\overline{v}(x,t)}(\lambda_2)$ $u^{\epsilon_k} \rightharpoonup \overline{u} = \int \lambda_1 d\nu$ and $v^{\epsilon_k} \rightharpoonup \overline{v} = \int \lambda_2 d\nu$.

Remarks. We will call these Young measures L^η-Young measures to emphasize the dependence of the representation and convergence properties on the function η.

As mentioned above, L^η–Young measures may not be probability measures, since some mass may leak off to infinity. But the uniform bound (12) and the growth of η imply that $\nu_{x,t}$ has mass equal to one for a.e. x, t. Also, this bound and the growth rate of η bound the strength of possible concentrations. Thus, to avoid problems with concentrations, one must use functions with sub–η growth. In particular, in Tartar's equation below, we are restricted to using entropy, entropy flux pairs ϕ, ψ with sub–η growth: i.e., $|\phi/\eta| + |\psi/\eta| \to 0$ as $|u| + |v| \to \infty$. This hypothesis of sub-η growth which appears throughout the above lemma can be weakened slightly, but for simplicity we do not consider this further. For more information and proof of the lemma see the references in the penultimate paragraph preceding lemma (2).

4 Compensated Compactness and Tartar's Equation

In this section we obtain estimates for two classes of entropy, entropy flux pairs and then show via the div-curl lemma that they satisfy Tartar's equation.

As introduced by Lax [10], a pair of functions ϕ, ψ is an entropy, entropy flux pair for a system of hyperbolic conservation laws $U_t + F(U)_x = 0$ if all smooth solutions U also satisfy $\phi(U)_t + \psi(U)_x = 0$. It is sufficient if ϕ, ψ satisfy the linear hyperbolic partial differential equation $\nabla\phi\nabla F = \nabla\psi$. In our case we have

$$\phi_v = \psi_u \tag{13}$$

$$\sigma'\phi_u = \psi_v. \tag{14}$$

We gain information about the viscosity approximations by composing them with entropy, entropy flux pairs (solutions of (13), (14)). Let $\phi^\epsilon = \phi(u^\epsilon, v^\epsilon)$ and $\psi^\epsilon = \psi(u^\epsilon, v^\epsilon)$. To use the div-curl lemma, we must show that $\{\phi^\epsilon_t + \psi^\epsilon_x\}_{\epsilon>0}$

lies in a compact subset of $H^{-1}_{loc}(\mathbb{R}^2)$. Multiplying (1), (2) by $\nabla\phi$, it suffices to show that $\epsilon u_{xx}\phi^\epsilon_u = \epsilon(u_x\phi^\epsilon_u)_x - \epsilon(\phi^\epsilon_{uu}u_x^2 + \phi^\epsilon_{uv}u_x v_x)$ is in a compact subset of $H^{-1}_{loc}(\mathbb{R}^2)$. If u^ϵ, v^ϵ are uniformly bounded, then so are ϕ^ϵ_u, ϕ^ϵ_{uu} and ϕ^ϵ_{uv} and the desired compactness follows from Murat's lemma and the second energy estimate. In this case, however, the diffusion matrix is not diagonal and a priori L^∞ estimates are not available. In particular, we cannot use Lax's asymptotic form for the entropies, as DiPerna did, since the standard growth estimates for solutions of linear hyperbolic problems grow exponentially as $|u|$ or $|v|$ tend to infinity and we have only L^p-like bounds on u and v.

Instead, we show the existence of two classes of entropy, entropy flux pairs with slow growth near infinity: i.e., we obtain growth bounds (which depend on σ') for ϕ and ψ and the first two derivatives of ϕ. This turns out to be sufficient since the second energy estimate bounds ϵu_x^2 and $\epsilon\sigma'(v)v_x^2$. The first class of entropy, entropy flux pairs consists of half-plane supported entropies which were first introduced and successfully used by Serre [14] to generalize DiPerna's weak* trace lemma and subsequently used by Shearer [15] with slow growth bounds to prove L^p existence for (3), (4) (assuming genuine nonlinearity and using artificial viscosity). The second class consists of roughly cross quadrant supported entropies and are introduced here to assist in the case of loss of genuine nonlinearity at a single point (unlike half plane supported entropies which are supported in a pair of adjacent quadrants, these are supported mostly in a pair of alternating quadrants).

