An arbitrary Lagrangian-Eulerian discontinuous Galerkin method for conservation laws: Entropy stability

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Abstract In Klingenberg, Schnücke and Xia (Math. Comp. Available via `http://dx.doi.org/10.1090/mcom/3126`) an arbitrary Lagrangian-Eulerian Discontinuous Galerkin (ALE-DG) method to solve conservation laws has been developed and analyzed. In this paper, the ALE-DG method will be briefly presented. Furthermore, the semi-discrete method will be discretized by the so-called ϑ-method. The ϑ-method is a generalization of the forward or backward Euler step. In particular, the method degenerates to the forward Euler step for $\vartheta = 0$ and to the backward Euler step for $\vartheta = 1$. The corresponding fully-discrete ϑ-P^k-ALE-DG method for scalar conservation laws will be analyzed with respect to entropy stability, where P^k denotes the space of polynomials of degree k which is used on a reference cell. The main results are a cell entropy inequality for the fully-discrete ϑ-P^k-ALE-DG method with respect to the square entropy function, when ϑ has a lower bound given by a mesh parameter depending constant, and a cell entropy inequality for the fully-discrete ϑ-P^0-ALE-DG method with respect to the Kružkov entropy functions.

Key words: Arbitrary Lagrangian-Eulerian discontinuous Galerkin method, conservation laws, entropy stability.

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1 Introduction

The present paper investigates an arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method to solve one dimensional conservation laws

$$\partial_t u + \partial_x f(u) = 0 \text{ in } \Omega \times (0,T), \qquad u(x,0) = u_0(x) \text{ in } \Omega \tag{1}$$

with periodic boundary conditions. The function $u_0 : \Omega \to \mathbb{R}$ is sufficiently smooth and compactly supported and the flux function $f : \mathbb{R} \to \mathbb{R}$ is at least one times continuously differentiable. This method was introduced and analyzed by Klingenberg et al. in [9].

The Arbitrary Lagrangian-Eulerian (ALE) approach has been rigorously described by Donea et al. in [5]. It is a kind of compromise between the Lagrangian and Eulerian approach. These two approaches are the two commonly used descriptions of motions in computational fluid dynamics. In the Lagrangian approach is the motion described by mesh generating points, which are moving with the fluid. This approach could produce distortions in the mesh. The distortions lead to numerical artifacts. This has been discussed by Donea et al. in [5]. The Eulerian description, in contrast to the Lagrangian approach, based on a fixed static mesh. Hence, numerical artifacts by geometric distortions are avoided in this approach. Nevertheless, a drawback of the Eulerian approach is the loss of specific properties of the physical model. For instance Springel discussed in [11] the lack of the Galilean invariance in cosmological hydrodynamical simulations. Furthermore, in the same paper, Springel showed by numerical simulations with a second order finite volume moving mesh method that the Galilei invariance is preserved when the mesh moves almost with the fluid.

The main idea of the ALE approach is to described the fluid motion almost as in the Lagrangian approach and if distortions with a destabilizing effect occur, the description of motion moves closer to the Eulerian approach. The implementation and mathematical description of the ALE approach ensure by a mapping which connects the physical domain with a suitable reference configuration. The mapping provides a description of the grid velocity field. In addition, the test function space is defined by the mapping, in the context of Galerkin methods. In general the mapping is globally defined. This is quite unattractive for discontinuous Galerkin methods, since these methods lose their local structure, when a global defined ALE mapping is used. Furthermore, if the ALE approach is combined with numerical schemes, which are derived by the method of lines approach, and the Jacobi matrix of the mapping depends on spatial variables a geometric error could appear by an unsuitable choice of the time integration method. This geometric error destabilizes the numerical scheme. The geometric error does not appear, if the ALE method satisfies the geometric conservation law (GCL). The error and the GCL have been analyzed by Guillard and Farhat in [7].

