Mathematische Annalen (2022) 382:1031–1046
https://doi.org/10.1007/s00208-020-02071-9

Mathematische Annalen

Zero relaxation time limits to a hydrodynamic model of two carrier types for semiconductors

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Received: 15 January 2020 / Revised: 6 August 2020 / Published online: 25 August 2020

Abstract

In this paper, we study the zero relaxation time limits to a one dimensional hydrodynamic model of two carrier types for semiconductors. First, we introduce the flux approximation coupled with the classical viscosity method to obtain the uniform L^p_{loc}, $p \geq 1$, bound of the approximation solutions $\rho_i^{\varepsilon,\delta}$ and other estimates of $(u_i^{\varepsilon,\delta}, E^{\varepsilon,\delta})$ with the help of the high energy estimates (Jungel and Peng Comm Partial Differ Equ 58:1007–1033, 1999). Then, we apply the compensated compactness method coupled with the scaled variables technique (Marcati and Natalini Arch Ration Mech Anal 129:129–145, 1995) to prove the zero-relaxation-time limits with arbitrarily large initial data, and arbitrary adiabatic exponents $\gamma_i > 1$.

Mathematics Subject Classification 35L65 · 76N10 · 65M12 · 78A35

1 Introduction

In this paper, we study the zero relaxation time limits to the following one-dimensional hydrodynamic model of two carrier types for semiconductors

$$\begin{cases} \rho_{it} + (\rho_i u_i)_x = 0, \\ (\rho_i u_i)_t + (\rho_i (u_i)^2 + P_i(\rho_i))_x = \rho_i E - \frac{a_i(x)\rho_i u_i}{\tau_i}, \quad i = 1, 2, \\ E_x = \rho_1 + \rho_2 - b(x), \end{cases} \tag{1.1}$$

Communicated by Y. Giga.

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in the region $(-\infty, +\infty) \times [0, \infty]$, with bounded initial data

$$(\rho_i, u_i)|_{t=0} = (\rho_{i0}(x), u_{i0}(x)), \quad \lim_{|x|\to\infty} (\rho_{i0}(x), u_{i0}(x)) = (0, 0), \quad \rho_{i0}(x) \geq 0 \tag{1.2}$$

and a condition at $-\infty$ for the electric potential

$$\lim_{x\to-\infty} E(x, t) = E_0, \quad \text{for a.e.} \quad t \in (0, \infty), \tag{1.3}$$

where E_0 is a fixed constant, (ρ_1, u_1) and (ρ_2, u_2) are the (density, velocity) pairs for electrons $(i = 1)$ and holes $(i = 2)$ respectively, E is the electric potential and the given function $b(x)$ represents the impurity doping profile, and $a_i(x) \geq 0$ are damping coefficients (cf. [1,3,6,17,20] and the references cited therein). The pressure-density relations are $P_i(\rho_i) = \frac{1}{\gamma_i}(\rho_i)^{\gamma_i}$, where $\gamma_i > 1$ correspond to the adiabatic exponents, $\tau_i > 0$ are the momentum relaxation times.

When damping coefficients $a_i(x) = 1$, the global existence of entropy solutions of the initial-boundary value problem of (1.1) was first studied by using the viscosity method [3] and the Godunov scheme method [20]), respectively, where the adiabatic exponents γ_i are limited in the region $(1, \frac{5}{3}]$ to ensure the uniform L^∞ estimates of the approximation solutions.

The global solutions of the Cauchy problem of (1.1) was obtained in [6,17], where the approximation solutions were constructed by the Lax-Friedrichs scheme and the Godunov scheme. Due to the lack of a technique to obtain the a-priori L^∞ estimate, it is a long-standing open problem to study the Cauchy problem of (1.1) by using the vanishing viscosity method.

Recently, in [11], the author introduced the following classical viscosity method coupled with the flux approximation to study the problem (1.1)-(1.3).

Consider

$$\begin{cases} \rho_{it} + ((\rho_i - 2\delta)u_i)_x = \varepsilon\rho_{ixx}, \\ (\rho_i u_i)_t + (\rho_i (u_i)^2 - \delta(u_i)^2 + S_i(\rho_i, \delta))_x \\ \qquad = \varepsilon(\rho_i u_i)_{xx} + (\rho_i - 2\delta)E - \frac{a_i(x)(\rho_i - 2\delta)u_i}{\tau_i}, \\ E_x = (\rho_1 - 2\delta) + (\rho_2 - 2\delta) - b(x) \end{cases} \tag{1.4}$$

with the initial data

$$(\rho_i^{\varepsilon,\delta}(x, 0), u_i^{\varepsilon,\delta}(x, 0)) = (\rho_{i0}(x) + 2\delta, u_{i0}(x)) * G^\varepsilon, \tag{1.5}$$

where $(\rho_{i0}(x), u_{i0}(x))$ are given in (1.2), $\delta > 0$ denotes a regular perturbation constant, the perturbation pressures

$$S_i(\rho_i, \delta) = \int_{2\delta}^{\rho_i} \frac{t - 2\delta}{t} P_i'(t)dt, \tag{1.6}$$

G^ε is a mollifier such that $(\rho_i^{\varepsilon,\delta}(x,0), u_i^{\varepsilon,\delta}(x,0))$ are smooth and

$$\lim_{|x|\to\infty}(\rho_i^{\varepsilon,\delta}(x,0), u_i^{\varepsilon,\delta}(x,0)) = (2\delta, 0), \quad \lim_{|x|\to\infty}(\rho_{ix}^{\varepsilon,\delta}(x,0), u_{ix}^{\varepsilon,\delta}(x,0)) = (0,0). \tag{1.7}$$

The existence result of solutions in [11] is summarized as follows:

