MATHEMATICS OF COMPUTATION
Volume 86, Number 305, May 2017, Pages 1203–1232
http://dx.doi.org/10.1090/mcom/3126
Article electronically published on June 20, 2016

ARBITRARY LAGRANGIAN-EULERIAN DISCONTINUOUS GALERKIN METHOD FOR CONSERVATION LAWS: ANALYSIS AND APPLICATION IN ONE DIMENSION

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Abstract. In this paper, we develop and analyze an arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method with a time-dependent approximation space for one dimensional conservation laws, which satisfies the geometric conservation law. For the semi-discrete ALE-DG method, when applied to nonlinear scalar conservation laws, a cell entropy inequality, L^2 stability and error estimates are proven. More precisely, we prove the sub-optimal $(k+\frac{1}{2})$ convergence for monotone fluxes and optimal $(k+1)$ convergence for an upwind flux when a piecewise P^k polynomial approximation space is used. For the fully-discrete ALE-DG method, the geometric conservation law and the local maximum principle are proven. Moreover, we state conditions for slope limiters, which ensure total variation stability of the method. Numerical examples show the capability of the method.

1. Introduction

Grid deformation methods are unavoidable in many applications in fluid dynamics. For instance, these kinds of methods are used for aeroelastic analysis of wings in engineering (cf. Robinson et al. [26]) or to describe star-formations and galaxies in astrophysics (cf. Kereš et al. [17]). In this paper a grid deformation method based on a discontinuous Galerkin (DG) discretization will be presented. To describe and analyze the method we consider the following simple model problem:

$$\partial_t u + \partial_x f(u) = 0, \quad \text{in } \Omega \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

with periodic boundary conditions. The set Ω is an open interval in $\mathbb{R}$, the initial data u_0 is considered to be periodic or compactly supported and f is a sufficiently smooth flux function.

In order to describe the method, we assume that the grid points are explicitly given for the upcoming time level, based on some grid moving methodology. Then, the cells of the partitions for the current and next time level can be connected by local affine linear mappings. In the finite volume context a technique using a local affine mapping was used by Fazio and LeVeque [10]. The mappings yield time dependent test functions for the DG discretization. Moreover, the grid is static if the

Received by the editor June 5, 2015 and, in revised form, September 25, 2015.
2010 *Mathematics Subject Classification.* Primary 65M12, 65M15, 65M60; Secondary 35L65.
Key words and phrases. Arbitrary Lagrangian-Eulerian discontinuous Galerkin method, hyperbolic conservation laws, geometric conservation law, cell entropy inequality, error estimates, maximum principle, slope limiter conditions.
The third author is the corresponding author. The research of the third author was supported by NSFC grants No. 11371342 and No. 11471306.

linear mappings are constant. In this case the motion of a fluid is described by the Eulerian description of motion. On the other hand, it is described by the Lagrangian description if the linear mappings describe approximately the motion of the particles in a fluid. Thus, our method belongs to the class of arbitrary Lagrangian-Eulerian (ALE) methods (cf. Donea et al. [7]). Thence, we call our method the arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method.

The Runge-Kutta DG method, in the context of static grids, has been developed and analyzed by Cockburn, Shu et al. in a series of papers (cf. [3–5] and a review article [6]). ALE-DG methods for equations with compressible viscous flows have been developed by Lomtev et al. [22], Persson et al. [24] and Nguyen [23]. In their papers, the focus lies on the implementation and performance of the methods in aeroelastic applications. However, there are also some theoretical aspects about ALE methods in the literature. These discussions are mostly about the significance of the geometric conservation law (GCL) for ALE methods. This law governs the geometric parameters of a grid deformation method, such that the method preserves constant states. The terminology GCL was introduced by Lombard and Thomas in [21]. It is well known that there is a lack of stability in a grid deformation method, if there is no GCL satisfied. For instance in [9] Grandmont, Guillard and Farhat have proven that for monotone ALE methods the GCL is a necessary and sufficient condition to obtain the local maximum principle for the method. Moreover, in [12] Guillard and Farhat have proven that the GCL is a necessary condition to ensure that the time discretization of the method is high order accurate for ALE-finite volume methods. Further, in [20] Lesoinne and Farhat have analyzed the relevance and implementation of geometric conservation laws for different ALE methods. They have shown that the GCL is not trivially satisfied for ALE-finite element methods with a Runge-Kutta time discretization. Thus, in particular for ALE-DG methods, it is important to pay attention to the GCL. We are able to prove that our ALE-DG method preserves constant states for any Runge-Kutta method. Therefore, our method satisfies the GCL.

It is well known that solutions of hyperbolic conservation laws are in general discontinuous, even if the initial data is chosen smooth. Discontinuities are the cause of numerical artifacts like spurious oscillations in high order methods for hyperbolic conservation laws. Without taming these artifacts a numerical method will become unstable. A possible way to stabilize DG methods has been introduced by Cockburn and Shu in [2, 5]. They constructed slope limiters such that the method stays high order accurate and the cell average values of the DG solution become total variation stable. By following Cockburn and Shu's approach we obtain conditions for slope limiters which stabilize our ALE-DG method. Furthermore, in numerical test examples we validate our conditions. Discontinuities are not the only source of instabilities in a numerical method. It is necessary that the method preserve bounds. In general it is not easy to prove that a high order method preserves bounds, even for methods on static grids. In [31] X. Zhang and Shu developed a limiter for static grids which ensures that the revised solution of a high order method preserves bounds. We prove that this limiter works for our ALE-DG method, too. Moreover, for scalar conservation laws we obtain the local maximum principle as X. Zhang and Shu did for high order methods on static grids.

Another peculiarity of hyperbolic conservation laws is that weak solutions are in general not unique. A weak solution has to satisfy an entropy inequality to be the unique physically relevant or entropy solution. For scalar conservation laws it is well known that there exists a unique entropy solution (cf. Kruzkov [18]). We prove for scalar conservation laws that our semi-discrete ALE-DG method satisfies a discrete version of the square entropy inequality. Thus, in particular the method is L^2 stable. Additionally, we prove for smooth solutions of scalar conservation laws the sub-optimal $(k + \frac{1}{2})$ convergence for the semi-discrete ALE-DG method with monotone numerical fluxes and the optimal $(k + 1)$ convergence for the method with an upwind numerical flux if a piecewise P^k polynomial approximation space is used.

For DG methods on static grids, there are already many results in the literature about the a priori error for smooth solutions of hyperbolic conservation laws. In the following, we will list a few results. The first a priori error estimate for a DG method has been proven by LeSaint and Raviart [19]. In [16] Johnson and Pitkřanta have proven that for linear conservation laws the discontinuous Galerkin a priori error behaves as $\mathcal{O}\left(h^{k+1}\right)$, and in [25] Peterson has proven that the result of Johnson and Pitkäranta is the optimal a priori error for any DG method for hyperbolic conservation laws. Further, nonlinear scalar conservation laws and symmetrizable systems have been considered by Zhang and Shu in [28], [29] and [30]. They have proven for DG methods with a second and third order total variation diminishing (TVD) Runge-Kutta time discretization that the a priori error behaves as $\mathcal{O}\left(h^{k+\frac{1}{2}} + (\triangle t)^{\sigma}\right)$, $\sigma = 2, 3$, in the general case and as $\mathcal{O}\left(h^{k+1} + (\triangle t)^{\sigma}\right)$, $\sigma = 2, 3$, by applying an upwind numerical flux.