We now derive the necessary estimates. Since $\sigma' > 0$ we can define $z(v) = \int_0^v \sqrt{\sigma'(s)}ds$, a smooth, monotonically increasing function with a smooth inverse $v(z)$. We change to a Riemann coordinate system by defining $w_1 = u + z(v)$ and $w_2 = u - z(v)$. As in [15] we also make a change of dependent variables

$$\phi = \tfrac{1}{2}(\sigma')^{-1/4}[\Phi + \Psi] \qquad (15)$$

$$\psi = \tfrac{1}{2}(\sigma')^{1/4}\left[\Phi - \Psi\right] \tag{16}$$

and obtain

$$\Phi_{w_1} = a\Psi \tag{17}$$

$$\Psi_{w_2} = -a\Phi \tag{18}$$

where $a = a(w_1 - w_2) = \sigma''(v(\frac{w_1-w_2}{2}))/8(\sigma'(v(\frac{w_1-w_2}{2}))^{3/2}$. To get half plane supported entropies (in the w_1, w_2 plane), we choose a point $(\overline{w_1}, \overline{w_2})$ and solve (17), (18) with Goursat data given on the lines $w_1 = \overline{w_1}$ and $w_2 = \overline{w_2}$; i.e., we set $\Phi(\overline{w_1}, w_2) = g(w_2)$ and $\Psi(w_1, \overline{w_2}) = h(w_1)$. For example, set $h = 0$ and let g be supported in $w_2 > \overline{w}_2$. Then solving the Goursat problem for Φ and Ψ we can use (15), (16) to get

$$\phi(w_1, w_2) = \tfrac{1}{2}(\sigma')^{-1/4}\left[g(w_2) + \int_{\overline{w}_2}^{w_2} G(w_1, w_2, w)g(w)dw\right] \tag{19}$$

$$\psi(w_1, w_2) = \tfrac{1}{2}(\sigma')^{1/4}\left[g(w_2) + \int_{\overline{w}_2}^{w_2} H(w_1, w_2, w)g(w)dw\right] \tag{20}$$

where the kernels G, H depend on $(\overline{w_1}, \overline{w_2})$. Clearly, ϕ, ψ are also supported in the half plane $w_2 \geq \overline{w}_2$ whenever the initial data g is. We will also make use of an integral operator which can be used to derive a solution of (17), (18) and the representations (19), (20). The integral operator $\mathcal{A}$ acting on a function $f \in L^1_{loc}(\mathrm{R}^2)$ is defined as

$$(\mathcal{A}f)(w_1, w_2) = -\int_{\overline{w}_1}^{w_1}\int_{\overline{w}_2}^{w_2} a(\xi - w_2)a(\xi - \eta)f(\xi, \eta)d\eta d\xi. \tag{21}$$

We are also interested in a second class of entropies obtained by solving the linear hyperbolic problem for Φ, Ψ with continuous, compactly supported initial data on a noncharacteristic line of the form $w_1 - w_2 = \xi_0 =$ a constant (i.e., $\Phi(\xi_0 + w, w) = g(w)$ and $\Psi(\xi_0 + w, w) = h(w)$) and rewriting the solution in terms of ϕ and ψ as before.

Lemma 3 *We assume the hypotheses H1-H4. Then the entropy, entropy flux pairs ϕ_i, ψ_i given by (15), (16) satisfy Tartar's equation (22) where Φ, Ψ satisfy*

(17), (18) and have either continuous, compactly supported Goursat data g_i, h_i or continuous, compactly supported Cauchy data on the line $w_1 - w_2 = \xi_0$.

$$< \nu, \phi_1\psi_2 - \phi_2\psi_1 > \; = \; < \nu, \phi_1 >< \nu, \psi_2 > - < \nu, \phi_2 >< \nu, \psi_1 > \quad (22)$$

The measure $\nu = \nu_{x,t}$ is an L^q-Young measure associated with a subsequence of the viscosity solutions and $< \nu, \phi >= \int \phi d\nu$.