The ALE-DG method in [9] is derived by local affine linear ALE mappings. Hence, the method has a local structure like the DG methods for static grids and it has been proven that the method to solve one dimensional conservation laws satisfies the GCL for any first order time discretization method or high order single step method in which the stage order is equal or higher than first order. Moreover, for the semi-discrete method a cell entropy inequality with respect to the square entropy function and a priori error estimates have been proven. For the time integration the total variation diminishing (TVD) Runge-Kutta methods, which were introduced by Shu in [10], are adopt. Hence, the ALE-DG method degenerates to the Runge-Kutta discontinuous Galerkin (RK-DG) method on a static non moving mesh. The RK-DG method was developed by Cockburn, Shu et al. in a series of papers [2, 3, 4] and is designed for the Eulerian description of fluid motion. Over the last decades this method has become quite popular in computational fluid dynamics. The TVD Runge-Kutta methods are convex combinations of the forward Euler step. Hence, a stability result for the forward Euler step could be extent by an adequate time step regulation. This feature of the TVD Runge-Kutta methods has been proven by Gottlieb and Shu in [6]. According to this property of the TVD Runge-Kutta methods it has been proven that the fully-discrete ALE-DG method satisfies a local maximum principle and the average values of the ALE-DG solution are total variation stable.

The next step is the analysis of the fully-discrete ALE-DG method with respect to entropy stability. Unfortunately, Chavent and Cockburn proved in [1] that the P^1-DG method to solve scalar conservation laws with a linear flux function on static grids is unconditionally $\mathrm{L}^\infty\left(0,T;\mathrm{L}^2\left(0,1\right)\right)$-unstable for any CFL restriction, when the forward Euler step is used. Hence, we cannot expect entropy stability for the forward Euler P^k-ALE-DG method, if $k \geq 1$. In particular, the entropy stability for the P^k-ALE-DG method with a TVD Runge-Kutta cannot be proven by Gottlieb and Shu's theorem and needs to be investigated separated from the forward Euler P^k-ALE-DG method.

Jiang and Shu analyzed in [8] fully-discrete DG methods with respect to entropy stability. They applied the ϑ-method for the time integration of the semi-discrete DG method and proved for $\frac{1}{2} \leq \vartheta \leq 1$ and polynomials of arbitrary degree a cell entropy inequality with respect to the square entropy function. The ϑ-method for the ordinary differential equation $\partial_t u = \mathscr{L}(u,t)$ is given by

$$u^{n+1} = u^n + \Delta t \mathscr{L}\left(u^{n+\vartheta}, t_{n+\vartheta}\right), \tag{2a}$$

$$u^{n+\vartheta} := (1-\vartheta)\, u^n + \vartheta u^{n+1}, \quad t_{n+\vartheta} := (1-\vartheta)\, t_n + \vartheta t_{n+1}. \tag{2b}$$

In this paper, the ϑ-method is applied for the time integration of the semi-discrete ALE-DG method and the corresponding ϑ-P^k-ALE-DG method is analyzed with respect to entropy stability in the sense of the square entropy and the Kružkov entropy functions.

This paper is organized as follows: It starts with a briefly presentation of the ALE-DG method in section 2. Afterward, in the same section, two entropy inequalities are proven for the fully-discrete method. It will be completed with some concluding remarks in section 3.

2 The ALE-DG method

This section is started with a summary of the main ingredients to describe the ALE-DG method. Let $\Omega \subseteq \mathbb{R}$ be an open interval. It need to be assumed that it exists for any time level $t = t_n$ a partition of the domain Ω with

$$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{K_j^n}, \quad K_j^n := \left(x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}}\right), \quad \Delta_j^n := x^n_{j+\frac{1}{2}} - x^n_{j-\frac{1}{2}}.$$

This assumption enables to define time-dependent straight lines for all $j = 1, \ldots, N$

$$x_{j-\frac{1}{2}}(t) := x^n_{j-\frac{1}{2}} + \omega^n_{j-\frac{1}{2}}(t - t_n), \quad \omega^n_{j-\frac{1}{2}} := \frac{1}{\Delta t}\left(x^{n+1}_{j-\frac{1}{2}} - x^n_{j-\frac{1}{2}}\right),$$

where Δt is specified by the partition of the time interval $(0,T)$. The straight lines provide for any $t \in [t_n, t_{n+1}]$ and all $j = 1, \ldots, N$ time-dependent cells

$$K_j(t) := \left(x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)\right), \quad \Delta_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t).$$