Theorem 1 *(Existence:) Let the initial data $\rho_{i0}(x) \in L^\infty(I\!R) \cap L^1(I\!R), u_{i0}(x) \in L^\infty(I\!R)$, the doping profile $b(x) \in L^1(I\!R)$ and the damping coefficients $a_i(x) \geq 0$. Then,*

(I) *for fixed $\varepsilon, \delta, \tau_i > 0$ and $\gamma_i > 1$, the problem (1.4)–(1.7) has a global smooth solution $(\rho_i^{\varepsilon,\delta,\tau_i}, u_i^{\varepsilon,\delta,\tau_i}, E^{\varepsilon,\delta,\tau_i})$ satisfying*

$$\begin{cases} 2\delta \leq \rho_i^{\varepsilon,\delta,\tau_i}, \quad w_i(\rho_i^{\varepsilon,\delta,\tau_i}, u_i^{\varepsilon,\delta,\tau_i}) \leq M(1+t), \\ \qquad z_i(\rho_i^{\varepsilon,\delta,\tau_i}, u_i^{\varepsilon,\delta,\tau_i}) \leq M(1+t) \\ |\rho_i^{\varepsilon,\delta,\tau_i}(\cdot,t) - 2\delta|_{L^1(R)} \leq M, \\ \lim_{|x|\to\infty}(\rho_i^{\varepsilon,\delta,\tau_i}(x,t), u_i^{\varepsilon,\delta,\tau_i}(x,t)) = (2\delta, 0), \\ \lim_{|x|\to\infty}(\rho_{ix}^{\varepsilon,\delta,\tau_i}(x,t), m_{ix}^{\varepsilon,\delta,\tau_i}(x,t)) = (0,0) \end{cases} \tag{1.8}$$

and

$$2\delta \leq \rho_i^{\varepsilon,\delta,\tau_i} \leq M(t), \quad |u^{\varepsilon,\delta,\tau_i}| \leq M(t), \quad |E^{\varepsilon,\delta,\tau_i}| \leq M, \tag{1.9}$$

where w_i, z_i are Riemann invariants of (1.4)

$$w_i(\rho_i, u_i) = \frac{1}{\theta_i}(\rho_i)^{\theta_i} + u_i, \quad z_i(\rho_i, u_i) = \frac{1}{\theta_i}(\rho_i)^{\theta_i} - u_i, \quad \theta_i = \frac{\gamma_i - 1}{2}, \tag{1.10}$$

M is a positive constant and $M(t)$ a positive function of t. Both M and $M(t)$ are independent of $\varepsilon, \delta, \tau_i$, but $M(t)$ could tend to $+\infty$ as the time $t \to +\infty$;

(II) *there exists a subsequence (still labelled) $(\rho^{\varepsilon,\delta,\tau_i}(x,t), u^{\varepsilon,\delta,\tau_i}(x,t), E^{\varepsilon,\delta,\tau_i}(x,t))$, which converges pointwisely to the bounded functions $(\rho^{\tau_i}(x,t), u^{\tau_i}(x,t), E^{\tau_i}(x,t))$ as δ, ε tend to zero, and $(\rho^{\tau_i}(x,t), u^{\tau_i}(x,t), E^{\tau_i}(x,t))$ is a weak entropy solution of the problem (1.1)–(1.3).*

In this paper, we study the relaxation limits of $(\rho^{\varepsilon,\delta,\tau_i}(x,t), u^{\varepsilon,\delta,\tau_i}(x,t), E^{\varepsilon,\delta,\tau_i}(x,t))$, as $\delta, \varepsilon, \tau_i$ tend to zero. Since our proof on the above bipolar hydrodynamic model is completely same to the following unipolar model for semiconductor devices, to avoid the use of knotty mathematical symbols, we only consider

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = \varepsilon\rho_{xx}, \\ (\rho u)_t + (\rho u^2 - \delta u^2 + S(\rho,\delta))_x = \varepsilon(\rho u)_{xx} + (\rho - 2\delta)E - \frac{1}{\tau}a(x)(\rho - 2\delta)u, \\ E_x = (\rho - 2\delta) - b(x) \end{cases} \tag{1.11}$$

with (1.5)-(1.7), where the subscript i is deleted.

As did in [15], we introduce the scaled variables in (1.11)

$$N^\tau(x,\xi) = \rho^{\varepsilon,\delta}\left(x, \frac{\xi}{\tau}\right), \quad J^\tau(x,\xi) = \frac{1}{\tau}m^{\varepsilon,\delta}\left(x, \frac{\xi}{\tau}\right), \quad \Upsilon^\tau(x,\xi) = E^{\varepsilon,\delta}\left(x, \frac{\xi}{\tau}\right), \tag{1.12}$$

where $m^{\varepsilon,\delta} = \rho^{\varepsilon,\delta}u^{\varepsilon,\delta}$, then (1.11) is rewritten as

$$\begin{cases} N^\tau_\xi + ((N^\tau - 2\delta)U^\tau)_x = \frac{\varepsilon}{\tau}N^\tau_{xx}, \\ \tau^2 J^\tau_\xi + (\tau^2(N^\tau(U^\tau)^2 - \delta(U^\tau)^2) + S(N^\tau,\delta))_x \\ = \varepsilon\tau J^\tau_{xx} + (N^\tau - 2\delta)\Upsilon^\tau - a(x)(N^\tau - 2\delta)U^\tau, \\ \Upsilon^\tau_x = (N^\tau - 2\delta) - b(x), \end{cases} \tag{1.13}$$

where $U^\tau(x,\xi) = \frac{J^\tau(x,\xi)}{N^\tau(x,\xi)} = \frac{1}{\tau}u^{\varepsilon,\delta}(x, \frac{\xi}{\tau})$.

