Global entropy solutions for systems modelling polymer flooding in enhanced oil recovery

Christian Klingenberg, Yun-guang Lu, and Changfeng Xue

ABSTRACT. In this paper, we obtain the existence of global entropy solutions for the Cauchy problem of the nonstrictly hyperbolic systems modelling polymer flooding in enhanced oil recovery, under a more flexible condition on the function $\beta(T)$, which models the adsorption of the polymer on rock. This work improves the previous results in the paper " Existence of Global Weak Entropy Solutions to Some Nonstrictly Hyperbolic Systems" (SIAM J. Math. Anal., 45(2013), 3592-3610), where $\beta(T)$ is limited to meas $\{T : \beta''(T) = 0\} = 0$ or $\beta(T) = bT$ for a nonnegative constant b.

1. Introduction

In this short paper, we are concerned with the existence of entropy solutions of the Cauchy problem for the nonstrictly hyperbolic systems modelling polymer flooding in enhanced oil recovery

$$(1.1) \quad \begin{cases} S_t + f(S,T)_x = 0, \\ (ST + \beta(T))_t + (Tf(S,T) + \alpha(T))_x = 0, \end{cases}$$

with bounded measurable smooth initial data

$$(1.2) \quad (S(x,0), T(x,0)) = (S_0(x), T_0(x)), \quad 0 \le S_0(x) \le \bar{S}, \lim_{|x| \to \infty} S_0(x) = \bar{S},$$

where $\bar{S}$ is a positive constant.

System (1.1) first appeared in [**Fa**], and the existence of a weak solution of the Cauchy problem (1.1)-(1.2) was well studied in [**Ba, JB, KK, AR, Te, IT, TW, KR, FR, Frid, GP, GO, CNGM, Lu1, Lu2**] when $\beta(T) = 0$ and $\alpha(T) = 0$. When $\alpha(T) = 0$ and $\beta(T) \ne 0$, the Riemann problem of (1.1) was resolved in [**JW**].

The existence of a weak solution of the Cauchy problem (1.1)-(1.2) for general functions $\beta(T)$ and $\alpha(T)$ was proved when the following conditions (A) and (B), or (A) and (C) on $f(S,T), \alpha(T)$ and $\beta(T)$ are satisfied

(A) meas $\{S : f_{SS}(S,T) = 0\} = 0$ for any fixed T;

(B) $\beta'(T) \ge 0$, meas $\{T : \beta''(T) = 0\} = 0$ or $\beta(T) = bT, b > 0$

or

(C) $\beta(T) = 0$ and $\alpha(T) = 0$.

In this paper, instead of the condition (B) or (C), we obtain the existence results under more flexible conditions on $\alpha(T)$ and $\beta(T)$:

(D) $\alpha(T) \in C^1, \beta(T) \in C^1, \beta'(T) \geq 0, |T\alpha'(T)| \leq M(\delta + \beta'(T))$ and $|\beta''(T)| \leq M, |T\beta''(T)| \leq M(\delta + \beta'(T))$ for a positive constant M and any fixed $\delta > 0$, where M could depend on T in any compact set of $(-\infty, \infty)$.

Remark 1. It is easy to check that the following functions $\alpha(T), \beta(T)$ satisfy the condition (D):

$$(1.3) \qquad \alpha(T) = \begin{cases} 0, & \text{for} \quad T \leq 0, \\ \frac{1}{\alpha_0} T^{\alpha_0}, & \text{for} \quad 0 \leq T \leq 1, \\ g(T), & \text{for} \quad T > 1 \end{cases}$$

and

$$(1.4) \qquad \beta(T) = \begin{cases} 0, & \text{for} \quad T \leq 0, \\ \frac{1}{\beta_0} T^{\beta_0}, & \text{for} \quad 0 \leq T \leq 1, \\ T - 1 + \frac{1}{\beta_0}, & \text{for} \quad T > 1, \end{cases}$$

where $g(T)$ is an arbitrary C^1 smooth function satisfying $g(1) = \frac{1}{\alpha_0}, g'(1) = 1$ and α_0, β_0 are constants satisfying $\alpha_0 \geq \beta_0 - 1, \beta_0 \geq 2$.

Mainly, we have the following existence results in this paper.