The organization of the paper is as follows: In Section 2 we develop our ALE-DG method in one dimension. First, we develop the semi-discrete ALE-DG scheme and prove the cell entropy inequality as well as the L^2 stability. Afterwards, the error estimates are proven for the method with monotone numerical fluxes and an upwind numerical flux. Then, in Section 2.4 we discuss the fully-discrete ALE-DG method. Further, the geometric conservation law and the local maximum principle are proven. Conditions for the slope limiter are derived, too. Section 3 contains numerical results for linear and nonlinear problems to demonstrate the accuracy and capabilities of the method. Finally, some concluding remarks are given in Section 4.

2. The arbitrary Lagrangian-Eulerian discontinuous Galerkin method

In this section, we develop and analyze an arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method for solving conservation laws.

2.1. **The semi-discrete ALE-DG discretization.**

In order to describe the method, we need to take the motion of the grid into account. We assume that there are given points $\left\{x^n_{j-\frac{1}{2}}\right\}_{j=1}^N$ at time level t_n and $\left\{x^{n+1}_{j-\frac{1}{2}}\right\}_{j=1}^N$ at t_{n+1}, such that

$$\Omega = \bigcup_{j=1}^N \left[x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}}\right] \quad \text{and} \quad \Omega = \bigcup_{j=1}^N \left[x^{n+1}_{j-\frac{1}{2}}, x^{n+1}_{j+\frac{1}{2}}\right].$$

Note that the first point and the last point stay the same for the compactly supported problem and could move at the same speed for the periodic boundary problem. Next, we connect the points $x^n_{j-\frac{1}{2}}$ and $x^{n+1}_{j-\frac{1}{2}}$ by rays

$$x_{j-\frac{1}{2}}(t) := x^n_{j-\frac{1}{2}} + \omega_{j-\frac{1}{2}}(t-t_n), \quad \text{for all} \quad t \in [t_n, t_{n+1}], \tag{2.1}$$

where

$$\omega_{j-\frac{1}{2}} := \frac{x^{n+1}_{j-\frac{1}{2}} - x^n_{j-\frac{1}{2}}}{t_{n+1}-t_n}. \tag{2.2}$$

The quantity $\omega_{j-\frac{1}{2}}$ describes the speed of motion in which the point $x^n_{j-\frac{1}{2}}$ moves to $x^{n+1}_{j-\frac{1}{2}}$. The rays (2.1) provide for all $t \in [t_n, t_{n+1}]$ time-dependent cells $K_j(t) := \left[x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)\right]$. The length of a time-dependent cell is denoted by

$$\triangle_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t).$$

Next, we introduce some assumptions:

(ω1): For all $j = 1, \cdots, N$ and $t \in [t_n, t_{n+1}]$,

$$\triangle_j(t) = (\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}})(t-t_n) + \triangle_j(t_n) > 0.$$

(ω2): There exists a constant C_0, independent of h, such that

$$\max_{(x,t)\in\Omega\times[0,T]} |\omega(x,t)| \le C_0.$$

(ω3): There exists a constant $C_{0,1}$, independent of h, such that

$$\max_{(x,t)\in\Omega\times[0,T]} |\partial_x(\omega(x,t))| \le C_{0,1}.$$

Note that the function $\omega : \Omega \times [0,T] \to \mathbb{R}$ is the grid velocity field. It is for any cell $K_j(t)$ given by

$$\omega(x,t) = \omega_{j+\frac{1}{2}} \frac{x - x_{j-\frac{1}{2}}(t)}{\triangle_j(t)} + \omega_{j-\frac{1}{2}} \frac{x_{j+\frac{1}{2}}(t) - x}{\triangle_j(t)}. \tag{2.3}$$

The length of the largest time-dependent cell is defined by $h(t) := \max_{1\le j\le N} \triangle_j(t)$. Moreover, for every time point the maximal cell length will be denoted by

$$h := \max_{t\in[0,T]} h(t). \tag{2.4}$$

In addition, we assume that the mesh is regular. Thus, there exists a constant $\rho > 0$, independent of h, such that

$$\triangle_j(t) \ge \rho h, \quad \forall j = 1, \cdots, N. \tag{2.5}$$

The equation (ω1) guarantees that the time-dependent cells $K_j(t)$ are well defined. Therefore, for any $t \in [t_n, t_{n+1}]$ the time-dependent cells $K_j(t)$ can be connected with a reference cell $[-1,1]$ by the mapping

$$\chi_j : [-1,1] \to K_j(t), \quad \chi_j(\xi,t) = \frac{\triangle_j(t)}{2}(\xi+1) + x_{j-\frac{1}{2}}(t). \tag{2.6}$$

The mapping yields a characterization of the grid velocity

$$\partial_t\big(\chi_j(\xi,t)\big) = \omega(\chi_j(\xi,t), t) \quad \text{for all} \quad (\xi,t) \in [-1,1] \times [t_n, t_{n+1}]. \tag{2.7}$$

Furthermore, for any $t \in [t_n, t_{n+1}]$ by the mapping a finite dimensional test function space can be defined by

$$\mathcal{V}_h(t) := \left\{ v_h \in \mathrm{L}^2(\Omega) \mid v_h\big(\chi_j(\cdot, t)\big) \in P^k\big([-1,1]\big),\ \forall j = 1, \cdots, N \right\}, \tag{2.8}$$

where $P^k([-1,1])$ denotes the space of polynomials in $[-1,1]$ of degree at most k. The space $\mathcal{V}_h(t)$ contains discontinuous functions. Hence, for a function $v_h \in \mathcal{V}_h(t)$, we denote the left as well as the right limit, the cell average and the jump in a point $x_{j-\frac{1}{2}}(t)$ as follows:

$$v_{h,j-\frac{1}{2}}^{-} = v_h\left(x_{j-\frac{1}{2}}^{-}(t), t\right) := \lim_{\varepsilon \to 0} v_h\left(x_{j-\frac{1}{2}}(t) - \varepsilon, t\right),$$

$$v_{h,j-\frac{1}{2}}^{+} = v_h\left(x_{j-\frac{1}{2}}^{+}(t), t\right) := \lim_{\varepsilon \to 0} v_h\left(x_{j-\frac{1}{2}}(t) + \varepsilon, t\right),$$

$$\{\!\{v_h\}\!\}_{j-\frac{1}{2}} := \frac{1}{2}\left(v_{h,j-\frac{1}{2}}^{+} + v_{h,j-\frac{1}{2}}^{-}\right) \quad \text{and} \quad [\![v_h]\!]_{j-\frac{1}{2}} := v_{h,j-\frac{1}{2}}^{+} - v_{h,j-\frac{1}{2}}^{-}.$$

In addition, for all $v, w \in \mathrm{L}^2(K_j(t))$ we denote the $\mathrm{L}^2(K_j(t))$ inner product by $(v,w)_{K_j(t)} := \int_{K_j(t)} vw\,dx$. The following transport equation will be essential for the upcoming.