When $a(x) = 1$, based on the uniform L^∞ bound assumption on the approximate solutions constructed by using the fractional step Lax-Friedrichs and Godounov schemes, the authors in [15] proved that the limit $(N(x,\xi), J(x,\xi), \Upsilon(x,\xi))$ of the sequence $(N^\tau(x,\xi), J^\tau(x,\xi), \Upsilon^\tau(x,\xi))$, as $\tau \downarrow 0^+$, is a solution of the following well-known drift-diffusion equations

$$\begin{cases} N_\xi + J_x = 0 \\ P(N)_x = N\Upsilon - a(x)J, \\ \Upsilon_x = N - b(x) \end{cases} \tag{1.14}$$

in the sense of distributions.

After giving up the attempt to obtain the uniform L^∞ bound on the approximate solutions, the authors in [6] constructed a family of positive and convex entropies to deduce the high energy estimates of solutions and the uniform L^p, $1 \le p < \infty$, estimates of approximate solutions, constructed by the Lax-Friedrichs and Godounov schemes. Based on the L^p estimates and the technical assumption $\gamma = 1 + \frac{2}{n}, n \ge 1$ being an integer, the zero relaxation limit of $(N^\tau(x,\xi), J^\tau(x,\xi), \Upsilon^\tau(x,\xi))$ was proved by using the compensated compactness method.

In this paper, we extend the results in [6] to any $\gamma > 1$ and an arbitrary bounded function $a(x) \geq 1$, by using the viscosity method given in (1.11). The new technique we used is to introduce a perturbation parameter $\delta > 0$ on the flux functions so that we may obtain the uniformly positive lower bound on the viscosity solutions $\rho^{\varepsilon,\delta,\tau}(x,t) \geq 2\delta$. With the help of this key explicit bound, we may choose the suitable relation among δ, the viscosity parameter ε and the relaxation time τ to obtain the necessary estimates (for instance, the estimates in Lemma 7) and to prove the the zero relaxation limit for any $\gamma > 1$.

Mainly we have the following theorem:

Theorem 2 *(Relaxation Limit:) Let $a(x) \geq 1$ be bounded. Suppose all conditions in Theorem 1 are true and the initial data $(\rho_0(x), u_0(x))$ tend to zero sufficiently fast as $|x| \to \infty$, such that $f(\rho_0(x), u_0(x)) \in L^1(I\!R)$ for any continuous function $f(\rho, u)$ satisfying $f(0,0) = 0$. Let $\varepsilon = o(\tau^2 l(\tau, \delta))$, where*

$$l(\tau,\delta) = \begin{cases} (2\delta)^{\gamma-2}, & \text{for } \gamma \geq 2, \\ \tau^{\frac{2(2-\gamma)}{\gamma-1}}, & \text{for } 1 < \gamma < 2. \end{cases} \tag{1.15}$$

Then, there exists a subsequence (still labelled) $(N^\tau(x,\xi), J^\tau(x,\xi), \Upsilon^\tau(x,\xi))$, which satisfies on any set $Q_L = I\!R \times (0, L)$, $L > 0$

$$\begin{cases} N^\tau(x,\xi) \to N(x,\xi) \quad \text{a.e. in } L^p_{loc}(Q_L), \quad p \geq 1, \\ \frac{N^\tau(x,\xi)-2\delta}{N^\tau(x,\xi)} J^\tau(x,\xi) \rightharpoonup J(x,\xi) \quad \text{weakly in } L^2(Q_L), \\ \Upsilon^\tau(x,\xi)) \to \Upsilon(x,\xi) \quad \text{a.e. in } L^\infty_{loc}(Q_L), \end{cases} \tag{1.16}$$

as $\varepsilon \downarrow 0^+, \tau \downarrow 0^+, \delta \downarrow 0^+$, where the limit $(N(x,\xi), J(x,\xi), \Upsilon(x,\xi))$ is a weak solution of the drift-diffusion equations (1.14) in the sense of distributions.

2 Proof of the Main Theorem

In this section, we shall prove Theorem 2. First, following the ideas in [6], we construct the necessary convex entropy-entropy flux pairs of (1.13).

We rewrite the first two equations in (1.13) as follows

$$\begin{cases} N^\tau_\xi + ((N^\tau - 2\delta)U^\tau)_x = \frac{\varepsilon}{\tau} N^\tau_{xx}, \\ J^\tau_\xi + ((N^\tau(U^\tau)^2 - \delta(U^\tau)^2) + \frac{1}{\tau^2} S(N^\tau, \delta))_x \\ = \frac{\varepsilon}{\tau} J^\tau_{xx} + \frac{1}{\tau^2}(N^\tau - 2\delta)(\Upsilon^\tau - a(x)U^\tau). \end{cases} \tag{2.1}$$

For simplicity, we first omit all the superscripts ε, δ and τ.

A convex, physical entropy-entropy flux pair of the left-hand side of (2.1) is

$$\begin{cases} \eta^*(N,U) = \tau^2\left(\frac{1}{2}NU^2 + \frac{a_\tau^2 N^\gamma}{\gamma-1}\right) \\ q^*(N,U) = \tau^2\left(\frac{1}{2}NU^3 + \frac{\gamma a_\tau^2 N^\gamma U}{\gamma-1}\right) \end{cases} \tag{2.2}$$

and the general entropy-entropy flux pair satisfies

$$\eta(N,J) = N\int_{-1}^{1} g\left(\frac{J}{N} + A_\tau y N^\theta\right)(1-y^2)^\lambda dy \tag{2.3}$$

and

$$q(N,J) = N\int_{-1}^{1} g\left(\frac{J}{N} + A_\tau y N^\theta\right)\left(\frac{J}{N} + \theta A_\tau y N^\theta\right)(1-y^2)^\lambda dy, \tag{2.4}$$

where $\lambda = \frac{3-\gamma}{2(\gamma-1)}$, $\theta = \frac{\gamma-1}{2}$, $a_\tau = \frac{1}{\gamma^{\frac{1}{2}}\tau}$ and $A_\tau = \frac{1}{\theta\tau}$.