THEOREM 1.1. *Suppose $(S_0(x), T_0(x))$ are bounded, $0 \leq S_0(x) \leq \bar{S}$ and $|T_0(x)| \leq M$; $\ln \bar{S} - \ln S_0(x) \in L^1(R)$ and $T_0(x)$ is of bounded total variation; $\alpha(T), \beta(T)$ are suitable smooth functions satisfying (D); $f(S,T)$ satisfies (A) and $f(\bar{S}, T) = 0, |\frac{f(S,T)}{S}| \leq M, |\frac{f_T(S,T)}{S}| \leq M$. Then the Cauchy problem (1.1) and (1.2) has a weak entropy solution.*

2. The Proof of Theorem 1.1

To prove Theorem 1.1. we first study the smooth solutions for the following parabolic system

$$(2.1) \qquad \begin{cases} S_t + (\frac{(S-\delta)}{S} f(S,T))_x = \varepsilon S_{xx}, \\ (ST + \beta(T))_t + (\frac{(S-\delta)}{S} Tf(S,T) + \alpha(T))_x = \varepsilon(ST + \beta(T))_{xx}, \end{cases}$$

with initial data

$$(2.2) \qquad (S^{\varepsilon,\delta}(x,0), T^{\varepsilon,\delta}(x,0)) = (\delta + \frac{\bar{S} - \delta}{\bar{S}} S_0(x), T_0(x)),$$

where ε, δ are positive, small perturbation constants.

Then

$$(2.3) \qquad \begin{cases} \delta \leq S^{\varepsilon,\delta}(x,0) \leq \bar{S}, \ \lim_{|x|\to\infty} S^{\varepsilon,\delta}(x,0) = \bar{S} \\ 0 \leq \ln \bar{S} - \ln S^{\varepsilon,\delta}(x,0) \leq \ln \bar{S} - \ln S_0(x) \in L^1(R). \end{cases}$$

LEMMA 2.1. *If the conditions in Theorem 1.1 are satisfied, then for fixed $\varepsilon > 0, \delta > 0$, the global smooth solution $(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t))$, of the Cauchy problem (2.1) and (2.2) exists, and satisfies*

$$(2.4) \qquad \delta \leq S^{\varepsilon,\delta}(x,t) \leq \bar{S}, \ |T^{\varepsilon,\delta}(x,t)| \leq M, \ \lim_{|x|\to\infty} S^{\varepsilon,\delta}(x,t) = \bar{S}, \ \text{for fixed } t,$$

$$(2.5) \qquad \int_{-\infty}^{\infty} |T_x^{\varepsilon,\delta}|(x,t)dx \leq \int_{-\infty}^{\infty} |T_x^{\varepsilon,\delta}|(x,0)dx \leq M,$$

and

(2.6)
$$\int_{-\infty}^{\infty} \ln \bar{S} - \ln S^{\varepsilon,\delta}(x,t) dx + \varepsilon \int_0^t \int_{-\infty}^{\infty} \frac{1}{(S^{\varepsilon,\delta})^2} (S_x^{\varepsilon,\delta})^2 dxdt \le |\ln \bar{S} - \ln S_0(x)|_{L^1(R)} + Mt.$$

Proof of Lemma 2.1. Since $\delta \le S_0^{\varepsilon,\delta}(x) \le \bar{S}$ and $f(\bar{S}, T) = 0$, by applying the maximum principle to the first equation in (2.1), we have $\delta \le S^{\varepsilon,\delta}(x,t) \le \bar{S}$.

Substituting the first equation in (2.1) into the second, we may rewrite the second equation in (2.1) as

(2.7)
$$T_t + \frac{\frac{(S-\delta)}{S} f + \alpha'(T)}{S + \beta'(T)} T_x = \varepsilon T_{xx} + \varepsilon \frac{2S_x + \beta''(T)T_x}{S + \beta'(T)} T_x.$$

Then we have the estimates $|T^{\varepsilon,\delta}(x,t)| \le M$ by applying the maximum principle to (2.7).

Thus the existence of the viscosity solution for the Cauchy problem (2.1)-(2.2) can be obtained by the standard theory of semilinear parabolic systems (c.f. [**Lu**]).