The local grid velocity of the ALE-DG method is for all $t \in [t_n, t_{n+1})$ and $x \in K_j(t)$ given by

$$\omega(x,t) = \frac{1}{\Delta_j(t)}\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)\left(x - x_{j-\frac{1}{2}}(t)\right) + \omega^n_{j-\frac{1}{2}}. \quad (3)$$

The time-dependent cells can be connected with a reference cell $[0,1]$ by an affine linear mapping

$$\chi_j : [0,1] \to \overline{K_j(t)}, \quad \xi \mapsto \chi_j(\xi, t) := \Delta_j(t)\,\xi + x_{j-\frac{1}{2}}(t).$$

This mapping enables to define the following time-dependent finite dimensional test function space

$$\mathscr{V}_h(t) := \left\{ v \in \mathrm{L}^2(\Omega) : \; (v \circ \chi_j) \in P^k([0,1]) \right\},$$

where $P^k([0,1])$ denotes the space of polynomials in $[0,1]$ of degree at most k. The test functions $v \in \mathscr{V}_h(t)$ are discontinuous in the points $x_{j-\frac{1}{2}}(t)$. Hence, the limits in these points are defined by

$$v^{\pm}_{j-\frac{1}{2}} := \lim_{\varepsilon \to 0} v\left(x_{j-\frac{1}{2}}(t) \pm \varepsilon, t\right).$$

Finally, it should be mentioned that in [9] for sufficiently smooth functions $u : \Omega \times (0,T) \to \mathbb{R}$ the following ALE transport equation has been proven

$$\frac{d}{dt}\int_{K_j(t)} uv\,dx = \int_{K_j(t)} (\partial_t u)\,v\,dx + \int_{K_j(t)} (\partial_x(\omega u))\,v\,dx, \quad \forall v \in \mathscr{V}_h(t). \tag{4}$$

2.1 The semi-discrete ALE-DG discretization

At the beginning, the solution u of the problem (1) is approximated by the function

$$u_h(x,t) = \sum_{\ell=0}^{k} u_\ell^j(t)\,\phi_\ell^j(x,t) \in \mathscr{V}_h(t), \quad \forall t \in [t_n, t_{n+1}) \text{ and } x \in K_j(t),$$

where $\left\{\phi_0^j(x,t),...,\phi_k^j(x,t)\right\}$ is a basis of the space $\mathscr{V}_h(t)$ in the cell $K_j(t)$. The coefficients $u_0^j(t),...,u_k^j(t)$ are the unknowns of the ALE-DG method. In order to determine these coefficients, the equation (1) is multiplied by a test function $v \in \mathscr{V}_h(t)$ and the transport equation (4) as well as the integration by parts formula are applied. In general the function u_h is discontinuous in the cell interface points $x_{j-\frac{1}{2}}(t)$. Hence, in these points, the following Lax-Friedrichs flux is applied

$$\widehat{g}\left(\omega,u^-,u^+\right) := \widehat{g}_+\left(\omega,u^-\right) - \widehat{g}_-\left(\omega,u^+\right), \quad \widehat{g}_\pm(\omega,u) := \frac{1}{2}\left(\lambda_j(t)\,u \pm g(\omega,u)\right)$$

where $g(\omega,u) := f(u) - \omega u$ and

$$\lambda_j(t) := \max\left\{\left|\partial_u g(\omega(x,t),u)\right| : \ u \in [m,M],\ x \in K_j(t)\right\} \tag{5}$$

with $m := \min_{x\in\Omega} u_0(x)$ and $M := \max_{x\in\Omega} u_0(x)$. Finally, the semi-discrete ALE-DG method can be summarized as:

Problem 1 (Semi-discrete ALE-DG method). Seek a function $u_h \in \mathscr{V}_h(t)$, such that for all $v \in \mathscr{V}_h(t)$ and $j = 1,\cdots,N$ holds

$$\begin{aligned} 0 = {} & \frac{d}{dt}\int_{K_j(t)} u_h v\,dx - \int_{K_j(t)} g(\omega,u_h)(\partial_x v)\,dx \\ & + \widehat{g}\left(\omega^n_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} - \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}}. \end{aligned} \tag{6}$$

The time discretization method for the problem (6) needs to be chosen carefully, since according to Guillard and Farhat [7] the geometric conservation needs to be respected. However, in [9] it has been proven that the ALE-DG method satisfies the geometric conservation law for any single step method with stage order equal or higher than first order. Hence, there is a lot of freedom in the choice of a time discretization method for the ALE-DG method.