Lemma 2.1. *Let $u \in \mathrm{W}^{1,\infty}\big(0,T;\mathrm{H}^1(\Omega)\big)$. Then for all $v_h \in \mathcal{V}_h(t)$ the following transport equation holds:*

$$\frac{d}{dt}\Big(u, v_h\Big)_{K_j(t)} = \Big(\partial_t u, v_h\Big)_{K_j(t)} + \Big(\partial_x(\omega u), v_h\Big)_{K_j(t)}. \tag{2.9}$$

Proof. Let $\phi_0(\xi), \cdots, \phi_k(\xi)$ be a basis of the polynomial space $P^k([-1,1])$. Then in any cell $K_j(t)$ the functions

$$\widehat{\phi}_\ell(x,t) := \phi_\ell\left(\frac{2\left(x - x_{j-\frac{1}{2}}(t)\right)}{\triangle_j(t)} - 1\right), \quad x \in K_j(t), \tag{2.10}$$

represent a basis of the test function space $\mathcal{V}_h(t)$. It is easy to verify that the functions (2.10) satisfy the equation

$$\partial_t\left(\widehat{\phi}_\ell(x,t)\right) + \omega(x,t)\,\partial_x\left(\widehat{\phi}_\ell(x,t)\right) = 0, \quad x \in K_j(t). \tag{2.11}$$

Let $u \in \mathrm{W}^{1,\infty}\big(0,T;\mathrm{H}^1(\Omega)\big)$ and $v_h \in \mathcal{V}_h(t)$. By (2.10) the function v_h can be written as

$$v_h(x,t) = \sum_{\ell=0}^{k} v_h^\ell \widehat{\phi}_\ell(x,t), \quad x \in K_j(t) \text{ and } v_h^\ell \in \mathbb{R}.$$

Next, by the identity (2.11),

$$\partial_t(v_h(x,t)) + \omega(x,t)\,\partial_x(v_h(x,t)) = 0, \quad x \in K_j(t). \tag{2.12}$$

Therefore, by the Reynolds transport theorem and (2.12),

$$\begin{aligned}\frac{d}{dt}(u, v_h)_{K_j(t)} &= (\partial_t(uv_h), 1)_{K_j(t)} + (\partial_x(\omega u v_h), 1)_{K_j(t)} \\ &= (\partial_t u, v_h)_{K_j(t)} + (\partial_x(\omega u), v_h)_{K_j(t)}.\end{aligned}$$

□

Finally, with all these ingredients we can start to describe the semi-discrete ALE-DG method. The description of the ALE-DG method for the time interval $[t_n, t_{n+1}]$ and the cell $K_j(t)$ starts by multiplying the equation (1.1a) with a test function $v_h \in \mathcal{V}_h(t)$. Next, we integrate the result over the cell $K_j(t)$ and apply the transport equation (2.9). Then, by an integration by parts we obtain

$$\begin{aligned}\frac{d}{dt}\left(u_h, v_h\right)_{K_j(t)} = \left(g\left(\omega, u_h\right), \partial_x v_h\right)_{K_j(t)} &- g\left(\omega_{j+\frac{1}{2}}, u_h\left(x_{j+\frac{1}{2}}(t), t\right)\right) v^-_{h,j+\frac{1}{2}} \\ &+ g\left(\omega_{j-\frac{1}{2}}, u_h\left(x_{j-\frac{1}{2}}(t), t\right)\right) v^+_{h,j-\frac{1}{2}},\end{aligned}$$

where $g(\omega, u_h) := f(u_h) - \omega u_h$ and $u_h \in \mathcal{V}_h(t)$ is an unknown approximation to the solution u of (1.1), which we try to determine by the ALE-DG method. Since u_h is discontinuous in the cell interface points $x_{j+\frac{1}{2}}(t)$, we replace the flux $g(\omega_{j+\frac{1}{2}}, u_h(x_{j+\frac{1}{2}}(t), t))$ by a numerical flux $\widehat{g}\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right)$, which is a single valued function defined in the cell interface points and depends on the values of the approximate solution u_h from both sides of the cell interfaces. In general the numerical flux $\widehat{g}\left(\omega_{j+\frac{1}{2}}(t), u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right)$ should be chosen as a monotone numerical flux which satisfies:

($\widehat{g}$1) Consistency: For any smooth function the identity $\widehat{g}(\omega, u, u) = g(\omega, u)$ holds.

($\widehat{g}$2) Monotonicity: The numerical flux function $\widehat{g}(\omega, \cdot, \cdot)$ is increasing in the second argument and decreasing in the third argument.

($\widehat{g}$3) Lipschitz continuity: For all $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ the inequality

$$\left|\widehat{g}(\omega, a_1, b_1) - \widehat{g}(\omega, a_2, b_2)\right| \le L^-_{\widehat{g}} |a_1 - a_2| + L^+_{\widehat{g}} |b_1 - b_2|$$

holds, where the Lipschitz constants $L^-_{\widehat{g}}$ and $L^+_{\widehat{g}}$ are independent of h.

Finally, the semi-discrete ALE-DG method can be written as follows: Find a function $u_h \in \mathcal{V}_h(t)$ such that

$$\begin{aligned}\frac{d}{dt}\left(u_h, v_h\right)_{K_j(t)} = \left(g\left(\omega, u_h\right), \partial_x v_h\right)_{K_j(t)} &- \widehat{g}\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} \\ &+ \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}},\end{aligned} \tag{2.13}$$

for all $v_h \in \mathcal{V}_h(t)$ and $j = 1, \cdots, N$.

2.2. **A cell entropy inequality and L^2 stability.** Weak solutions of equation (1.1) are in general not unique. The physically relevant unique entropy solution can be found by the entropy inequality

$$\partial_t \eta(u) + \partial_x F(u) \le 0, \text{ in } \Omega \times (0, T] \tag{2.14}$$

in the sense of distribution. The entropy $\eta : \mathbb{R} \to \mathbb{R}$ in (2.14) can be any convex function (cf. Di Perna [8]) if the flux function in (1.1) is convex. For a flux function $f \in \mathcal{C}^1(\mathbb{R})$ the entropy inequality (2.14) has to be true for all convex functions or η has to be the so-called Kruzkov entropy (cf. Kruzkov [18]). Further, the entropy flux is given by $F(u) := \int^u \eta'(v) f'(v)\, dv$.

By integrating the entropy inequality (2.14) over the cell $K_j(t)$ and applying the transport equation (2.9) with $v_h = 1$ we obtain

$$\begin{aligned} 0 \geq {} & \frac{d}{dt}(\eta(u), 1)_{K_j(t)} + F\left(u_{j+\frac{1}{2}}\right) - \omega_{j+\frac{1}{2}}\eta\left(u_{j+\frac{1}{2}}\right) \\ & - F\left(u_{j-\frac{1}{2}}\right) + \omega_{j-\frac{1}{2}}\eta\left(u_{j-\frac{1}{2}}\right), \end{aligned} \tag{2.15}$$

where $u_{j-\frac{1}{2}} := u\left(x_{j-\frac{1}{2}}(t), t\right)$. In the following, we show that in every cell $K_j(t)$ the ALE-DG method satisfies an inequality, which is consistent with (2.15) for smooth functions. Thus, we have a cell entropy inequality for the ALE-DG method like Jiang and Shu [15] for the DG methods on static grids.