By using (2.7) and a technique from [**Se**] or [**Lu**], we may obtain the proof of (2.5). To prove (2.6), multiplying the first equation in (2.1) by $-\frac{1}{S}$, we have

(2.8)
$$(\ln \bar{S} - \ln S)_t - \frac{1}{S}(\frac{(S-\delta)}{S} f(S,T))_x = (\ln \bar{S} - \ln S)_{xx} - \varepsilon \frac{1}{S^2} S_x^2,$$

where

(2.9)
$$\frac{1}{S}(\frac{(S-\delta)}{S} f(S,T))_x = \frac{1}{S}(\frac{(S-\delta)}{S} f(S,T))_S S_x + \frac{1}{S}\frac{(S-\delta)}{S} f(S,T)_T T_x$$

$$= (\int_{\bar{S}}^{S} \frac{1}{\tau}(\frac{(\tau-\delta)}{\tau} f(\tau,T))_\tau d\tau)_x - \int_{\bar{S}}^{S} \frac{1}{\tau}(\frac{(\tau-\delta)}{\tau} f(\tau,T)_T)_\tau d\tau T_x + \frac{1}{S}\frac{(S-\delta)}{S} f(S,T)_T T_x$$

and

(2.10)
$$|\int_{\bar{S}}^{S} \frac{1}{\tau}(\frac{(\tau-\delta)}{\tau} f(\tau,T)_T)_\tau d\tau + \frac{1}{S}\frac{(S-\delta)}{S} f(S,T)_T| \le M.$$

So, integrating (2.8) in $R \times [0,t]$, we obtain the proof of (2.6) due to (2.5).

LEMMA 2.2. *If the condition (D) is satisfied, then there exists a subsequence (still labelled $T^{\varepsilon,\delta}(x,t)$) such that*

(2.11)
$$T^{\varepsilon,\delta}(x,t) \to T(x,t)$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$. Furthermore, if the condition (A) is satisfied, then there exists a subsequence (still labelled $S^{\varepsilon,\delta}(x,t)$) such that

(2.12)
$$S^{\varepsilon,\delta}(x,t) \to S(x,t)$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$.

Proof of Lemma 2.2. Since $|\frac{T\beta''(T)}{\delta+\beta'(T)}| \le M$ for any fixed $\delta > 0$, we choose n to be a large odd number such that $|\frac{T\beta''(T)}{\delta+\beta'(T)}| \le n - 3$, and multiply (2.7) by nT^{n-1} to obtain

(2.13)
$$(T^n)_t + nT^{n-1}\frac{\frac{(S-\delta)}{S} f + \alpha'(T)}{S+\beta'(T)} T_x$$
$$= \varepsilon(T^n)_{xx} - \varepsilon n(n-1)T^{n-2}(T_x)^2 + \varepsilon nT^{n-1}\frac{2S_x+\beta''(T)T_x}{S+\beta'(T)} T_x.$$

Since

$$(2.14) \quad \begin{cases} |\frac{\frac{(S-\delta)}{S}f+\alpha'(T)}{S+\beta'(T)}T| \leq M, \ nT^{n-1}\frac{2S_x}{S+\beta'(T)}T_x| \leq nT^{n-2}(T_x)^2 + nT^n(\frac{S_x}{S})^2, \\ \\ |nT^{n-1}\frac{\beta''(T)T_x}{S+\beta'(T)}T_x| \leq n(n-3)T^{n-2}|(T_x)^2, \end{cases}$$

we have from (2.13) and (2.14) that

$$(2.15) \qquad \varepsilon nT^{n-2}(T_x)^2 \leq -(T^n)_t + nT^{n-2}M|T_x| + \varepsilon(T^n)_{xx} + T^n(\frac{S_x}{S})^2.$$

Let $K \subset R \times R^+$ be an arbitrary compact set and choose $\phi \in C_0^\infty(R \times R^+)$ such that $\phi_K = 1, 0 \leq \phi \leq 1$. Multiplying (2.15) by ϕ and integrating over $R \times R^+$, we obtain immediately

$$(2.16) \qquad \varepsilon T^{n-2}(T_x^{\varepsilon,\delta})^2 \quad \text{are bounded in } L_{loc}^1(R \times R^+).$$

Thus, the terms in (2.13)

$$(2.17) \quad nT^{n-1}\frac{\frac{(S-\delta)}{S}f+\alpha'(T)}{S+\beta'(T)}T_x + \varepsilon n(n-1)T^{n-2}(T_x)^2 - \varepsilon nT^{n-1}\frac{2S_x+\beta''(T)T_x}{S+\beta'(T)}T_x$$

are bounded in $L_{loc}^1(R \times R^+)$ and

$$(2.18) \qquad \varepsilon(T^n)_{xx} \quad \text{are compact in } H_{loc}^{-1}(R \times R^+),$$

which deduce that

$$(2.19) \qquad ((T^{\varepsilon,\delta})^n)_t \ \text{ or } \ ((T^{\varepsilon,\delta})^n)_t + c_x \quad \text{are compact in } H_{loc}^{-1}(R \times R^+)$$

for any constant c, by using the Murat's lemma (c.f. [**Mu**]).