We choose $g(x) = x^{2k}$ in (2.3). Since

$$\begin{aligned}\left(\frac{J}{N} + A_\tau y N^{\frac{\gamma-1}{2}}\right)^{2k} &= \sum_{i=0}^{2k} C_{2k}^i \left(\frac{J}{N}\right)^i A_\tau^{2k-i} y^{2k-i} N^{(2k-i)(\gamma-1)/2} \\ &= \sum_{i=0}^{2k} C_{2k}^i A_\tau^{2k-i} y^{2k-i} N^{(2k-i)(\gamma-1)/2-i} J^i,\end{aligned}$$

where $C_k^i = \frac{k!}{(k-i)!i!}$ and

$$\int_{-1}^{1} y^i (1-y^2)^\lambda d\,y = 0, \quad \text{for } i \text{ odd}, \tag{2.5}$$

we deduce that

$$\begin{aligned}\eta(N,J) &= N\int_{-1}^{1} \sum_{i\le 2k,\ i\ even} C_{2k}^i A_\tau^{2k-i} y^{2k-i}(1-y^2)^\lambda N^{(2k-i)(\gamma-1)/2-i} J^i d\,y \\ &= N\int_{-1}^{1} \sum_{i=0}^{k} C_{2k}^{2i} A_\tau^{2(k-i)} y^{2(k-i)}(1-y^2)^\lambda N^{(k-i)(\gamma-1)-2i} J^{2i} d\,y. \end{aligned} \tag{2.6}$$

Hence

$$\eta(N,J) = \sum_{i=0}^{k} \beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i}, \tag{2.7}$$

where

$$\beta_i^{(k)} = C_{2k}^{2i} \int_{-1}^{1} y^{2(k-i)}(1-y^2)^{\lambda} d\,y, \quad 0 \le i \le k, \tag{2.8}$$

$$\alpha_i^{(k)} = (k-i)(\gamma-1) - 2i + 1, \quad 0 \le i \le k, \tag{2.9}$$

which are constants independent of τ.

Similarly, we have from (2.4) that

$$\begin{aligned} q(N,J) = &\frac{J}{N}\eta(N,J) \\ &+ \int_{-1}^{1} \sum_{i\le 2k,\ i\ odd} \theta C_{2k}^{i} A_{\tau}^{2k-i+1} y^{2k-i+1}(1-y^2)^{\lambda} N^{(2k-i+1)\theta-i+1} J^i dy. \end{aligned} \tag{2.10}$$

When $i=0$, we replace the following function in (2.7)

$$\beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} = \beta_0^{(k)} A_\tau^{2k} N^{k(\gamma-1)+1} \tag{2.11}$$

with

$$\beta_0^{(k)} A_\tau^{2k} (N^{k(\gamma-1)} - (2\delta)^{k(\gamma-1)})N \tag{2.12}$$

because N is also an entropy, and let

$$\eta_\tau^{(k)}(N,J) = \sum_{i=1}^{k} \beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i} + \beta_0^{(k)} A_\tau^{2k}(N^{k(\gamma-1)} - (2\delta)^{k(\gamma-1)})N. \tag{2.13}$$

Then the corresponding flux

$$\begin{aligned} q_\tau^{(k)}(N,J) = \frac{J}{N}\sum_{i=1}^{k} \beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i} + \beta_0^{(k)} A_\tau^{2k}(N^{k(\gamma-1)} \\ -(2\delta)^{k(\gamma-1)})J + R_\tau^{(k)}, \end{aligned} \tag{2.14}$$

where

$$R_\tau^{(k)} = \int_{-1}^{1} \sum_{i\le 2k,\ i\ odd} \theta C_{2k}^{i} A_{\tau}^{2k-i+1} y^{2k-i+1}(1-y^2)^{\lambda} N^{(2k-i+1)\theta-i+1} J^i d\,y. \tag{2.15}$$

Obviously, $q_\tau^{(k)}(N,J)$ goes to zero when J tends to zero, $0 < \beta_i^{(k)} \le 2C_{2k}^{2i}$ and

$$\frac{\partial \eta_\tau^{(k)}}{\partial J} = \sum_{i=1}^{k} 2i\beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i-1}, \tag{2.16}$$

or equivalently

$$\frac{\partial \eta_\tau^{(k)}}{\partial J} = 2\sum_{i=0}^{k-1}(i+1)\beta_{i+1}^{(k)} A_\tau^{2(k-i-1)} N^{\alpha_{i+1}^{(k)}} J^{2i+1}. \tag{2.17}$$

Now, we prove Theorem 2 by the following several Lemmas.

Lemma 3 *For any positive integer k, the viscosity-flux approximation solutions of (1.13) satisfy the entropy inequalities :*

$$\begin{aligned}
&\frac{d}{d\xi}\int_{I\!R} \sum_{i=1}^{k} \beta_i^{(k)} A^{2(k-i)} \tau^{2i} \left(N^{\alpha_i^{(k)}} J^{2i}\right)(\xi, x) \\
&\quad + \beta_0^{(k)} A^{2k} (N^{k(\gamma-1)} - (2\delta)^{k(\gamma-1)}) N(\xi, x) dx \\
&\quad \le 2\sum_{i=0}^{k-1} (i+1)\beta_{i+1}^{(k)} A^{2(k-i-1)} \tau^{2i} \int_{I\!R} \frac{N-2\delta}{N} \Big(\Upsilon N^{\alpha_{i+1}^{(k)}+1} J^{2i+1} \\
&\quad - a(x) N^{\alpha_{i+1}^{(k)}} J^{2(i+1)}\Big)(\xi, x) dx.
\end{aligned} \tag{2.18}$$

Proof of Lemma 3 Multiplying system (2.1) by $(\frac{\partial \eta_\tau^{(k)}}{\partial N}, \frac{\partial \eta_\tau^{(k)}}{\partial J})$, we may obtain the proof of Lemma 3. □