Proposition 2.2. *The solution u_h of the semi-discrete ALE-DG method given by (2.13) satisfies the following cell entropy inequality:*

$$\begin{aligned} 0 \geq {} & \frac{d}{dt}(\eta(u_h), 1)_{K_j(t)} + H\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) \\ & - H\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right), \end{aligned} \tag{2.16}$$

where $\eta(u) := \frac{u^2}{2}$ is the square entropy and

$$H(\omega, u^-, u^+) := -\int^{u^-} f(v)\, dv + \frac{\omega}{2}(u^-)^2 + \widehat{g}(\omega, u^-, u^+)\, u^-.$$

Proof. By applying the transport equation (2.9) the equation (2.13) can be written for all test functions $v_h \in \mathcal{V}_h(t)$ as follows:

$$\begin{aligned} 0 = {} & (\partial_t u_h, v_h)_{K_j(t)} + (\partial_x(\omega u_h v_h), 1)_{K_j(t)} - (f(u_h), \partial_x v_h)_{K_j(t)} \\ & + \widehat{g}\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} - \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}}. \end{aligned} \tag{2.17}$$

For equation (2.17) we can choose $v_h = u_h$ as a test function. Then, by the Reynolds transport theorem,

$$\begin{aligned} 0 = {} & \frac{1}{2}\frac{d}{dt}(u_h, u_h)_{K_j(t)} + \frac{1}{2}\left(\partial_x\left(\omega u_h^2\right), 1\right) - (f(u_h), \partial_x u_h)_{K_j(t)} \\ & + \widehat{g}\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) u^-_{h,j+\frac{1}{2}} - \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) u^+_{h,j-\frac{1}{2}}. \end{aligned} \tag{2.18}$$

Next, we define the quantities

$$G(\omega, u) := \int^u f(v)\, dv - \frac{\omega}{2}u^2$$

and

$$H(\omega, u^-, u^+) := -G(\omega, u^-) + \widehat{g}(\omega, u^-, u^+).$$

Then, equation (2.18) can be rewritten as

$$\begin{aligned} 0 = {} & \frac{1}{2}\frac{d}{dt}(u_h, u_h)_{K_j(t)} + H\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) \\ & - H\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) + \Theta_{j-\frac{1}{2}}, \end{aligned}$$

where

$$\Theta_{j-\frac{1}{2}} := G\left(\omega_{j-\frac{1}{2}}, u^{+}_{h,j-\frac{1}{2}}\right) - G\left(\omega_{j-\frac{1}{2}}, u^{-}_{h,j-\frac{1}{2}}\right)$$
$$- \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^{-}_{h,j-\frac{1}{2}}, u^{+}_{h,j-\frac{1}{2}}\right) [\![u_h]\!]_{j-\frac{1}{2}}.$$

The function $G(\omega, u)$ is differentiable in the second argument. Thus, by the mean value theorem there exists a $\vartheta \in [u^{-}_{h,j-\frac{1}{2}}, u^{+}_{h,j-\frac{1}{2}}]$ such that

$$G\left(\omega, u^{+}_{h,j-\frac{1}{2}}\right) - G\left(\omega, u^{-}_{h,j-\frac{1}{2}}\right) = g(\omega, \vartheta) [\![u_h]\!]_{j-\frac{1}{2}}.$$

Hence, by the properties ($\widehat{g}1$) and ($\widehat{g}2$) of the numerical flux $\widehat{g}(\omega, \cdot, \cdot)$ it follows that $\Theta_{j-\frac{1}{2}} \geq 0$. □

We would like to mention that for static grids, which means $\omega = 0$, we obtain the same cell entropy inequality as in [15]. In addition, the cell entropy inequality (2.16) implies the L^2 stability of the semi-discrete ALE-DG method.

Corollary 2.3. *The solution u_h of the semi-discrete ALE-DG method given by* (2.13) *satisfies for all $t \in [0, T]$ the inequality*

$$\|u_h(t)\|_{\mathrm{L}^2(\Omega)} \leq \|u_h(0)\|_{\mathrm{L}^2(\Omega)}.$$

2.3. **A priori error estimates.** In this section, we present a priori error estimates for the ALE-DG method for smooth solutions of (1.1). Therefore, we follow the approach of Zhang and Shu (cf. [28], [29] and [30]). However, since we apply time-dependent cells in our method, there are some differences in our proof. First of all, as in the proof of the cell entropy inequality (2.15), we cannot apply the ALE-DG solution u_h as a test function in equation (2.13). Thus, we have to apply the equivalent equation (2.17) for the proof. Further, we have to apply the transport equation (2.9) to manage the differentiation of the time-dependent volume integrals. Finally, we compensate the nonlinear nature of the flux function $f(u)$ by Taylor expansion as Zhang and Shu did. Therefore, we need an a priori assumption given by

$$\max_{t\in[0,T]} \|u - u_h\|_{\mathrm{L}^\infty(\Omega)} \leq C_1 h, \tag{2.19}$$

where the constant C_1 is independent of u_h and h. For the utilization of Taylor expansion, we need to ensure that the flux function $f(u)$ and its derivatives are bounded. Since we consider scalar conservation laws (1.1), the maximum principle guarantees that the flux function $f(u)$ itself and up to third order derivatives are bounded. To evaluate the numerical flux function $\widehat{g}(\omega, \cdot, \cdot)$ with Taylor expansion we proceed again as Zhang and Shu (cf. [28]) and apply a quantity $\widehat{a}(\widehat{g}; u)$ to measure the difference between the numerical flux function $\widehat{g}(\omega, u^-, u^+)$ and the flux $g(\omega, u)$. The quantity is for any piecewise smooth function $v \in \mathrm{L}^2(\Omega)$ defined by

$$\widehat{a}(\widehat{g}; v) := \begin{cases} [\![v]\!]^{-1} \left(g(\omega, \{\!\{v\}\!\}) - \widehat{g}(\omega, v^-, v^+)\right), & \text{if} \quad [\![v]\!] \neq 0, \\ |g'(\omega, \{\!\{v\}\!\})|, & \text{if} \quad [\![v]\!] = 0. \end{cases} \tag{2.20}$$

This quantity was introduced by Harten in [13]. Moreover, Zhang and Shu (cf. [28]) proved the following lemma for the quantity above.

Lemma 2.4. *Suppose the numerical flux function $\widehat{g}$ has the properties $(\widehat{g}1)$ - $(\widehat{g}3)$. Then for any piecewise smooth function $v \in \mathrm{L}^2(\Omega)$ the quantity $\widehat{a}(\widehat{g}; v)$ given by* (2.20) *is nonnegative and bounded by the constant $C_2 := \frac{1}{2}\left(L_{\widehat{g}}^- + L_{\widehat{g}}^+\right)$. In addition, the inequality*

$$\frac{1}{2}\left|g'\left(\omega, \{\!\{v\}\!\}\right)\right| \leq \widehat{a}\left(\widehat{g}; v\right) + C_3 \left|[\![v]\!]\right|$$

holds, where the constant C_3 only depends on the maximum of $|f''|$.