Proof. The proof for the class of entropies with Goursat data is the same as in lemmas 2 and 3 in [15]. This is because we have the same energy estimates and the same hypotheses H1-H4 except for hypothesis H2, the loss of genuine nonlinearity at one point. But this is not a problem as noted in the remark after lemma 2 in [15].

We prove the result for the other entropy pairs by reducing to the Goursat data case. Namely, compactly supported Cauchy data produces two quarter planes on which the solution is identically zero. In each of the other quarter planes we show that the solution agrees with a solution with continuous, compactly supported Goursat data. Hence it will satisfy slow growth estimates and consequently also Tartar's equation.

Assume that the support of the Cauchy data on the line $w_1 - w_2 = \xi_0$ lies between the points $(\overline{w}_1, \overline{w}_2)$ and $(\hat{w}_1, \hat{w}_2)$ where $\overline{w}_1 - \overline{w}_2 = \xi_0 = \hat{w}_1 - \hat{w}_2$ and $\overline{w}_1 < \hat{w}_1$ and $\overline{w}_2 < \hat{w}_2$. Clearly the characteristics for the hyperbolic problem (17), (18) are $w_1 =$ constant, $w_2 =$ constant. Hence the solution Φ, Ψ is identically zero in the quadrants $w_1 > \hat{w}_1$, $w_2 > \hat{w}_2$ and $w_1 < \overline{w}_1$, $w_2 < \overline{w}_2$. We use this solution to get initial data for a Goursat problem in the quadrant $w_1 < \hat{w}_1$, $w_2 > \overline{w}_2$; consider continuous, compactly supported Goursat initial data g, h satisfying $g(w_2) = \Phi(\hat{w}_1, w_2)$ for $w_2 > \overline{w}_2$ and $h(w_1) = \Psi(w_1, \overline{w}_2)$ for $w_1 < \hat{w}_1$. Then the solution $\hat{\Phi}, \hat{\Psi}$ to (17), (18) with this Goursat data g, h has slow growth. By uniqueness of the Goursat problem, $\hat{\Phi} = \Phi$ and $\hat{\Psi} = \Psi$ in the quadrant $w_1 < \hat{w}_1$, $w_2 > \overline{w}_2$ and so Φ, Ψ also have slow growth in this quadrant. The same argument works in the other quadrant and consequently the associated ϕ, ψ satisfy Tartar's equation. $\Box$

5 Half Plane Supported Entropy Pairs and Reduction

Half plane supported entropy pairs were first introduced and used in Serre [14] to generalize DiPerna's weak* trace lemma which is the key lemma used to show the support of a Young measure reduces to a point.

Half plane supported entropies were designed to exploit the structure in Tartar's equation. If the entropy pairs (ϕ_1, ψ_1), (ϕ_2, ψ_2) are supported on opposite halves of the plane, then either their supports don't intersect and the left hand side of Tartar's equation is zero (which has consequences for the right hand side), or the supports intersect in a (possibly narrow) band which then can be used to get a weak* trace of the Young measure.

In general, a measure on the plane does not have a well defined restriction to a line (its trace is not well defined). However, if $\mu(x, y)$ is a non-negative, finite measure on the x, y plane one can define a (possibly nonunique) weak* trace. For example, let $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(x/\epsilon)$ be a nonnegative family of approximate delta functions on R. Multiplying by ϕ_ϵ one gets $\phi_\epsilon\mu$ which localizes μ to a band containing the y-axis. Furthermore, either μ is identically zero in some band containing the y-axis, or $< \mu, \phi_\epsilon >$ is never zero, and one can normalize the product to obtain a family of probability measures μ_ϵ defined by $< \mu_\epsilon, f >= \frac{<\mu, \phi_\epsilon f>}{<\mu, \phi_\epsilon>}$ where f is a bounded, continuous function with limit zero at infinity. By Alaoglu's theorem, there is a subsequence converging weak* to a measure μ^* which is a weak* trace of μ along the y-axis. Different subsequences may converge to different measures; hence the weak* trace may not be unique. If μ is compactly supported, then μ^* has mass one; if not, then it may have mass anywhere between 0 and 1, as some mass may "leak" off to infinity as $\epsilon \to 0$. Observe that because of the normalization, μ^* contains only information about the most singular part of μ living near the y-axis.