Finally, since $((T^{\varepsilon,\delta})^n)_x$ or $c_t + ((T^{\varepsilon,\delta})^n)_x$ are bounded in $L_{loc}^1(R \times R^+)$ and so compact in $H_{loc}^{-1}(R \times R^+)$, we may apply the div-curl lemma in the compensated compactness theory [**Ta**] to the following special pairs of functions

$$(2.20) \qquad\qquad (c, (T^{\varepsilon,\delta})^n), \quad ((T^{\varepsilon,\delta})^n, c)$$

to obtain

$$(2.21) \qquad\qquad \overline{(T^{\varepsilon,\delta})^n} \cdot \overline{(T^{\varepsilon,\delta})^n} = \overline{(T^{\varepsilon,\delta})^{2n}},$$

which deduces the pointwise convergence of $(T^{\varepsilon,\delta})^n(x,t) \to T^n(x,t)$ a.e. on any bounded and open set $\Omega \subset R \times R^+$, and so the conclusion (2.11) immediately because n is a odd number, where $\overline{f(\theta^{\varepsilon,\delta})}$ denotes the weak-star limit of $f(\theta^{\varepsilon,\delta})$.

Thanks to the pointwise convergence of $T^{\varepsilon,\delta}(x,t)$, we may use the Div-Curl Lemma on the scalar conservation equation (the first equation in (2.1)) with a space-time discontinuous flux [**KT, Lu**] to obtain the pointwise convergence of $S^{\varepsilon,\delta}(x,t)$ in (2.12). **Lemma 2.2 is proved.**

Proof of Theorem 1.1. Letting ε, δ go to zero, we may easily prove from (2.1) that the limit $(S(x,t), T(x,t))$ satisfies

(2.22)
$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty S\phi_t + f(S,T)\phi_x dxdt + \int_{-\infty}^\infty S_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty ST + \beta(T)\phi_t + (Tf(S,T)+\alpha(T))\phi_x dxdt + \int_{-\infty}^\infty S_0(x)T_0(x)\phi(x,0)dx = 0 \end{cases}$$

for all test function $\phi \in C_0^1(R \times R^+)$.

Let $C = ST + \beta(T)$. Since $\beta'(T) \geq 0$, then for any fixed $S \in (0, \bar{S}]$, there exists a smooth, inverse function $T = \theta(S, C)$ and system (2.1) can be rewritten as

$$(2.23) \qquad \begin{cases} S_t + (\frac{(S-\delta)}{S} f(S, \theta(S, C)))_x = \varepsilon S_{xx}, \\ C_t + (\frac{(S-\delta)}{S} \theta(S, C) f(S, \theta(S, C)) + \alpha(\theta(S, C)))_x = \varepsilon C_{xx}. \end{cases}$$

Let $(\eta(S, C), q(S, C))$ be any convex entropy-entropy flux pair of the system

$$(2.24) \qquad \begin{cases} S_t + f(S, \theta(S, C))_x = 0, \\ C_t + (\theta(S, C) f(S, \theta(S, C)) + \alpha(\theta(S, C)))_x = 0. \end{cases}$$

Multiplying the first and second equations of the system (2.24) by $\frac{\partial \eta(S,C)}{\partial S}$ and $\frac{\partial \eta(S,C)}{\partial C}$, respectively, then adding the result, we have

(2.25)

$$\eta(S, C)_t + q(S, C)_x - \delta \frac{\partial \eta(S,C)}{\partial S} (\frac{f(S,\theta(S,C))}{S})_x - \delta \frac{\partial \eta(S,C)}{\partial C} (\frac{\theta(S,C) f(S,\theta(S,C))}{S})_x$$

$$= \varepsilon \eta(S, C)_{xx} - \varepsilon(\eta_{SS} S_x^2 + 2\eta_{SC} S_x C_x + \eta_{CC} C_x^2) \leq \varepsilon \eta(S, C)_{xx}.$$