The capability of the ALE-DG method with a third order TVD Runge-Kutta method for problems with a compressible flow have been shown by numerical experiments for the inviscid Burgers' equation and Euler equations in [9]. In particular, it has been shown numerically that the method is able to reach the optimal rate of convergence and can handle strong singularities like shock waves.

2.2 Cell entropy inequalities

In this section, cell entropy inequalities for the fully-discrete ϑ-P^k-ALE-DG method are discussed, where the ϑ-P^k-ALE-DG results from a discretization of the semi-discrete formulation (6) with the ϑ-method (2). The corresponding method can be written on the reference cell $(0,1)$ as:

Problem 2 (The ϑ-P^k-ALE-DG method). For a given function $\widehat{u}_h^n \in \mathscr{V}_h(t_n)$ seek a function $\widehat{u}_h^{n+1} \in \mathscr{V}_h(t_{n+1})$, such that for all $\widehat{v} \in P^k([0,1])$ and $j = 1,\cdots,N$ holds

$$
\begin{aligned}
0 = &\int_0^1 \Delta_j^{n+1}\widehat{u}_h^{n+1}\widehat{v}\,d\xi - \int_0^1 \Delta_j^n \widehat{u}_h^n v\,d\xi - \int_0^1 g\left(\widehat{\omega}\left(t_{n+\vartheta}\right),\widehat{u}_h^{n+\vartheta}\right)\left(\partial_\xi \widehat{v}\right)d\xi \\
&+\widehat{g}\left(\omega_{j+\frac{1}{2}}^n,\widehat{u}_{h,j+\frac{1}{2}}^{n+\vartheta,-},\widehat{u}_{h,j+\frac{1}{2}}^{n+\vartheta,+}\right)\widehat{v}_{j+\frac{1}{2}}^{-} - \widehat{g}\left(\omega_{j-\frac{1}{2}}^n,\widehat{u}_{h,j-\frac{1}{2}}^{n+\vartheta,+},\widehat{u}_{h,j-\frac{1}{2}}^{n+\vartheta,+}\right)\widehat{v}_{j-\frac{1}{2}}^{+},
\end{aligned}
$$

where $\widehat{u}_h := u_h \circ \chi_j$, $\widehat{v} := v \circ \chi_j$, $\widehat{\omega} = \omega \circ \chi_j$ and $t_{n+\vartheta}$ is defined as in (2b).

At first, a cell entropy inequality with respect to the square entropy function $\eta(u) = \frac{1}{2}u^2$ is proven. The proof based on the upcoming discrete variation on the ALE transport equation (4).

Lemma 1. *Let $u : [0,1]\times[0,T] \to \mathbb{R}$ be a sufficiently smooth function and $\eta(u) = \frac{1}{2}u^2$. Then holds*

$$
\begin{aligned}
&\int_0^1 \Delta_j^{n+1}u^{n+1}u^{n+\vartheta}\,d\xi - \int_0^1 \Delta_j^n u^n u^{n+\vartheta}\,d\xi \\
=& \int_0^1 \Delta_j^{n+1}\eta\left(u^{n+1}\right)d\xi - \int_0^1 \Delta_j^n \eta\left(u^n\right)d\xi \\
&+\Delta t \int_0^1 \left(\partial_\xi \widehat{\omega}\left(t_{n+\vartheta}\right)\right)\eta\left(u^{n+\vartheta}\right)d\xi \\
&+\int_0^1 \left[\vartheta^2 \Delta_j^n - (1-\vartheta)^2 \Delta_j^{n+1}\right]\eta\left(u^{n+1}-u^n\right)d\xi,
\end{aligned}
\tag{7}
$$

where $u^{n+\vartheta}$ and $t_{n+\vartheta}$ are defined as in (2b).