Lemma 4 *For any positive integer k,*

$$\int_{I\!R} \sum_{i=1}^{k} \tau^{2i} \left(N^{\alpha_i^{(k)}} J^{2i}\right)(\xi, x) dx \le M(\xi); \tag{2.19}$$

$$\begin{cases} \int_{I\!R} (N^{k(\gamma-1)} - (2\delta)^{k(\gamma-1)}) N(\xi, x) dx \le M(\xi), \\ \int_{I\!R} (N - (2\delta)) N^{k(\gamma-1)}(\xi, x) dx \le M(\xi); \end{cases} \tag{2.20}$$

$$\sum_{i=0}^{k-1} \int_0^{\xi} \int_{I\!R} \tau^{2i} \frac{N-2\delta}{N} \left(N^{\alpha_{i+1}^{(k)}} J^{2(i+1)}\right)(\xi, x) dx dt \le M(\xi), \tag{2.21}$$

where $M(\xi)$ is a positive bounded function ξ, which could go to infinite as ξ goes to infinite.

Proof of Lemma 4 First, by using the first equation and the third equation in (1.13), we may obtain (cf. [11,15])

$$\int_{I\!R} N(\xi, x) - 2\delta dx \le M, \quad |\Upsilon(\xi, x)|_{L^\infty(I\!R \times I\!R^+)} \le M. \tag{2.22}$$

Choose $k=1$ in (2.18), which is corresponding to the convex, physical entropy given in (2.2). By (2.9), $\alpha_1^{(1)}=-1$. So, we have from (2.18) that

$$\begin{aligned}
&\frac{d}{d\xi}\int_{I\!R}\beta_1^{(1)}\tau^2\left(N^{-1}J^2\right)(\xi,x)+\beta_0^{(1)}A^2(N^{\gamma-1}-(2\delta)^{\gamma-1})N(\xi,x)dx\\
&\quad\le 2\beta_1^{(1)}\int_{I\!R}\frac{N-2\delta}{N}\left(\Upsilon J-a(x)N^{-1}J^2\right)(\xi,x)dx\\
&\quad\le 2\beta_1^{(1)}\int_{I\!R}\frac{\Upsilon^2}{2}(N-2\delta)-\frac{1}{2}a(x)N^{-2}J^2(N-2\delta)dx,
\end{aligned}\tag{2.23}$$

which deduces

$$\int_{I\!R}\tau^2\left(N^{-1}J^2\right)(\xi,x)dx\le M,\quad \int_{I\!R}(N^{\gamma-1}-(2\delta)^{\gamma-1})N(\xi,x)dx\le M\tag{2.24}$$

and

$$\int_0^{\xi}\int_{I\!R}(N-2\delta)N^{-2}J^2dxdt\le M.\tag{2.25}$$

From the second inequality in (2.24), we may obtain

$$\int_{I\!R}(N-2\delta)N^{\gamma-1}(\xi,x)dx\le M.\tag{2.26}$$

In fact, when $\gamma>1$, we have

$$(N-2\delta)N^{\gamma-1}\le N^{\gamma}-(2\delta)^{\gamma}=(N^{\gamma-1}-(2\delta)^{\gamma-1})N+(2\delta)^{\gamma-1}(N-2\delta),\tag{2.27}$$

which deduces (2.26).

Let $k=2$ in (2.18). Since $\alpha_1^{(2)}=\gamma-2$, $\alpha_2^{(2)}=-3$, we have from (2.18) that

$$\begin{aligned}
&\frac{d}{d\xi}\int_{I\!R}\beta_1^{(2)}A^2\tau^2\left(N^{\gamma-2}J^2\right)(\xi,x)+\beta_2^{(2)}\tau^4\left(N^{-3}J^4\right)(\xi,x)\\
&\qquad+\beta_0^{(2)}A^4(N^{2(\gamma-1)}-(2\delta)^{2(\gamma-1)})N(\xi,x)dx\\
&\quad\le 2\beta_1^{(2)}A^2\int_{I\!R}\frac{N-2\delta}{N}\left(\Upsilon N^{\gamma-1}J-a(x)N^{\gamma-2}J^2\right)(\xi,x)dx\\
&\qquad+4\beta_2^{(2)}\tau^2\int_{I\!R}\frac{N-2\delta}{N}\left(\Upsilon N^{-2}J^3-a(x)N^{-3}J^4\right)(\xi,x)dx\\
&\quad\le 2\beta_1^{(2)}A^2\int_{I\!R}\frac{1}{2}(N-2\delta)N^{\gamma-1}\Upsilon^2-\frac{1}{2}(N-2\delta)a(x)N^{\gamma-3}J^2dx\\
&\qquad+4\beta_2^{(2)}\tau^2\int_{I\!R}\frac{1}{2}(N-2\delta)N^{-2}\Upsilon^2J^2-\frac{1}{2}(N-2\delta)a(x)N^{-4}J^4dx
\end{aligned}$$

$$\leq M - \beta_1^{(2)} A^2 \int_{I\!R} a(x)(N-2\delta)N^{\gamma-3}J^2 dx$$
$$-2\beta_2^{(2)}\tau^2 \int_{I\!R} a(x)(N-2\delta)N^{-4}J^4 dx \tag{2.28}$$

due to the second inequality in (2.22), (2.25) and (2.26).