We now show in a series of lemmas that for a given arbitrary point x, t, the L^q Young measure at x, t ($\nu = \nu_{x,t}$) is supported in a rectangle. In the present case, the L^q-Young measures are not necessarily compactly supported and genuine nonlinearity is not assumed, so the following lemmas from [14] have been modified in [15] and here to account for this. In particular, in the case when a Young measure is compactly supported, the minimal rectangle containing the support plays an important role in showing the support reduces to a point. In the noncompactly supported case, the four lines defined below play a somewhat analogous role.

Define supp(ϕ, ψ) =support$(\phi) \cup$support(ψ) and let

$$\begin{aligned} w_2^- &= \inf \{w_2 \in \mathrm{R} : \text{there is an entropy pair } (\phi, \psi) \text{ with supp}(\phi, \psi) \text{ in} \\ &\qquad \mathrm{R} \times (-\infty, w_2] \text{ and not both } < \nu, \phi >, < \nu, \psi > \text{ are zero}\} \\ w_2^+ &= \sup \{w_2 \in \mathrm{R} : \text{there is an entropy pair } (\phi, \psi) \text{ with supp}(\phi, \psi) \text{ in} \\ &\qquad \mathrm{R} \times [w_2, +\infty) \text{ and not both } < \nu, \phi >, < \nu, \psi > \text{ are zero}\} \end{aligned}$$

Note that w_2^- and w_2^+ may take the values $-\infty$ or $+\infty$. We also define w_1^+, w_1^- analogously. The four lines of interest are $w_1 = w_1^+$, $w_1 = w_1^-$, $w_2 = w_2^+$ and $w_2 = w_2^-$.

If the supporting half planes of two entropy pairs don't intersect, then their products are zero and the left hand side of Tartar's equation is zero. It is useful that this is often still true when the supporting half planes intersect in a band.

Let $\alpha_0 \notin \{w_2^-, w_2^+\}$ and choose $0 < \epsilon_0 < \frac{1}{3} distance(\alpha_0, \{w_2^-, w_2^+\})$. Define the interval $I = (\alpha_0 - \epsilon_0, \alpha_0 + \epsilon_0)$.

Lemma 4 *For any $\overline{\alpha}_1, \overline{\alpha}_2 \in I$ and any entropy pairs with*

$$supp(\phi_1, \psi_1) \subset \mathrm{R} \times (-\infty, \overline{\alpha}_1]$$

$$supp(\phi_2, \psi_2) \subset \mathbb{R} \times [\overline{\alpha}_2, +\infty)$$

we have

$$< \nu, \phi_1\psi_2 - \phi_2\psi_1 > = 0$$

Remark. For the proof, see [15], lemma 4.

To apply this lemma we define $\alpha_1 = \alpha_0 - \epsilon$, $\alpha_2 = \alpha_0 + \epsilon$ with $0 < \epsilon < \epsilon_0$. Then let g_1 and g_2 be continuous, compactly supported functions with $|g_i| \leq 2\epsilon$ $i = 1, 2$ and satisfying:

$$\begin{aligned}
g_1(w_2) = \quad & w_2 - \alpha_1 \quad \text{for} \quad \alpha_1 \leq w_2 \leq \alpha_2 \\
& 0 \quad \text{for} \quad w_2 \leq \alpha_1 \ \text{ or } \ w_2 \geq \alpha_2 + \epsilon \\
g_2(w_2) = \quad & \alpha_2 - w_2 \quad \text{for} \quad \alpha_1 \leq w_2 \leq \alpha_2 \\
& 0 \quad \text{for} \quad w_2 \geq \alpha_2 \ \text{ or } \ w_2 \leq \alpha_1 - \epsilon
\end{aligned}$$