Proof. First of all, by a simple algebraic manipulation follows

$$
\begin{aligned}
&\int_0^1 \Delta_j^{n+1}u^{n+1}u^{n+\vartheta}\,d\xi - \int_0^1 \Delta_j^n u^n u^{n+\vartheta}\,d\xi \\
=& \int_0^1 \Delta_j^{n+1}\eta\left(u^{n+1}\right)d\xi - \int_0^1 \Delta_j^n \eta\left(u^n\right)d\xi
\end{aligned}
$$

$$+\int_0^1 \left(\Delta_j^{n+1}-\Delta_j^n\right)\left((2\vartheta-1)\,\eta\left(u^{n+1}\right)+(1-\vartheta)\,u^{n+1}u^n\right)d\xi$$
$$+(2\vartheta-1)\int_0^1 \Delta_j^n\eta\left(u^{n+1}-u^n\right)d\xi. \tag{8}$$

Next, it should be noted that $\partial_x\omega\left(t_{n+\vartheta}\right)\Delta_j^{n+\vartheta}=\partial_\xi\widehat{\omega}\left(t_{n+\vartheta}\right)$. Hence, by the formula (3) follows

$$\Delta t\partial_\xi\widehat{\omega}\left(t_{n+\vartheta}\right)=\Delta t\left(\omega_{j+\frac{1}{2}}^n-\omega_{j-\frac{1}{2}}^n\right)=\left(\Delta_j^{n+1}-\Delta_j^n\right). \tag{9}$$

Moreover, the identity (9) and the integration by substitution formula provide

$$\int_0^1 \left(\Delta_j^{n+1}-\Delta_j^n\right)\left((2\vartheta-1)\,\eta\left(u^{n+1}\right)+(1-\vartheta)\,u^{n+1}u^n\right)d\xi$$
$$-(1-\vartheta)^2\int_0^1\left(\Delta_j^{n+1}-\Delta_j^n\right)\eta\left(u^{n+1}-u^n\right)d\xi$$
$$=\Delta t\int_0^1\left(\partial_\xi\,\omega\left(t_{n+\vartheta}\right)\right)\eta\left(u^{n+\vartheta}\right)d\xi. \tag{10}$$

Finally, the discrete transport equation (7) follows by combining the equations (8) and (10). □

The discrete transport equation (7) provides only a cell entropy inequality, if it can be ensured that

$$\int_0^1\left[\vartheta^2\Delta_j^n-(1-\vartheta)^2\Delta_j^{n+1}\right]\eta\left(u^{n+1}-u^n\right)d\xi\geq 0. \tag{11}$$

In fact it follows from a simple calculation that ϑ needs to satisfy

$$0<\frac{\sqrt{\Delta_j^{n+1}}}{\sqrt{\Delta_j^n}+\sqrt{\Delta_j^{n+1}}}\leq\vartheta\leq 1. \tag{12}$$

It should be noted that on a static mesh the equation Δ_j^n=Δ_j^{n+1} is satisfied. Hence, in this case, (12) yields the restriction $\frac{1}{2}\leq\vartheta\leq 1$. This is the same restriction as in Jiang and Shu's paper [8]. However, the restriction (12) ensures the upcoming entropy inequality with respect to the square entropy function for the ϑ-P^k-ALE-DG method.

Proposition 1. *Suppose* ϑ *satisfies the restriction* (12). *Then the solution of the* ϑ-P^k*-ALE-DG method satisfies with respect to the square entropy function* $\eta\left(u\right)=\frac{1}{2}u^2$ *the cell entropy inequality*

$$0\geq\int_{K_j^{n+1}}\eta\left(u_h^{n+1}\right)dx-\int_{K_j^n}\eta\left(u_h^n\right)dx$$

$$+\Delta t\left(H\left(\omega^n_{j+\frac{1}{2}},u^{n+\vartheta,-}_{h,j+\frac{1}{2}},u^{n+\vartheta,+}_{h,j+\frac{1}{2}}\right)-H\left(\omega^n_{j-\frac{1}{2}},u^{n+\vartheta,-}_{h,j-\frac{1}{2}},u^{n+\vartheta,+}_{h,j-\frac{1}{2}}\right)\right),$$

where $H(\omega,u^-,u^+):=-\int^{u^-}f(u)\,du+\omega\eta(u^-)+\widehat{g}(\omega,u^-,u^+)\,u^-$. *Furthermore, holds* $\left\|u^n_h\right\|_{\mathrm{L}^2(\Omega)}\leq\left\|u^0_h\right\|_{\mathrm{L}^2(\Omega)}$.