Let $\frac{\partial \eta}{\partial v_i}$ be the partial derivatives of the function η with respect to the first and the second variables. Then we have

$$\delta \frac{\partial \eta(S,C)}{\partial S} (\frac{f(S,\theta(S,C))}{S})_x + \delta \frac{\partial \eta(S,C)}{\partial C} (\frac{\theta(S,C) f(S,\theta(S,C))}{S})_x$$

$$= \delta \frac{\partial \eta(S,ST+\beta(T))}{\partial v_1} ((\frac{f(S,T)}{S})_S S_x + (\frac{f(S,T)}{S})_T T_x)$$

$$+ \frac{\partial \eta(S,ST+\beta(T))}{\partial v_2} ((\frac{T f(S,T)}{S})_S S_x + (\frac{T f(S,T)}{S})_T T_x)$$

(2.26)

$$= \delta \frac{\partial}{\partial x} (\int_0^S \frac{\partial \eta(S,ST+\beta(T))}{\partial v_1} (\frac{f(S,T)}{S})_S + \frac{\partial \eta(S,ST+\beta(T))}{\partial v_2} (T \frac{f(S,T)}{S})_S dS)$$

$$- \delta \frac{\partial}{\partial T} (\int_0^S \frac{\partial \eta(S,ST+\beta(T))}{\partial v_1} (\frac{f(S,T)}{S})_S + \frac{\partial \eta(S,ST+\beta(T))}{\partial v_2} (T \frac{f(S,T)}{S})_S dS) T_x$$

$$+ \delta (\frac{\partial \eta(S,ST+\beta(T))}{\partial v_1} (\frac{f(S,T)}{S})_T + \frac{\partial \eta(S,ST+\beta(T))}{\partial v_2} (\frac{T f(S,T)}{S})_T) T_x.$$

Since T_x is locally bounded in $L^1(R \times R^+)$, letting ε, δ go to zero, we may prove from (2.25) and (2.26) that the following entropy condition, for the limit functions $(S(x, t), T(x, t))$ and any convex entropy-entropy flux pair (η, q),

$$(2.27) \qquad \int_0^\infty \int_{-\infty}^\infty \eta(S, ST + \beta(T)) \phi_t + q(S, ST + \beta(T)) \phi_x \phi dx dt \geq 0$$

holds, where $\phi \in C_0^\infty(R \times R^+ - \{t = 0\})$ is a non-negative test function. So, we complete **the proof of Theorem 1.1**.

References

[AR] A. Aw and M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math., **60** (2000), 916-938.

[Ba] T. Barkve, *The Riemann Problem for a Nonstrictly Hyperbolic System Modeling Nonisothermal, Two-Phase Flow in a Porous Medium*, SIAM Journal on Applied Mathematics, **49** (1989), 784-798.

[CNGM] G. M. Coclite, N. De Nitti, M. Garavello and F. Marcellini, *Vanishing viscosity for a 2×2 system modelling congested vehicular traffic*, preprint, 2020.

[Fa]   F. J. Fayers, *Some theoretical results concerning the displacement of a viscous oil by a hot fluid in a porous medium* , J. Fluid. Mech.,**13** (1962), 65-76.

[FR]   H. Freistuehler, *On the Cauchy problem for a class of hyperbolic systems of conservation laws*, J. Diff. Eqs., **112** (1994), 170-178.

[Frid] H. Frid, *Asymptotic Stability of Non-planar Riemann Solutions for Multi-D Systems of Conservation Laws With Symmetric Nonlinearities*, J. Hyperbolic Differential Equations, **1** (2004), 567-579.

[GP]   M. Garavello and B. Piccoli, *Traffic flow on a road network using the Aw-Rascle model*, Comm. Partial Differential Equations, **31** (2006), 243-275.

[GO]   M. Godvik and H. Hanche-Olsen, *Existence of Solutions for the AW-Rascle Traffic Flow Model with Vacuum*, J. Hyperbolic Differ. Equ., **5** (2008), 45-64.

[IT]   E. Isaacson and B. Temple, *Analysis of a singular hyperbolic system of conservation laws*, J. Differential Equations, **65** (1986), 250-268.

[JW]   T. Johansen and R. Winther, *The solution of the Riemann problem for a hyperbolic system of conservation laws modelling polymerflooding*, SIAM J. Math. Anal., **19** (1988), 541-566.