Then we may obtain from (2.28) that

$$\int_{I\!R} \tau^2 \left(N^{\gamma-2}J^2\right)(\xi,x) + \tau^4\left(N^{-3}J^4\right)(\xi,x)dx \leq M(\xi), \tag{2.29}$$
$$\int_{I\!R} (N^{(2(\gamma-1)} - (2\delta)^{2(\gamma-1)})N(\xi,x)dx \leq M(\xi), \tag{2.30}$$
$$\int_0^\xi \int_{I\!R} (N-2\delta)N^{\gamma-3}J^2 dx d\xi \leq M(\xi) \tag{2.31}$$

and

$$\int_0^\xi \int_{I\!R} \tau^2(N-2\delta)N^{-4}J^4 dx d\xi \leq M(\xi). \tag{2.32}$$

Similarly to the proof of (2.26), we may obtain from (2.30) that

$$\int_{I\!R} (N-2\delta)N^{2(\gamma-1)}(\xi,x)dx \leq M(\xi). \tag{2.33}$$

Now, we complete the proof of Lemma 4 by induction. Suppose the conclusions in Lemma 4 are true for k. By simple calculations, when k is replaced by $k+1$, the right-hand side of (2.18) replaced by

$$2\sum_{i=0}^{k}(i+1)\beta_{i+1}^{(k+1)}A^{2(k-i)}\tau^{2i}\int_{I\!R}\frac{N-2\delta}{N}\left(\Upsilon N^{\alpha_{i+1}^{(k+1)}+1}J^{2i+1}\right.$$
$$\left.-a(x)N^{\alpha_{i+1}^{(k+1)}}J^{2(i+1)}\right)(\xi,x)dx$$
$$\leq \sum_{i=0}^{k}(i+1)\beta_{i+1}^{(k+1)}A^{2(k-i)}\tau^{2i}\int_{I\!R}(N-2\delta)\Upsilon^2 N^{\alpha_{i+1}^{(k+1)}+1}J^{2i}$$
$$-a(x)(N-2\delta)N^{\alpha_{i+1}^{(k+1)}-1}J^{2(i+1)}dx. \tag{2.34}$$

When $i=0$, since $\alpha_1^{(k+1)}+1 = k(\gamma-1)$, we have

$$\int_{I\!R}(N-2\delta)\Upsilon^2 N^{\alpha_1^{(k+1)}+1}dx \leq M(\xi) \tag{2.35}$$

due to (2.20).

Since

$$\begin{aligned}\alpha_{i+1}^{(k+1)}+2&=(k-i)(\gamma-1)-2(i+1)+1+2\\&=(k-i)(\gamma-1)-2i+1=\alpha_i^{(k)},\end{aligned}\tag{2.36}$$

we have for any $1\le i\le k$,

$$\sum_{i=1}^{k}(i+1)\beta_{i+1}^{(k+1)}A^{2(k-i)}\tau^{2i}\int_{I\!R}(N-2\delta)\Upsilon^2N^{\alpha_{i+1}^{(k+1)}+1}J^{2i}dx\le M(\xi)\tag{2.37}$$

due to (2.21). Finally if we replace k with $k+1$ and integrate (2.18) on $[0,\xi)$, we have

$$\begin{aligned}&\int_{I\!R}\sum_{i=1}^{k+1}\beta_i^{(k+1)}A^{2(k+1-i)}\tau^{2i}\left(N^{\alpha_i^{(k+1)}}J^{2i}\right)(\xi,x)\\&\quad+\beta_0^{(k+1)}A^{2(k+1)}(N^{(k+1)(\gamma-1)}-(2\delta)^{(k+1)(\gamma-1)})N(\xi,x)dx\\&\le M-\sum_{i=0}^{k}(i+1)\beta_{i+1}^{(k+1)}A^{2(k-i)}\tau^{2i}\int_0^\xi\int_{I\!R}a(x)(N-2\delta)N^{\alpha_{i+1}^{(k)}-1}J^{2(i+1)}dxd\xi.\end{aligned}\tag{2.38}$$

Thus the conclusions in Lemma 4 are true for any k. **Lemma 4 is proved**. □

For any $p\ge1$, we can choose k sufficiently large such that $k(\gamma-1)>p$, thus we have from (2.20) and (2.22) that

Lemma 5

$$N\in L^p(Q_L),\quad p\ge1.\tag{2.39}$$

Moreover, we have

Lemma 6

$$(N-2\delta)\frac{J}{N}\in L^2(Q_L).\tag{2.40}$$

Proof of Lemma 6 We have

$$(N-2\delta)^2\frac{J^2}{N^2}\le\begin{cases}(N-2\delta)\frac{J^2}{N^2}, & \text{for}\quad 0\le N-2\delta\le1,\\ (N-2\delta)\frac{J^2}{N}\le(N-2\delta)N^{\alpha_1^{(k)}-1}J^2, & \text{for}\quad N-2\delta\ge1,\end{cases}\tag{2.41}$$

because $\alpha_1^{(k)}=(k-1)(\gamma-1)-1>0$ for large k. Then we obtain the proof of (2.40) by using (2.25) and (2.21). **Lemma 6 is proved**. □

Lemma 7

$$\tau^2 J \to 0, \quad \tau^2\left(\frac{J^2}{N} - \delta\frac{J^2}{N^2}\right) \to 0, \quad in \quad L^2_{loc}(I\!R \times I\!R^+). \tag{2.42}$$

Proof of Lemma 7 First, by using the following estimates, given in Theorem 1, on the viscosity solutions

$$2\delta \le \rho, \quad \rho^\theta - u \le M(1+t), \quad \rho^\theta + u \le M(1+t), \tag{2.43}$$

we have

$$2\delta \le N, \quad N^\theta - \tau\frac{J}{N} \le M\left(\frac{\xi}{\tau} + 1\right), \quad N^\theta + \tau\frac{J}{N} \le M\left(\frac{\xi}{\tau} + 1\right) \tag{2.44}$$

or

$$\tau(2\delta)^\theta \le \tau N^\theta \le M(\xi + 1), \quad |\tau^2\frac{J}{N}| \le M(\xi + 1). \tag{2.45}$$

Thus we have

$$\tau^4 J^2 \le 2\tau^4(N - 2\delta)^2\frac{J^2}{N^2} + 2(2\delta)^2\tau^4\frac{J^2}{N^2}, \tag{2.46}$$

where the first term tends to zero in $L^1_{loc}(I\!R \times I\!R^+)$ and the second tends to zero in $L^\infty_{loc}(I\!R \times I\!R^+)$ due to (2.40) and the second estimate in (2.45), when $\tau \to 0, \delta \to 0$.