Proof. The ϑ-P^k-ALE-DG can be written as follows

$$\begin{aligned}0=&\int_0^1\Delta^{n+1}_j\eta\left(\widehat{u}^{n+1}_h\right)d\xi-\int_0^1\Delta^n_j\eta\left(\widehat{u}^n_h\right)d\xi\\&+\int_0^1\left[\vartheta^2\Delta^n_j-(1-\vartheta)^2\Delta^{n+1}_j\right]\eta\left(\widehat{u}^{n+1}_h-\widehat{u}^n_h\right)d\xi\\&-\Delta t\int_0^1 f\left(\widehat{u}^{n+\vartheta}_h\right)\left(\partial_\xi\widehat{u}^{n+\vartheta}_h\right)d\xi+\Delta t\int_0^1\partial_\xi\left(\widehat{\omega}\left(t_{n+\vartheta}\right)\eta\left(\widehat{u}^{n+\vartheta}_h\right)\right)d\xi\\&+\Delta t\left(\widehat{g}\left(\omega^n_{j+\frac{1}{2}},\widehat{u}^{n+\vartheta,-}_{h,j+\frac{1}{2}},\widehat{u}^{n+\vartheta,+}_{h,j+\frac{1}{2}}\right)\widehat{u}^{n+\vartheta,-}_{h,j+\frac{1}{2}}-\widehat{g}\left(\omega^n_{j-\frac{1}{2}},\widehat{u}^{n+\vartheta,-}_{h,j-\frac{1}{2}},\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}\right)\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}\right),\end{aligned}$$

when the test function $\widehat{v}=\widehat{u}^{n+\vartheta}_h$ and the discrete transport equation (7) are applied. The next steps ensues similar as in the proof of the entropy inequality for the semi-discrete ALE-DG method in [9]. First of all, the integration by substitution formula and the function $H(\omega,u^-,u^+)$ are applied to write the method as

$$\begin{aligned}0\geq&\int_0^1\Delta^{n+1}_j\eta\left(\widehat{u}^{n+1}_h\right)d\xi-\int_0^1\Delta^n_j\eta\left(\widehat{u}^n_h\right)d\xi+\Theta^{n+\vartheta}_{j-\frac{1}{2}}\\&+\Delta t\left(H\left(\omega^n_{j+\frac{1}{2}},\widehat{u}^{n+\vartheta,-}_{h,j+\frac{1}{2}},\widehat{u}^{n+\vartheta,+}_{h,j+\frac{1}{2}}\right)-H\left(\omega^n_{j-\frac{1}{2}},\widehat{u}^{n+\vartheta,-}_{h,j-\frac{1}{2}},\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}\right)\right),\end{aligned}\quad(13)$$

where

$$\Theta^{n+\vartheta}_{j-\frac{1}{2}}:=\Delta t\left(g\left(\omega^n_{j-\frac{1}{2}},\theta^{n+\vartheta}_j\right)-\widehat{g}\left(\omega^n_{j-\frac{1}{2}},\widehat{u}^{n+\vartheta,-}_{h,j-\frac{1}{2}},\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}\right)\right)[\![\widehat{u}^{n+\vartheta}_h]\!]_{j-\frac{1}{2}}$$

with a value $\theta^{n+\vartheta}_j$ between $\widehat{u}^{n+\vartheta,-}_{h,j-\frac{1}{2}}$ and $\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}$ and $[\![\widehat{u}^{n+1}_h]\!]_{j-\frac{1}{2}}:=\widehat{u}^{n+\vartheta,+}_{h,j-\frac{1}{2}}-\widehat{u}^{n+\vartheta,-}_{h,j-\frac{1}{2}}$. It should be noted that (13) is an inequality, since it has been assumed that ϑ satisfies the restriction (12) and thus the inequality (11) is satisfied, too. Moreover, the term $\Theta^{n+\vartheta}_{j-\frac{1}{2}}$ is non-negative, since the method is considered with a monotone and consistent numerical flux. Next, the inequality (13) is transformed to the physical domain by the integration by substitution formula. The inequality on physical domain provides the desired cell entropy inequality. Finally a summation of the cell entropy inequality over all cells yields the L^2-stability, since we consider the problem (1) with periodic boundary conditions. □