We derive two pairs of entropies by using two pairs of Goursat axes defined by the points $(\overline{w_1}, \alpha_1)$, $(\overline{w_1}, \alpha_2)$ and using g_1, h and g_2, h with $h = 0$ as respective Goursat initial data. The two entropy pairs ϕ_i, ψ_i are given by the formulas (19), (20) where $\overline{w}_2$ is replaced by α_i and the kernels $G = G_i$, $H = H_i$ depend on α_i. These pairs are supported on half planes which overlap in a narrow band. Thus the quadratic form $\phi_1\psi_2 - \phi_2\psi_1$ is nonzero only for w_2 in the interval $[\alpha_1, \alpha_2]$. Define $\Delta_\epsilon = \Delta_\epsilon(\alpha_0, w_2) = g_1(w_2)g_2(w_2)$ and note that $\Delta_\epsilon = 0$ when $|w_2 - \alpha_0| \geq \epsilon$.

Lemma 5 *We assume $a, a' \in L^\infty$ and $a \in L^2$ (it suffices to assume H3). For the entropy pairs ϕ_i, ψ_i, $i = 1, 2$, defined above we have*

$$\phi_1\psi_2 - \phi_2\psi_1 = -\frac{1}{2}\epsilon\Delta_\epsilon a(w_1 - \alpha_0) + \epsilon^2\Delta_\epsilon E(w_1, w_2, \alpha_0)$$

where the error term E is supported in the strip $\alpha_1 \leq w_2 \leq \alpha_2$ and E is bounded by a constant independent of $w_1, w_2, \alpha_0, \epsilon < \epsilon_0$.

Proof: By integrating the differential equations (17), (18) with $\alpha_1 \le w_2 \le \alpha_2$, we have

$$\Phi_i \pm \Psi_i \;=\; A_i \mp B_i + C_i \mp D_i \tag{23}$$

where $A_i = g_i$, $B_i = \int_{\alpha_i}^{w_2} a(w_1 - w)g_i(w)dw$, $C_i = \mathcal{A}_i\Phi_i$, and $D_i = \int_{\alpha_i}^{w_2} a(w_1 - w)\mathcal{A}_i\Phi_i(w_1, w)dw$. Then there is a constant C such that $|B_i| + |C_i| \le C\epsilon|w_2 - \alpha_i|$ and $|D_i| \le C\epsilon^2|w_2 - \alpha_i|$, for any w_1, w_2, α_0 and $\epsilon < \epsilon_0$ (see [15], lemma 5). Then $\phi_1\psi_2 - \phi_2\psi_1 = (A_1B_2 - A_2B_1)/2 + \Delta_\epsilon O(\epsilon^2)$.

Using $a(w_1 - w) = a(w_1 - \alpha_0) + a'(\eta)(\alpha_0 - w)$ in B_i we get $A_1B_2 - A_2B_1 = -\epsilon\Delta_\epsilon a(w_1 - \alpha_0) + \Delta_\epsilon O(\epsilon^2)$. Since $a' \in L^\infty$, the error term is uniform in w_1, w_2, α_0. $\square$

Lemma 6 *If $f = f(w_2)$ is any continuous, compactly supported function such that its support does not contain either w_2^+ or w_2^-, then*

$$< \nu, fa > \;=\; 0$$

where $< \nu, fa > = \int f(w_2)a(w_1 - w_2)d\nu(w_1, w_2)$.

Remark. The same result holds for any continuous compactly supported function $f(w_1)$ with support not containing w_1^+ or w_1^-.