2.3.1. *Projections, interpolation properties and inverse inequalities.* First of all, we present two projections. The L^2 projection $\mathcal{P}_h(u)$ of a function $u \in \mathrm{L}^2(\Omega)$ into $\mathcal{V}_h(t)$ is for all $v_h \in \mathcal{V}_h(t)$ defined by

$$\left(\mathcal{P}_h(u), v_h\right)_{K_j(t)} = \left(u, v_h\right)_{K_j(t)}. \tag{2.21}$$

In addition, if $k \geq 1$, we define the Gauss-Radau projections $\mathcal{P}_h^{\pm}(u)$ of a function $u \in \mathrm{L}^2(\Omega)$ into $\mathcal{V}_h(t)$ for all $v_h \in \mathcal{V}_h(t)$ with the property $v_h(\chi_j(\cdot, t)) \in \mathbb{P}^{k-1}\big([-1,1]\big)$ by

$$\left(\mathcal{P}_h^{\pm}(u), v_h\right)_{K_j(t)} = \left(u, v_h\right)_{K_j(t)} \tag{2.22a}$$

and

$$\mathcal{P}_h^+(u)\left(x_{j-\frac{1}{2}}^+(t)\right) = u\left(x_{j-\frac{1}{2}}^+(t)\right), \quad \mathcal{P}_h^-(u)\left(x_{j+\frac{1}{2}}^-(t)\right) = u\left(x_{j+\frac{1}{2}}^-(t)\right). \tag{2.22b}$$

For the L^2 projection we have the following lemma.

Lemma 2.5. *Let $u \in \mathrm{L}^2(\Omega)$, $\mathcal{P}_h(u)$ be the L^2 projection of u and $v_h \in \mathcal{V}_h(t)$. Suppose for any cell $K_j(t)$ that the function v_h can be written as*

$$v_h(x,t) := \sum_{\ell=0}^{k} v_h^{\ell}(t)\, \widehat{\phi}_{\ell}(x,t),$$

where $v_h^0, \ldots, v_h^k \in \mathrm{H}^1(0,T)$ and $\widehat{\phi}_0, \ldots, \widehat{\phi}_k$ are given by (2.10). *Then we have*

$$\Big(u - \mathcal{P}_h(u), \partial_t v_h\Big)_{K_j(t)} = 0. \tag{2.22}$$

Proof. By (2.6) and (2.3) for any cell $K_j(t)$ the grid velocity can be rewritten as

$$\omega(\chi_j(\xi,t),t) = \frac{1}{2}\left((1-\xi)\omega_{j-\frac{1}{2}} + (1+\xi)\omega_{j+\frac{1}{2}}\right), \quad \xi \in [-1,1].$$

Hence, $\omega\partial_x v_h \in \mathcal{V}_h$, and therefore $\partial_t v_h \in \mathcal{V}_h$, since $\partial_t\widehat{\phi}_{\ell} = -\omega\partial_x\widehat{\phi}_{\ell}$, $\ell = 0, \ldots, k$, by (2.11). Thus, (2.21) yields (2.22). □

In addition, we apply the following auxiliary lemma.

Lemma 2.6. *Let $u \in \mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^1(\Omega)\right)$ and $\mathcal{Q}_h$ be either $\mathcal{P}_h$, $\mathcal{P}_h^-$ or $\mathcal{P}_h^+$. Then*

$$\partial_t\left(\mathcal{Q}_h(u)\right) + \omega\partial_x\left(\mathcal{Q}_h(u)\right) = \mathcal{Q}_h\left(\partial_t(u)\right) + \mathcal{Q}_h\left(\omega\partial_x(u)\right). \tag{2.23}$$

Proof. In order to prove (2.23), we will apply Legendre polynomials. Each Legendre polynomial L_{ℓ}, $\ell = 0, \cdots, k$, is an ℓ-th degree polynomial and can be expressed by Rodrigues' formula (cf. Abramowitz and Stegun [14]). In addition, the Legendre polynomials satisfy

$$\left(L_{\ell}, L_{\ell'}\right)_{[-1,1]} = \frac{2}{2\ell+1}\delta_{\ell\ell'}, \quad L_{\ell}(-1) = (-1)^{\ell} \quad \text{and} \quad L_{\ell}(1) = 1.$$

Furthermore, the Legendre polynomials supply an orthogonal basis of the space $P^k([-1,1])$. Therefore, the functions

$$\widehat{L}_\ell(x,t) := L_\ell\left(\frac{2\left(x - x_{j-\frac{1}{2}}(t)\right)}{\triangle_j(t)} - 1\right), \quad x \in K_j(t), \tag{2.24}$$

represent for any cell $K_j(t)$ an orthogonal basis of the test function space $\mathcal{V}_h(t)$. Hence, for the cell $K_j(t)$ the L^2 projection of a function $u \in \mathrm{L}^2(\Omega)$ can be written as

$$\mathcal{P}_h(u) := \sum_{\ell=0}^{k} c_\ell(u,t)\,\widehat{L}_\ell(x,t), \quad c_\ell(u,t) := \left(\frac{2\ell+1}{\triangle_j(t)}\right)\left(u, \widehat{L}_\ell(x,t)\right)_{K_j(t)}.$$

Similarly, for the cell $K_j(t)$ the Gauss-Radau projections of a function $u \in \mathrm{L}^2(\Omega)$ can be written as

$$\mathcal{P}_h^{\pm}(u) := \sum_{\ell=0}^{k} r_\ell^{\pm}(u,t)\,\widehat{L}_\ell(x,t),$$

where the coefficients are given by

$$r_\ell^{\pm}(u,t) := \left(\frac{2\ell+1}{\triangle_j(t)}\right)\left(u, \widehat{L}_\ell(x,t)\right)_{K_j(t)}, \quad \ell = 0, \cdots, k-1,$$

and

$$r_k^{+}(u,t) := (-1)^k u\left(x_{j-\frac{1}{2}}^{+}(t), t\right) - \sum_{\ell=0}^{k-1} (-1)^{k+\ell} r_\ell^{+}(u,t),$$

$$r_k^{-}(u,t) := u\left(x_{j+\frac{1}{2}}^{-}(t), t\right) - \sum_{\ell=0}^{k-1} r_\ell^{-}(u,t).$$

Next, we start to prove equality (2.23). Let u be an element of the space $\mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^1(\Omega)\right)$. It is easy to verify that

$$\frac{d}{dt}\left(u\left(x_{j+\frac{1}{2}}^{\pm}(t), t\right)\right) = \partial_t\left(u\left(x_{j+\frac{1}{2}}^{\pm}(t), t\right)\right) + \omega_{j+\frac{1}{2}}\partial_x\left(u\left(x_{j+\frac{1}{2}}^{\pm}(t), t\right)\right). \tag{2.25}$$

Likewise, it is easy to verify that

$$\partial_x(\omega(x,t)) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{\triangle_j(t)} = \frac{\triangle_j'(t)}{\triangle_j(t)}, \quad x \in K_j(t),$$

where $\triangle_j'(t)$ is the derivative with respect to t of the cell length $\triangle_j(t)$.

Therefore, we obtain by the transport equation (2.9) and (2.11)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{2\ell+1}{\triangle_j(t)}\left(u, \widehat{L}_\ell\right)_{K_j(t)}\right) &= \frac{2\ell+1}{\triangle_j(t)}\left(\partial_t u + \partial_x(\omega u), \widehat{L}_\ell\right)_{K_j(t)} \\
&\quad - \left(\frac{2\ell+1}{\triangle_j(t)}\right)\left(\frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{\triangle_j(t)}\right)\left(u, \widehat{L}_\ell\right)_{K_j(t)} \\
&= \frac{2\ell+1}{\triangle_j(t)}\left(\partial_t u + \omega\partial_x u, \widehat{L}_\ell\right)_{K_j(t)}.
\end{aligned} \tag{2.26}$$

Thus, by (2.25) and (2.26) we obtain

$$\partial_t(c_\ell(u,t)) = c_\ell(\partial_t(u), t) + c_\ell(\omega\partial_x(u), t) \tag{2.27}$$

and

$$\partial_t \left(r_\ell^\pm (u,t)\right) = r_\ell^\pm \left(\partial_t (u), t\right) + r_\ell^\pm \left(\omega \partial_x (u), t\right), \tag{2.28}$$

where the functions $(c_\ell (u,t))$ and $\left(r_\ell^\pm (u,t)\right)$ are the coefficients of the projections $\mathcal{P}_h (u)$ and $\mathcal{P}_h^\pm (u)$.