The result in Proposition 1 holds merely for the square entropy function. Nevertheless, for the piecewise constant ϑ-P^0-ALE-DG method an entropy inequality with respect to the Kružkov entropy functions can be proven. Henceforth, the upcoming notation is used

$$\overline{u}_j(t) := \frac{1}{\Delta_j(t)} \int_{K_j(t)} u_h(t)\, dx, \quad \Delta_j^{n+1-\vartheta} := \vartheta \Delta_j^n + (1-\vartheta)\Delta_j^{n+1}. \tag{14}$$

The following identity follows from a simple calculation

$$\Delta_j^{n+1}\overline{u}_j^{n+1} - \Delta_j^n \overline{u}_j^n = \Delta_j^{n+1-\vartheta}\left(\overline{u}_j^{n+1} - \overline{u}_j^n\right) + \Delta t\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\overline{u}_j^{n-\vartheta}, \tag{15}$$

since $\Delta_j^{n+1} - \Delta_j^n = \Delta t\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)$. The equation (15) provides the upcoming formulation of the ϑ-P^0-ALE-DG method

$$\begin{aligned} 0 = \overline{u}_j^{n+1} - \overline{u}_j^n &+ \frac{\Delta t}{\Delta_j^{n+1-\vartheta}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, \overline{u}_j^{n+\vartheta}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, \overline{u}_{j-1}^{n+\vartheta}\right)\right) \\ &- \frac{\Delta t}{\Delta_j^{n+1-\vartheta}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_{j+1}^{n+\vartheta}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_j^{n+\vartheta}\right)\right). \end{aligned} \tag{16}$$

In the following, an entropy inequality with respect to the Kružkov entropy functions $\eta_\ell(u) := |u-\ell|$, $\ell \in \mathbb{R}$, is presented for the method (16).

Proposition 2. *Suppose the CFL condition*

$$\left(\lambda_j^{n+\vartheta} + \frac{1}{2}\max_{t\in[t_n,t_{n+1}]}\left\{\left|\partial_x\omega(x,t)\Delta_j(t)\right| :\ x \in K_j(t)\right\}\right)\frac{\Delta t}{\Delta_j^{n+1-\vartheta}} \le 1, \tag{17}$$

where the parameters $\lambda_j^{n+\vartheta}$ *and* $\Delta_j^{n+1-\vartheta}$ *are given by* (5) *and* (14) *respectively. Then the solution of the scheme* (16) *satisfies the cell entropy inequality*

$$\eta_\ell\left(\overline{u}_j^{n+1}\right) - \eta_\ell\left(\overline{u}_j^n\right) + \frac{\Delta t}{\Delta_j^{n+1-\vartheta}}\left(H_\ell\left(\omega, \overline{u}_j^{n+\vartheta}, \overline{u}_{j+1}^{n+\vartheta}\right) - H_\ell\left(\omega, \overline{u}_{j-1}^{n+\vartheta}, \overline{u}_j^{n+\vartheta}\right)\right) \le 0,$$

where $\eta_\ell(u) := |u-\ell|$, $\ell \in \mathbb{R}$, *are the Kružkov entropy functions and for all* $a,b \in [m,M]$, $H_\ell(\omega,a,b)$ *is given by*

$$\begin{aligned} H_\ell(\omega,a,b) := &\frac{1}{2}\int_\ell^a \eta_\ell'(v)\left(\lambda_j^n + f'(v) - \omega_{j-\frac{1}{2}}^n\right) dv \\ &- \frac{1}{2}\int_\ell^b \eta_\ell'(v)\left(\lambda_j^n - f'(v) + \omega_{j+\frac{1}{2}}^n\right) dv. \end{aligned}$$