Proof: Note that $\delta_\epsilon(w_2 - \alpha_0) = 3\Delta(w_2, \alpha_0)/4\epsilon^3$ is an approximate delta function. Using lemmas 4 and 5 and integrating $f(\alpha_0)d\alpha_0$ against 0, we get

$$\begin{aligned}
0 &= < \nu, -3\frac{\phi_1\psi_2 - \phi_2\psi_1}{2\epsilon^4} > \\
&= \int f(\alpha_0)\int a(w_1 - w_2)\delta_\epsilon(w_2 - \alpha_0)d\nu d\alpha_0 - \frac{3}{2}\int f(\alpha_0)\int \Delta_\epsilon E/\epsilon^2 d\nu d\alpha_0 \\
&= \int\int f(\alpha_0)\delta_\epsilon(w_2 - \alpha_0)d\alpha_0\, a(w_1 - w_2)d\nu - \frac{3}{2}\int\int f(\alpha_0)\Delta_\epsilon E/\epsilon^2\, d\alpha_0 d\nu.
\end{aligned}$$

For fixed $\epsilon > 0$ all terms are in L^∞ and f is compactly supported so Fubini can be applied to both integrals. Now let $\epsilon \to 0$. Since $|\int f\delta_\epsilon d\alpha_0| \le \|f\|_{L^\infty}$, we can use dominated convergence to bring the limit inside the first integral in e portion of the line $w_1 - w_2 = \lambda_0$ lying above and to $_1^*, w_2^*)$. We show that the support of ν intersected with o do this, we solve the Cauchy problem

$$\Phi_{w_1} = a\Psi$$

$$\Psi_{w_2} = -a\Phi$$

supported arbitrary initial data given on P. Since the *constant*, it follows that the entropies are well defined nion of the half planes $w_1 \geq w_1^*$, $w_2 \geq w_2^*$. Since the ion is confined to P, we can choose initial data so that Tartar's equation becomes $< \nu, \phi_1\psi_2 >=< \nu, \phi_1 ><$ arbitrary on P. Hence the support of ν is at most a t of the quadrant $w_1 < w_1^*$, $w_2 < w_2^*$.

shows that the support of ν in the complement of at most a point, where $w_1^{**} < w_1^-$, $w_2^{**} < w_2^-$, and ining these results, we see that ν is supported in a

mass.

nimal rectangle with edges parallel to the coordinate support of ν. We show that the support of ν is a point le is nondegenerate and arrive at a contradiction. The the weak* trace introduced and used by DiPerna in trace of ν along an edge of $\mathcal{R}$ and show this trace is ss near one of the boundary, contradicting minimality

e $w_1 - w_2 = \lambda_0$ where $a = 0$. If $\mathcal{L}$ does not intersect $\neq 0$ on $\mathcal{R}$ and we can use slowly growing entropies and the arguments in [14] chapter 5 or [15] to show that ν is supported on a point. If $\mathcal{L}$ intersects $\mathcal{R}$ then either it passes through the two opposite vertices or there is at least one edge which $\mathcal{L}$ does not intersect. We consider the two cases separately.

Suppose $\mathcal{L}$ does not intersect the top edge which is included in the line $w_2 = w_2^T$. Since ν is supported in a minimal rectangle, lemmas 4 and 6 can be proved without the assumptions concerning w_2^-, w_2^+ (see [14]). If δ_ϵ is defined as in the proof of lemma 6, then by the minimality of $\mathcal{R}$, $< \nu, \delta_\epsilon > \neq 0$ for all small ϵ. Taking an appropriate subsequence ϵ_k we can define a probability measure μ^T (a weak* trace of ν) along the top edge of $\mathcal{R}$ by

$$< \mu^T, f > \;=\; \lim_{k\to\infty} \frac{< \nu, \delta_{\epsilon_k} f >}{< \nu, \delta_{\epsilon_k} >}$$

where $f = f(w_1)$ is any continuous function. Setting $f^T(w_1) = a(w_1 - w_2^T)$ and noting that $|< \nu, \delta_{\epsilon_k}(f^T - a) >| \leq \epsilon \|a'\|_{L^\infty} < \nu, \delta_{\epsilon_k} >$, we have

$$< \mu^T, f^T > \;=\; \lim_{k\to\infty} \frac{< \nu, \delta_{\epsilon_k} a >}{< \nu, \delta_{\epsilon_k} >} + \epsilon_k \frac{< \nu, \delta_{\epsilon_k} E >}{< \nu, \delta_{\epsilon_k} >} \tag{24}$$

$$= \lim_{k\to\infty} \frac{< \nu, \frac{-3}{2\epsilon^4}(\phi_1\psi_2 - \phi_2\psi_1) >}{< \nu, \delta_{\epsilon_k} >} \;=\; 0. \tag{25}$$