Moreover,

$$\tau^4(N\frac{J^2}{N^2} - \delta\frac{J^2}{N^2})^2 \le 2\tau^4(N - 2\delta)^2\frac{J^4}{N^4} + 2\tau^4(\delta)^2\frac{J^4}{N^4}, \tag{2.47}$$

where the second term tends to zero due to (2.45), and the first satisfies

$$\tau^4(N - 2\delta)^2\frac{J^4}{N^4} \le \begin{cases} \tau^4(N - 2\delta)\frac{J^4}{N^4}, & \text{for} \quad 0 \le N - 2\delta \le 1, \\ \tau^4(N - 2\delta)\frac{J^4}{N^3} \le \tau^4(N - 2\delta)N^{\alpha_2^{(k)} - 1}J^4, & \text{for} \quad N - 2\delta \ge 1, \end{cases} \tag{2.48}$$

due to $\alpha_2^{(k)} - 1 = (k-2)(\gamma - 1) - 4 > -3$ for a large k.

Since $\tau^4(N - 2\delta)\frac{J^4}{N^4}$ tends to zero due to (2.32), and $\tau^4(N - 2\delta)N^{\alpha_2^{(k)} - 1}J^4$ tends to zero due to the estimate (2.21), **so we complete the proof of Lemma 7.** □

Lemma 8

$$\frac{\varepsilon}{\tau}l(\tau, \delta)N_x^2 \in L^1_{loc}(I\!R \times I\!R^+), \quad \varepsilon\tau^3 l(\tau, \delta)J_x^2 \in L^1_{loc}(I\!R \times I\!R^+), \tag{2.49}$$

where

$$l(\tau,\delta)=\begin{cases}(2\delta)^{\gamma-2}, & \text{for } \gamma\geq 2,\\ \tau^{\frac{2(2-\gamma)}{\gamma-1}}, & \text{for } \gamma<2.\end{cases} \tag{2.50}$$

Proof of Lemma 8 By simple calculations, we have from the convex entropy given in (2.2)

$$\begin{aligned}&\frac{\varepsilon}{\tau}(N_x,J_x)\cdot\nabla^2\eta^\star(N,J)\cdot(N_x,J_x)^T\\&=\varepsilon\tau\left[\left(\frac{J^2}{N^3}+a_\tau^2\gamma N^{\gamma-2}\right)N_x^2-2\frac{J}{N^2}N_xJ_x+\frac{1}{N}J_x^2\right]\\&=\varepsilon\tau\left[\frac{J^2}{N^3}N_x^2-2\frac{J}{N^2}N_xJ_x+\frac{1}{N}J_x^2\right]+\frac{\varepsilon}{\tau}N^{\gamma-2}N_x^2\geq\frac{\varepsilon}{\tau}N^{\gamma-2}N_x^2\end{aligned} \tag{2.51}$$

and

$$\begin{aligned}&\frac{\varepsilon}{\tau}(N_x,J_x)\cdot\nabla^2\eta^\star(N,J)\cdot(N_x,J_x)^T\\&=\varepsilon\tau\left[\left(\frac{J^2}{N^3}+a_\tau^2\gamma N^{\gamma-2}\right)N_x^2-2\frac{J}{N^2}N_xJ_x+\frac{1}{\frac{J^2}{N^3}+a_\tau^2\gamma N^{\gamma-2}}\left(\frac{J}{N^2}\right)^2J_x^2\right]\\&\quad+\varepsilon\tau\left(\frac{1}{N}-\frac{1}{\frac{J^2}{N^3}+a_\tau^2\gamma N^{\gamma-2}}\left(\frac{J}{N^2}\right)^2\right)J_x^2]\\&\geq\varepsilon\tau\left(\frac{1}{N}-\frac{1}{\frac{J^2}{N^3}+a_\tau^2\gamma N^{\gamma-2}}\left(\frac{J}{N^2}\right)^2\right)J_x^2\\&=\varepsilon\tau\frac{N^\gamma}{\tau^2J^2+N^{\gamma+1}}J_x^2\end{aligned} \tag{2.52}$$

due to $a_\tau^2\gamma=\frac{1}{\tau^2}$. Thus, multiplying system (2.1) by $(\frac{\partial\eta^\star}{\partial N},\frac{\partial\eta^\star}{\partial J})$ and using the results of Lemma 4 for $k=1$, we have

$$\frac{\varepsilon}{\tau}N^{\gamma-2}N_x^2\in L^1_{loc}(I\!R\times I\!R^+),\quad \varepsilon\tau\frac{N^\gamma}{\tau^2J^2+N^{\gamma+1}}J_x^2\in L^1_{loc}(I\!R\times I\!R^+). \tag{2.53}$$

Moreover, when $\gamma\geq 2$, since $N\geq 2\delta$, we have $N^{\gamma-2}\geq(2\delta)^{\gamma-2}$; when $\gamma<2$, since $N\leq\frac{M}{\tau^{\frac{1}{\theta}}}$ due to the first estimate in (2.45), we have $N^{\gamma-2}\geq\tau^{\frac{2(2-\gamma)}{\gamma-1}}$. Thus we obtain the proof of the first part in (2.49).

Furthermore, from the estimates in (2.45), we have

$$\tau^4\frac{J^2}{N^2}\leq M,\quad \tau^2N^{\gamma-1}\leq M. \tag{2.54}$$

Then

$$\frac{N^{\gamma}}{\tau^2 J^2 + N^{\gamma+1}} \geq c\tau^2 N^{\gamma-2} \tag{2.55}$$

for a suitable constant $c > 0$. Thus we obtain the proof of the second part in (2.49). **Lemma 8 is proved.** □

To complete the proof of Theorem 2, from now on, we consider N and J with the superscript τ.