Let $\mathcal{Q}_h (u)$ be either $\mathcal{P}_h (u)$ or $\mathcal{P}_h^\pm (u)$. Then, for the cell $K_j(t)$ the projection $\mathcal{Q}_h (u)$ can be written as

$$\mathcal{Q}_h (u) = \sum_{\ell=0}^{k} q_\ell (u,t) \widehat{L}_\ell (x,t),$$

where the coefficients $q_\ell (u,t)$ are $c_\ell (u,t)$ or $r_\ell^\pm (u,t)$. Finally, by (2.11), (2.27) and (2.28) we obtain

$$\begin{aligned} \partial_t (\mathcal{Q}_h (u)) &= \sum_{\ell=0}^{k} \partial_t (q_\ell (u,t)) \widehat{L}_\ell (x,t) - \left(\sum_{\ell=0}^{k} q_\ell (u,t)\, \omega (x,t)\, \partial_x \left(\widehat{L}_\ell (x,t)\right)\right) \\ &= \mathcal{Q}_h (\partial_t (u)) + \mathcal{Q}_h (\omega \partial_x (u)) - \omega \partial_x (\mathcal{Q}_h (u)). \end{aligned} \qquad \square$$

Further, we will apply the following interpolation properties. For an arbitrary fixed function $u \in \mathrm{H}^{k+1}(\Omega)$ there are constants C_4 and C_5, which are independent of h, such that

$$\|u - \mathcal{Q}_h (u)\|_{\mathrm{L}^2(\Omega)}^2 \le C_4 \left\|\partial_x^{k+1} u\right\|_{\mathrm{L}^2(\Omega)}^2 h^{2k+2} \tag{2.29}$$

and

$$\left\|u - \mathcal{Q}_h (u)\right\|_{\mathrm{L}^2(\Gamma)}^2 \le C_5 \left\|\partial_x^{k+1} u\right\|_{\mathrm{L}^2(\Omega)}^2 h^{2k+1}, \tag{2.30}$$

where we have applied the norm

$$\|u\|_{\mathrm{L}^2(\Gamma)}^2 := \sum_{n,j=1}^{N} \left(\left| u\left(x_{j-\frac{1}{2}}^{+}(t), t\right)\right|^2 + \left| u\left(x_{j-\frac{1}{2}}^{-}(t), t\right)\right|^2 \right).$$

Moreover, we will apply for all $v_h \in \mathcal{V}_h(t)$ the inverse and trace inequality

$$h^2 \|\partial_x (v_h)\|_{\mathrm{L}^2(\Omega)}^2 + h \|v_h\|_{\mathrm{L}^2(\Gamma)}^2 \le C_6 \|v_h\|_{\mathrm{L}^2(\Omega)}^2, \tag{2.31}$$

where the constant C_6 is independent of h and v_h. These inequalities can be proven by well known results of basic approximation theory (cf. Ciarlet [1]), since we assume that the mesh is regular.

2.3.2. *A sub-optimal error estimate by using monotone fluxes.* In this section, we state an a priori error estimate for the semi-discrete ALE-DG method with a general monotone numerical flux.

Theorem 2.7. *Let* $u \in \mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^{k+1}(\Omega)\right)$ *be the exact solution of equation* (1.1), $f \in \mathcal{C}^2(\mathbb{R})$ *and* u_h *be the solution of the semi-discrete ALE-DG method* (2.13) *with a monotone numerical flux* $\widehat{g}$. *The initial data for the method is the* L^2 *projection of the function* u_0, *and the grid velocity satisfies the conditions* (ω1) *and* (ω2). *Then there exists a constant* C *independent of* u_h *and* h *such that there holds the error estimate*

$$\max_{t\in[0,T]} \|e_h\|_{\mathrm{L}^2(\Omega)} \le C h^{k+\frac{1}{2}},$$

where $e_h = u - u_h$ *and* h *is given by* (2.4).

Proof. We define the quantities

$$\psi_h := u - \mathcal{P}_h(u) \quad \text{and} \quad \varphi_h := u_h - \mathcal{P}_h(u). \tag{2.32}$$

Then, the error function can be written as

$$e_h := u - u_h = \psi_h - \varphi_h. \tag{2.33}$$

The exact solution u and the approximation solution u_h satisfy the equation (2.13) and the equivalent equation (2.17). Hence, equation (2.17) supplies the following error equation:

$$\begin{aligned} 0 &= (\partial_t e_h, v_h)_{K_j(t)} + (\partial_x(\omega e_h v_h), 1)_{K_j(t)} - (f(u) - f(u_h), \partial_x v_h)_{K_j(t)} \\ &\quad + g\left(\omega_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} - g\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}} \\ &\quad - \widehat{g}\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} + \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}}. \end{aligned} \tag{2.34}$$

Taylor expansion on the flux function $f(u_h)$ up to second order provides the identity

$$f(u_h) = f(u) - f'(u)\, e_h + \frac{1}{2} f''(\Theta)(e_h)^2, \tag{2.35}$$

where Θ is a value between u and u_h. Since we assume that the exact solution u of (1.1) is sufficiently smooth, the following equation holds in the cell interface points $x_{j-\frac{1}{2}}(t)$:

$$[\![u_h]\!]_{j-\frac{1}{2}} = -[\![e_h]\!]_{j-\frac{1}{2}}. \tag{2.36}$$

Thus, by applying (2.20) we obtain in the cell boundary points $x_{j-\frac{1}{2}}(t)$:

$$\begin{aligned} &g\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - \widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) \\ &= g\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - g\left(\omega_{j-\frac{1}{2}}, \{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right) + \widehat{a}(\widehat{g}; u_h)_{j-\frac{1}{2}} [\![u_h]\!]_{j-\frac{1}{2}} \\ &= g'\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) \{\!\{e_h\}\!\}_{j-\frac{1}{2}} - \frac{1}{2} f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2 - \widehat{a}(\widehat{g}; u_h)_{j-\frac{1}{2}} [\![e_h]\!]_{j-\frac{1}{2}}, \end{aligned} \tag{2.37}$$

where $\Theta_{j-\frac{1}{2}}$ is a value between $u_{j-\frac{1}{2}}$ and $\{\!\{u_h\}\!\}_{j-\frac{1}{2}}$. Next, we apply φ_h as a test function and sum the error equation (2.34). Then, we obtain by (2.35) and (2.37)

$$\begin{aligned} 0 &= \sum_{j=1}^{N} (\partial_t e_h, \varphi_h)_{K_j(t)} + \sum_{j=1}^{N} (\partial_x(\omega e_h \varphi_h), 1)_{K_j(t)} - \sum_{j=1}^{N} (f'(u)\, e_h, \partial_x \varphi_h)_{K_j(t)} \\ &\quad + \frac{1}{2} \sum_{j=1}^{N} \left(f''(\Theta)(e_h)^2, \partial_x \varphi_h\right)_{K_j(t)} - \sum_{j=1}^{N} g'\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) \{\!\{e_h\}\!\}_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}} \\ &\quad + \frac{1}{2} \sum_{j=1}^{N} f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}} + \sum_{j=1}^{N} \widehat{a}(\widehat{g}; u_h)_{j-\frac{1}{2}} [\![e_h]\!]_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}}. \end{aligned} \tag{2.38}$$