[JB]   R. Juanes and M. J. Blunt, *Analytical Solutions to Multiphase First-Contact Miscible Models with Viscous Fingering*, Transport in Porous Media **64** (2006), 339-373.

[JPP] F. James, Y.-J. Peng and B. Perthame, *Kinetic formulation for chromatography and some other hyperbolic systems*, J. Math. Pure Appl., **74** (1995), 367-385.

[KK]   B. Keyfitz and H. Kranzer, *A system of nonstrictly hyperbolic conservation laws arising in elasticity*, Arch. Rat. Mech. Anal., **72** (1980), 219-241.

[KR]   C. Klingenberg and N. H. Risebro, *Stability of a resonant system of conservation laws modeling polymer flow with gravitation*, J. Diff. Eqs., **170** (2001), 344-380.

[KT]   K. H. Karlsen and J. D. Towers, *Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux*, Chinese Annals of Mathematics, Ser. B., **25** (2004), 287-318.

[Lu]   Y.-G. Lu, *Existence of Global Weak Entropy Solutions to Some Nonstrictly Hyperbolic Systems*, SIAM J. Math. Anal., **45** (2013), 3592-3610.

[Lu1] Y.-G. Lu, *Existence of Global Bounded Weak Solutions to a Non-Symmetric System of Keyfitz-Kranzer type*, J. Funct. Anal., **261** (2011), 2797-2815.

[Lu2] Y.-G. Lu, *Existence of global entropy solutions to general system of Keyfitz-Kranzer type*, J. Funct. Anal., **264** (2013), 2457-2468.

[Mu]   F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa, **5** (1978), 489-507.

[Se]   D. Serre, *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation*, J. Diff. Eqs., **68** (1987), 137-168.

[Ta]   T. Tartar, *Compensated compactness and applications to partial differential equations*, In: Research Notes in Mathematics, Nonlinear Analysis and Mechanics, Heriot-Watt symposium, Vol. **4**, ed. R. J. Knops, Pitman Press, London, 1979. Anal., **261** (2011), 2797-2815.

[Te]   B. Temple, *Global solution of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws*, Advances in Applied Mathematics **3** (1982), 335-375.

[TW]   A. Tveito and R. Winther, *Existence, uniqueness and continuous dependence for a system of conservation laws modelling polymer flooding* , SIAM J. Math. Anal.,**22** (1991), 905-933.

DEPARTMENT OF MATHEMATICS, WUERZBURG UNIVERSITY, GERMANY

DEPARTMENT OF MATHEMATICS, HANGZHOU NORMAL UNIVERSITY, HANGZHOU, CHINA
E-mail address: ylu2005@ustc.edu.cn

SCHOOL OF MATHEMATICS AND PHYSICS, YANCHENG INSTITUTE OF TECHNOLOGY, CHINA

Yun-guang Lu
K.K.Chen Institute for Advanced Studies
Hangzhou Normal University, China
email: ylu2005@ustc.edu.cn

May 8, 2021

Professor B. Temple
Editor
Applied Mathematics Letters

Dear Prof. Temple:

How are you? I wish you and your family all the best in this special time.

I would like to submit the manuscript entitled " Global entropy solutions for systems modelling polymer flooding in enhanced oil recovery" jointed with Prof. C. Klingenberg and Dr. C. Xue to Applied Mathematics Letters.

For the convenience to find the reviewers, I would be glad to list the following experts in this field for your reference:

1. Prof. Feimin Huang, Chinese Academy of Sciences, Beijing

fhuang@amt.ac.cn

2. Prof. Yue-jun Peng, Universit Clermont Auvergne, France

Yue-Jun.Peng@math.univ-bpclermont.fr;

3. Prof. Hermano Frid, IMPA, Brazil

hermano@impa.br

4. Prof. Yuusuke Sugiyama,

sugiyama.y@e.usp.ac.jp

5. Prof. Yong Zhou, Sun Yat-Sen University

zhouyong3@mail.sysu.edu.cn

6. Prof. Mingxuan Zhu, Qufu Normal University

mxzhu@mail.zjxu.edu.cn

7. Prof. Elder Villamizar Roa, Universidad Industrial de Santander

jvillami@uis.edu.co

8. Prof. Ping Zhang, Chinese Academy of Sciences, Beijing

zp@amss.ac.cn

Thank you very much for your consideration.

Sincerely yours,

Yun-guang Lu