However, f^T is continuous and nonzero on the top edge since $\mathcal{L}$ does not intersect that edge; this contradicts $< \mu^T, f^T >= 0$. Thus the rectangle must be a line and at least one end point of $\mathcal{R}$ is not on $\mathcal{L}$. The same argument as above shows that the trace of ν on that endpoint is zero, again contradicting minimality and we conclude the rectangle is a point.

We now consider the case when $\mathcal{L}$ intersects opposite vertices of the rectangle $\mathcal{R}$ which we denote as $\mathbf{w}^T = (w_1^T, w_2^T)$ and $\mathbf{w}^B = (w_1^B, w_2^B)$. The above argument shows that any weak* traces μ^T and μ^B of ν along the top and bottom edges of $\mathcal{R}$ satisfy $< \mu^T, f^T >=< \mu^B, f^B >= 0$. Since a is continuous and nonzero on the two edges except at the endpoints $\mathbf{w}^T$ and $\mathbf{w}^B$, we conclude that μ^T and μ^B are Dirac point masses supported at the respective endpoints.

Similarly, we can take weak* traces of ν along the left and right edges of $\mathcal{R}$ and denote them as μ^L and μ^R. They are also Dirac point masses supported respectively on the endpoints $\mathbf{w}^B$ and $\mathbf{w}^U$.

We construct four families of half plane supported entropy pairs, one family for each direction, north, south, east, west, which indicate the type of supporting half plane. We label them as E_N, F_N, E_S, F_S, E_E, F_E and E_W, F_W. We will use a sequence of approximate delta functions as Goursat initial data. For example, let $h = 0$ and let $g_\epsilon^N(w_2)$ be a family of approximate delta functions symmetrically centered at w_2^T and supported on $(w_2^T - \epsilon, w_2^T + \epsilon)$ with $g_\epsilon^N(w_2) = (w_2 - w_2^T + \epsilon)/\epsilon$ for $w_2 \in (w_2^T - \epsilon, w_2^T)$. Then from (19), (20) we have (for notational convenience we suppress the dependence of the entropy pairs on ϵ)

$$E_N(w_1, w_2) = \frac{1}{2}(\sigma')^{-1/4}\left[g_\epsilon^N(w_2) + \int_{(w_2^T-\epsilon)}^{w_2} G g_\epsilon^N dw\right] \qquad (26)$$

$$F_N(w_1, w_2) = \frac{1}{2}(\sigma')^{1/4}\left[g_\epsilon^N(w_2) + \int_{(w_2^T-\epsilon)}^{w_2} H g_\epsilon^N dw\right]. \qquad (27)$$

Observe that E_N, F_N are supported in the upper half plane or "north" half plane. The other three families are defined analogously with approximate delta initial data g_ϵ^S, h_ϵ^E, h_ϵ^W.

Note that G is bounded by a constant for all $(w_1, w_2) \in \mathcal{R}$. Thus,

$$\left|\frac{1}{2}(\sigma')^{-1/4}\int_{(w_2^T-\epsilon)}^{w_2} G g_\epsilon^N dw\right| \leq c\epsilon g_\epsilon^N(w_2)$$

when $w_2 \in (w_2^T - \epsilon, w_2^T)$ and $(w_1, w_2) \in \mathcal{R}$. Thus, we can take an appropriate subsequence as $\epsilon \to 0$ to obtain a weak* trace of ν:

$$< \mu^T, \frac{1}{2}(\sigma')^{-1/4} f > = \lim \frac{< \nu, E_N f >}{< \nu, g_\epsilon^N >}.$$

For brevity we will suppress ν and write $< E_N >$ in place of $< \nu, E_N >$ in the next series of equations which follow by using Tartar's equation successively, first to expand and allow terms to cancel, then to contract into a single product which is zero since the supports of the entropies do not intersect.