By the transport equation (2.9) we obtain

$$
\begin{aligned}
&-\sum_{j=1}^{N}\left(\partial_t\varphi_h,\varphi_h\right)_{K_j(t)}-\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right),1\right)_{K_j(t)}\\
&=-\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}-\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right),1\right)_{K_j(t)}. \qquad (2.39)
\end{aligned}
$$

Likewise, by the properties (2.21) as well as (2.22) of the L^2 projection and the transport equation (2.9) we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N}\left(\partial_t\psi,\varphi_h\right)_{K_j(t)}+\sum_{j=1}^{N}\left(\partial_x\left(\omega\psi\varphi_h\right),1\right)_{K_j(t)}\\
&=\sum_{j=1}^{N}\left(\partial_t\left(\psi\varphi_h\right),1\right)_{K_j(t)}+\sum_{j=1}^{N}\left(\partial_x\left(\omega\psi\varphi_h\right),1\right)_{K_j(t)}\\
&=\sum_{j=1}^{N}\frac{d}{dt}\left(\psi,\varphi_h\right)_{K_j(t)}=0. \qquad (2.40)
\end{aligned}
$$

Therefore, by (2.39) and (2.40) the equation (2.38) can be rewritten as

$$
\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}=a_1\left(\psi_h,\varphi_h\right)+a_2\left(e_h,\varphi_h\right)+a_3\left(\omega,\psi_h,\varphi_h\right), \qquad (2.41)
$$

where

$$
\begin{aligned}
a_1\left(\psi_h,\varphi_h\right)&:=-\sum_{j=1}^{N}\left(f'(u)\,\psi_h,\partial_x\varphi_h\right)_{K_j(t)},\\
a_2\left(e_h,\varphi_h\right)&:=\frac{1}{2}\sum_{j=1}^{N}\left(f''(\Theta)\left(e_h\right)^2,\partial_x\varphi_h\right)_{K_j(t)}\\
&\quad+\frac{1}{2}\sum_{j=1}^{N}f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2[\![\varphi_h]\!]_{j-\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
a_3\left(\omega,\psi_h,\varphi_h\right)&:=\frac{1}{2}\sum_{j=1}^{N}\left(f'(u),\partial_x\left(\varphi_h\right)^2\right)_{K_j(t)}\\
&\quad-\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right),1\right)_{K_j(t)}\\
&\quad-\sum_{j=1}^{N}g'\left(\omega_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)\{\!\{e_h\}\!\}_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}\\
&\quad+\sum_{j=1}^{N}\widehat{a}\left(\widehat{g};u_h\right)_{j-\frac{1}{2}}[\![e_h]\!]_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}.
\end{aligned}
$$

Henceforth, the quantities $a_1(\psi_h,\varphi_h)$, $a_2(e_h,\varphi_h)$ and $a_3(\omega,\psi_h,\varphi_h)$ can be estimated by (2.29), (2.30) and (2.31) as in the papers of Zhang and Shu [28–30].

Thus, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} \leq C_I\left(h^{2k+1} + \left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right), \tag{2.42}$$

where the constant C_I is independent of u_h, h and $t \in [0,T]$. Hence, by Gronwall's inequality and the identity $u_h(0) = \mathcal{P}_h(u_0)$ it follows that, for all $t \in [0,T]$,

$$\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} \leq \mathrm{e}^{2C_I T} h^{2k+1}. \tag{2.43}$$

Thus, for all $t \in [0,T]$ the error function e_h can be estimated as

$$\left\|e_h\right\|_{\mathrm{L}^2(\Omega)} \leq \left\|\psi_h\right\|_{\mathrm{L}^2(\Omega)} + \left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)} \leq C_{II} h^{2k+1},$$

where the constant C_{II} is independent of u_h and h. □

2.3.3. *An optimal error estimate by using an upwind numerical flux.* In order to achieve the optimal a priori error estimate for the ALE-DG method, we assume that $g'(\omega, v) \geq 0$. Thus, we can apply an upwind numerical flux function given by

$$\widehat{g}\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) := g\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}(t)\right), \quad \forall j = 1, \cdots, N. \tag{2.44}$$

This numerical flux provides the following a priori error estimate.

Theorem 2.8. *Let $u \in \mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^{k+2}(\Omega)\right)$ be the exact solution of equation (1.1). Suppose $f \in \mathcal{C}^2(\mathbb{R})$ and the grid velocity satisfies the conditions (ω1), (ω2) as well as (ω3). Further, the condition $g'(\omega, v) \geq 0$ is satisfied. Let u_h be the solution of the semi-discrete ALE-DG method (2.13) with the upwind flux (2.44). The initial data for the method is the Gauss-Radau projection $\mathcal{P}^-_h$ of u_0. Then there exists a constant C independent of u_h and h such that the following error estimate holds:*

$$\max_{t\in[0,T]} \left\|e_h\right\|_{\mathrm{L}^2(\Omega)} \leq C h^{k+1},$$

where $e_h := u - u_h$ and h is given by (2.4).

Proof. First of all, we define the quantities

$$\psi_h := u - \mathcal{P}^-_h(u) \quad \text{and} \quad \varphi_h := u_h - \mathcal{P}^-_h(u) \tag{2.45}$$

as in the proof of Theorem 2.7. Then, the ALE-DG scheme (2.17) yields the following error equation:

$$\begin{aligned} 0 = &\left(\partial_t e_h, v_h\right)_{K_j(t)} + \left(\partial_x\left(\omega e_h v_h\right), 1\right)_{K_j(t)} - \left(f(u) - f(u_h), \partial_x v_h\right)_{K_j(t)} \\ &+ g\left(\omega_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} - g\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}} \\ &- g\left(\omega_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}\right) v^-_{h,j+\frac{1}{2}} + g\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}\right) v^+_{h,j-\frac{1}{2}}. \end{aligned} \tag{2.46}$$

By Taylor expansion on the flux function $g(\omega, \cdot)$ up to second order,

$$\begin{aligned} &g\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - g\left(\omega_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}\right) \\ =&g'\left(\omega_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) e^-_{h,j-\frac{1}{2}} - \frac{1}{2} f''\left(\Theta^-_{j-\frac{1}{2}}\right)\left(e_{h,j-\frac{1}{2}}\right)^2 \end{aligned} \tag{2.47}$$

holds, where $\Theta^-_{j-\frac{1}{2}}$ is a value between $u_{j-\frac{1}{2}}$ and $u^-_{h,j-\frac{1}{2}}$. Next, we apply φ_h as a test function and sum the error equation (2.46). Then, we obtain by (2.35) and (2.47)