Lemma 9 *If $\varepsilon = o(\tau^2 l(\tau, \delta))$, then*

$$\frac{\varepsilon}{\tau} N^{\tau}_{xx} \quad \textit{and} \quad \varepsilon\tau J^{\tau}_{xx} + (N^{\tau} - 2\delta)\Upsilon^{\tau} - a(x)(N^{\tau} - 2\delta)U^{\tau} \tag{2.56}$$

lie in a compact set of $H^{-1}_{loc}(I\!R \times I\!R^+)$.

Proof of Lemma 9 First, for any given test function $\phi \in H^1_0(I\!R \times I\!R^+)$, if we choose ε to go zero sufficiently fast than τ and δ, more precisely, if $\varepsilon = o(\tau l(\tau, \delta))$, we have from the first estimate in (2.49) that

$$\left| \int_0^{\infty} \int_{-\infty}^{\infty} \frac{\varepsilon}{\tau} N^{\tau}_{xx} \phi dx dt \right| \leq \left(\frac{\varepsilon}{\tau l}\right)^{\frac{1}{2}} \left(\int \int_{\Omega} \frac{\varepsilon}{\tau} l (N^{\tau}_x)^2 d\Omega \right)^{\frac{1}{2}} \left(\int \int_{\Omega} (\phi_x)^2 d\Omega \right)^{\frac{1}{2}} \to 0, \tag{2.57}$$

where Ω is any bounded set in $I\!R \times I\!R^+$. Then $\frac{\varepsilon}{\tau} N^{\tau}_{xx}$ is compact in $H^{-1}_{loc}(I\!R \times I\!R^+)$. Similarly, if $\varepsilon = o(\tau^2 l(\tau, \delta))$, we may prove that $\varepsilon\tau J^{\tau}_{xx}$ is compact in $H^{-1}_{loc}(I\!R \times I\!R^+)$.

Second, from the estimates in (2.39) and (2.40), $(N^{\tau} - 2\delta)\Upsilon^{\tau} - a(x)(N^{\tau} - 2\delta)U^{\tau}$ are uniformly bounded in $L^2_{loc}(I\!R \times I\!R^+)$, and so compact in $H^{-1}_{loc}(I\!R \times I\!R^+)$ by using the Sobolev imbedding theorem. **Lemma 9 is proved.** □

Proof of Theorem 2 Since the conclusions in (2.39),(2.40),(2.42) and (2.56), we may apply the Div-Curl lemma (cf. [16,19]) to the following pairs of functions

$$(N^{\tau}, (N^{\tau} - 2\delta)U^{\tau}), \quad (\tau^2 J^{\tau}, \tau^2 (N^{\tau}(U^{\tau})^2 - \delta(U^{\tau})^2) + S(N^{\tau}, \delta)) \tag{2.58}$$

to obtain (cf. [6,12,15])

$$\overline{(N^{\tau})^{\gamma+1}} = N\overline{(N^{\tau})^{\gamma}}, \tag{2.59}$$

where we used

$$S(N^{\tau}, \delta)) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t) dt \to \overline{(N^{\tau})^{\gamma}}, \quad \text{as } \delta, \tau \to 0, \tag{2.60}$$

N is the weak limit of N^{τ} and $\overline{f(u^{\tau})}$ denotes the weak limit of $f(u^{\tau})$ in $L^p_{loc}(I\!R \times I\!R^+)$. Thus the technique, given in [6,14], deduces the pointwise convergence of N^{τ} in $L^p_{loc}(I\!R \times I\!R^+)$, $p \geq 1$.

Since the uniform estimate (2.40) of $\frac{N^\tau(x,\xi)-2\delta}{N^\tau(x,\xi)} J^\tau(x,\xi)$ in $L^2(Q_L)$, there exists a function $J(x,\xi) \in L^2(Q_L)$ such that the second limit in (1.16) is true.

Since the right-hand side of the third equation in (1.13), $(N^\tau - 2\delta) - b(x)$ is uniformly bounded in $L^1_{loc}(I\!R \times I\!R^+)$, and so compact in $W^{-1,\alpha}_{loc}(I\!R \times I\!R^+)$ by the Sobolev imbedding theorem, moreover, at the same time, the left-hand side of Υ^τ_x is bounded in $W^{-1,\infty}_{loc}(I\!R \times I\!R^+)$, thus Υ^τ_x is compact in $H^{-1}_{loc}(I\!R \times I\!R^+)$.

From the third equation and the first equation in (1.13), we have

$$\Upsilon^\tau_t = \int_{-\infty}^{x} N^\tau_t dx = \varepsilon N^\tau_x - (N^\tau - 2\delta)U^\tau, \tag{2.61}$$

which is compact in $H^{-1}_{loc}(I\!R \times I\!R^+)$ by (2.40) and the first estimate in (2.49). Then we may apply the Div-Curl lemma to the following pairs of functions

$$(0, \Upsilon^\tau), \quad (\Upsilon^\tau, 0) \tag{2.62}$$

to obtain $(\overline{\Upsilon^\tau})^2 = \overline{(\Upsilon^\tau)^2}$, which deduces the pointwise convergence of Υ^τ in $L^\infty_{loc}(I\!R \times I\!R^+)$.

Letting $\varepsilon, \delta, \tau$ in (1.13) go to zero, we may prove that the limit (N, J, Υ) satisfies the drift-diffusion equations (1.14) in the sense of distributions. Thus, **we complete the proof of Theorem 2.** □

Acknowledgements The authors thank the anonymous reviewers for their valuable comments. The third author (Y. Lu) is supported by the NSFC grant No. LY20A010023, a Qianjiang professorship of Zhejiang Province of China and a Humboldt renewed research fellowship of Germany, who is very grateful to the colleagues in University of Wuerzburg for their warm hospitality.

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