Proof. The integration by parts formula and the convexity of the functions η_ℓ provide

$$\begin{aligned} \left(\overline{u}_j^{n+1} - \overline{u}_j^n\right)\eta_\ell'\left(\overline{u}_j^{n+1}\right) \ge\ &\eta_\ell\left(\overline{u}_j^{n+1}\right) - \eta_\ell\left(\overline{u}_j^n\right) \\ &+ \int_{\overline{u}_j^{n+\vartheta}}^{\overline{u}_j^{n+1}}\left(v - \overline{u}_j^{n+\vartheta}\right)\eta_\ell''(v)\, dv. \end{aligned} \tag{18}$$

Next, the scheme (16) is multiplied by $\eta_\ell\left(\overline{u}_j^{n+1}\right)$ and the integration by parts formula, the functions $H_\ell\left(\omega,a,b\right)$, (9) and (18) are applied. This results in

$$\begin{aligned} 0 \geq\ & \eta_\ell\left(\overline{u}_j^{n+1}\right) - \eta_\ell\left(\overline{u}_j^{n}\right) + \Theta_j^{n+\vartheta} \\ & + \frac{\Delta t}{\Delta_j^{n+1}}\left(H_\ell\left(\omega, \overline{u}_j^{n+\vartheta}, \overline{u}_{j+1}^{n+\vartheta}\right) - H_\ell\left(\omega, \overline{u}_{j-1}^{n+\vartheta}, \overline{u}_j^{n+\vartheta}\right)\right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Theta_j^{n+\vartheta} :=\ & \left(1 - \frac{\Delta t}{\Delta_j^{n+1-\vartheta}} C\left(\lambda_j^{n+\vartheta}, \omega\left(t_{n+\vartheta}\right)\right)\right) \int_{\overline{u}_j^{n+\vartheta}}^{\overline{u}_j^{n+1}} \left(v - \overline{u}_j^{n+\vartheta}\right) \eta_\ell''\left(v\right) dv \\ & + \int_{\overline{u}_{j-1}^{n+\vartheta}}^{\overline{u}_j^{n+1}} \left(\widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, v\right) - \widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_{j-1}^{n+\vartheta}\right)\right) \eta_\ell''\left(v\right) dv \\ & \int_{\overline{u}_{j+1}^{n+\vartheta}}^{\overline{u}_j^{n+1}} \left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, v\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_{j+1}^{n+\vartheta}\right)\right) \eta_\ell''\left(v\right) dv, \end{aligned} \tag{20}$$

$$C\left(\lambda_j^{n+\vartheta}, \omega\left(t_{n+\vartheta}\right)\right) := \lambda_j^{n+\vartheta} + \frac{1}{2}\partial_x \omega\left(t_{n+\vartheta}\right) \Delta_j\left(t_{n+\vartheta}\right).$$

The inequality (19) is almost the desired cell entropy inequality. Nevertheless, it need to be ensured that the term $\Theta_j^{n+\vartheta}$ is non-negative. In fact the integrals in equation (20) are non-negative, since the functions η_ℓ are convex and the functions $\widehat{g}_\pm\left(\omega,u\right)$ are monotone increasing. It should be noted that η_ℓ'' are Dirac delta distributions. However, the products in all the integrals are well defined, since the delta distributions are multiplied with continuous functions. Furthermore, the term in front of the first integral in equation (20) is non-negative by the CFL condition (17). Hence, the term $\Theta_j^{n+\vartheta}$ is non-negative and the inequality (19) yields the desired cell entropy inequality. □

3 Conclusions

In this paper, an ALE-DG method for solving scalar conservation laws has been presented. A cell entropy inequality with respect to the Kružkov entropy functions has been proven for the fully discrete ϑ-P^0-ALE-DG method. Likewise, a cell entropy inequality with respect to the square entropy function has been proven for the fully discrete ϑ-P^k-ALE-DG method, when ϑ satisfies the restriction (12). Cell entropy inequalities are very useful in the analysis of numerical methods. Besides the convergence to the physical relevant solution, cell entropy inequalities provide certain stability properties and statements about the qualitative behavior of a numerical method. For instance, a cell entropy inequality with respect to the square entropy function provides the L^2-stability of the method and is the key to a priori error estimates. Hence, in future works, it is of particular interest to prove cell entropy inequalities or at least the L^2-stability for the ALE-DG method when other time integration methods like the explicit third order TVD-RK methods are adopted.

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