$$\begin{aligned}
& < E_N F_E - E_E F_N >< E_S F_W - E_W F_S > \\
& - < E_N F_W - E_W F_N >< E_S F_E - E_E F_S > \\
= & - < E_N >< F_E >< E_W >< F_S > - < E_E >< F_N >< E_S >< F_W > \\
& + < E_N >< F_W >< E_E >< F_S > + < E_W >< F_N >< E_S >< F_E > \\
= & < E_N F_S - E_S F_N >< E_E F_W - E_W F_E > \\
= & \ 0
\end{aligned}$$

Sending $\epsilon \to 0$ (and normalizing by dividing by $< \nu, h_\epsilon^E >$ and $< \nu, h_\epsilon^W >$) in the east and west entropy pairs in the equation above and noting that $< \mu^R, E_S > = < \mu^R, F_S > = 0$ (recall μ^R is supported on the point $\mathbf{w}^T$), we get

$$< \mu^R, (\sigma')^{1/4} E_N - (\sigma')^{-1/4} F_N >< \mu^L, (\sigma')^{1/4} E_S - (\sigma')^{-1/4} F_S > = 0 \qquad (28)$$

Note that integration against μ^R is just evaluation at $\mathbf{w}^T$.

Now let the initial data defining the north entropies be given as $h = 0$ and $g_\epsilon(w) = w - (w_2^T - \epsilon)$ when $w \in (w_2^T - \epsilon, w_2^T)$ with $|g_\epsilon| \leq 2\epsilon$, continuous and supported on $[w_2^T - \epsilon, w_2^T + \epsilon]$. Using the representations (15), (16) and (23) we have $< \mu^R, (\sigma')^{1/4} E_N - (\sigma')^{-1/4} F_N > /2 = -B - D =$

$$\Psi_\epsilon(w_1^T, w_2^T) \ = \ -[1 + O(\epsilon)] \int_{w_2^T - \epsilon}^{w_2^T} a(w_1^T - w)\left(w - (w_2^T - \epsilon)\right) dw \qquad (29)$$

To see that the error term above is correct note that $|\Phi_\epsilon| \leq c\|g_\epsilon\|_{L^\infty}$ when $w_2 \in (w_2^T - \epsilon, w_2^T)$ (see [15], lemma 5). Then

$$\mathcal{A}\Phi_\epsilon(w_1, w_2) \ = \ -\int_{\overline{w}_1}^{w_1} \int_{w_2^T - \epsilon}^{w_2} a(\xi - w_2) a(\xi - \eta) \Phi_\epsilon(\xi, \eta) d\eta d\xi$$

which we can estimate by using $|\Phi_\epsilon| \leq c\|g_\epsilon\|_{L^\infty}$ and integrating first in ξ, using Cauchy-Schwarz, then integrating in η to get

$$\begin{aligned}
|D| \ & = \ \left| \int_{w_2^T - \epsilon}^{w_2^T} a(w_1^T - w) \mathcal{A}\Phi_\epsilon(w_1^T, w) dw \right| \\
& \leq \ c\|a\|_{L^2}^2 \|g_\epsilon\|_{L^\infty} \left| \int_{w_2^T - \epsilon}^{w_2^T} a(w_1^T - w)\left(w - (w_2^T - \epsilon)\right) dw \right| \qquad (30)
\end{aligned}$$

Since the integrand in (30) above is nonzero and continuous on the interval $(w_2^T - \epsilon, w_2^T)$, we can rescale g_ϵ by dividing by $\int_{w_2^T-\epsilon}^{w_2^T} a(w_1^T - w)\left(w - (w_2^T - \epsilon)\right) dw$. Then $\Psi_\epsilon(w_1^T, w_2^T)$ converges to the nonzero constant -1 as $\epsilon \to 0$. A similar argument shows that the other term in the product in (28) is also nonzero which is a contradiction. Thus the minimal rectangle is at most a line; but as seen before, this is also not possible. The support of ν is a point.

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