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} = b_1\left(e_h,\varphi_h\right) + b_2\left(\omega,\varphi_h,\varphi_h\right) + b_3\left(\omega,\psi_h,\varphi_h\right), \tag{2.48}$$

where

$$\begin{aligned} b_1\left(e_h,\varphi_h\right) =& \frac{1}{2}\sum_{j=1}^{N}\left(f''\left(\Theta\right)\left(e_h\right)^2,\partial_x\varphi_h\right)_{K_j(t)} \\ &+\frac{1}{2}\sum_{j=1}^{N} f''\left(\Theta^-_{j-\frac{1}{2}}\right)\left(e^-_{h,j-\frac{1}{2}}\right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} b_2\left(\omega,\varphi_h,\varphi_h\right) =& \frac{1}{2}\sum_{j=1}^{N}\left(f'\left(u\right),\partial_x\left(\left(\varphi_h\right)^2\right)\right)_{K_j(t)} - \frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right),1\right)_{K_j(t)} \\ &+\sum_{j=1}^{N} g'\left(\omega_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)\varphi^-_{h,j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} b_3\left(\omega,\psi_h,\varphi_h\right) =& \sum_{j=1}^{N}\left(\partial_t\psi_h,\varphi_h\right)_{K_j(t)} + \sum_{j=1}^{N}\left(\partial_x\left(\omega\psi_h\varphi_h\right),1\right)_{K_j(t)} \\ &-\sum_{j=1}^{N}\left(f'\left(u\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)}. \end{aligned}$$

Henceforth, the quantities $b_1\left(e_h,\varphi_h\right)$, $b_2\left(\omega,\varphi_h,\varphi_h\right)$ and $b_3\left(\omega,\psi_h,\varphi_h\right)$ can be estimated by (2.29), (2.30) and (2.31) as in the papers of Zhang and Shu [28–30]. Thus, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} \leq C_I\left(h^{2k+2} + \left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right), \tag{2.49}$$

where the constant C_I is independent of u_h, h and $t \in [0,T]$. The final steps in the proof of Theorem 2.8 are exactly the same as in the proof of Theorem 2.7. □

Remark 2.1. If we assume $g'\left(\omega,v\right) \leq 0$, the result in Theorem 2.8 holds true, too. In this case, we have to apply the numerical flux function

$$\widehat{g}\left(\omega_{j-\frac{1}{2}},u^-_{h,j-\frac{1}{2}},u^+_{h,j-\frac{1}{2}}\right) := g\left(\omega_{j-\frac{1}{2}},u^+_{h,j-\frac{1}{2}}\right), \quad \forall j = 1,\cdots,N,$$

and the Gauss-Radau projection $\mathcal{P}^+_h$.

2.4. **The fully discrete ALE-DG method.** In this section, we consider and analyze the time discretization of the ALE-DG method.

2.4.1. *The geometric conservation law.* The geometric conservation law (GCL) governs the geometric parameters of a grid deformation method, such that the method preserves constant states. In other words, if we consider the equation (1.1) with the initial condition $u_h(x,0) \equiv 1$ for all $(x,t) \in \Omega \times [0,T]$, the approximate solution given by the ALE-DG method has to be $u_h(x,t) \equiv 1$, too.

For all $v_h \in \mathcal{V}_h(t)$ the ALE-DG scheme (2.13) degenerates to

$$\frac{d}{dt}\left(1, v_h\right)_{K_j(t)} = \left(\partial_x \omega, v_h\right)_{K_j(t)}, \quad \forall j = 1, \cdots, N, \tag{2.50}$$

if the approximate solution is given by $u_h(x,t) \equiv 1$. This equation is the geometric conservation law (GCL) for the ALE-DG method. Certainly, the equation (2.50) is a special case of the transport equation (2.9) and thus is satisfied.

However, the situation is slightly different when the method is discretized in time. A discrete version of the GCL is the discrete geometric conservation law (dGCL). In general it is not clear that the discrete geometric conservation law (dGCL) holds true whenever the GCL is satisfied. The method will not preserve constant states if there is no dGCL satisfied. This leads to a lack of stability and accuracy (cf. Grandmont, Guillard and Farhat [9] or Farhat and Geuzaine [12]). Fortunately, the forward Euler time discretization of the ALE-DG method satisfies a dGCL. This can be realized as follows. By applying the mapping (2.6) we rewrite the semi-discrete GCL condition (2.50) as

$$\frac{d}{dt}\Big(1, v_h J\Big)_{[-1,1]} = \Big(\partial_x \omega, v_h J\Big)_{[-1,1]}, \tag{2.51}$$

where $J(t) = \frac{\Delta_j(t)}{2}$ is the determinant of the Jacobian matrix. Note that $J(t) = \frac{\Delta_j(t)}{2}$ does not depend on x. Further, the definition of the mapping yields

$$\partial_x\big(\omega(x,t)\big) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{2}.$$

Thus, $\partial_x\big(\omega(x,t)\big)$ does not depend on x either. Hence, the dGCL condition becomes

$$\frac{d}{dt}\Big(J(t)\Big) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{2}.$$

Therefore, since $J(t)$ is linearly dependent on t, the dGCL can be easily satisfied for any first order or high order single step time discretization method, e.g. the forward Euler method or total variation diminishing (TVD) Runge-Kutta methods, also known as strong stability preserving (SSP) Runge-Kutta methods (cf. Gottlieb and Shu [11]).

Proposition 2.9. *The fully discrete ALE-DG method* (2.13) *with the approximation space* (2.8) *satisfies the discrete geometric conservation law for any first order time discretization method or high order single step method in which the stage order is equal to or higher than first order.*

2.4.2. *The local maximum principle.* In [31] X. Zhang and Shu developed for static grids a maximum-principle-satisfying limiter. We will prove that their limiter can also be applied to our ALE-DG method. However, our proof is slightly different from Zhang and Shu's proof, since we have to control an extra term resulting from the time-dependent cells. In this section, we consider the ALE-DG method with the Lax-Friedrichs flux. The Lax-Friedrichs flux is given by

$$\widehat{g}\left(\omega, u_h^-, u_h^+\right) := \frac{g\left(\omega, u_h^-\right) + g\left(\omega, u_h^+\right)}{2} - \frac{\lambda}{2}\left(u_h^+ - u_h^-\right), \tag{2.52}$$

where

$$\lambda := \max_{x\in\Omega}\left\{\left|\partial_{u_h} g\left(\omega\left(x,t\right), u_h\right)\right|\right\}. \tag{2.53}$$

The Lax-Friedrichs flux can be split up in an increasing function

$$\widehat{g}_+\left(\omega, u_h^-\right) := \frac{1}{2}\left(g\left(\omega, u_h^-\right) + \lambda u_h^-\right) \tag{2.54}$$

and a decreasing function

$$\widehat{g}_-\left(\omega, u_h^+\right) := \frac{1}{2}\left(g\left(\omega, u_h^+\right) - \lambda u_h^+\right). \tag{2.55}$$

Further, for all $x, y \in K_j\left(t\right)$ it holds that

$$\widehat{g}\left(\omega\left(x,t\right), a, b\right) - \widehat{g}\left(\omega\left(y,t\right), a, b\right) = -\left(\omega\left(x,t\right) - \omega\left(y,t\right)\right)\frac{a+b}{2}. \tag{2.56}$$

Henceforth, for any cell $K_j(t)$ the average value of the ALE-DG solution u_h is denoted by

$$\overline{u}_j(t) := \frac{1}{\triangle_j(t)}\int_{K_j(t)} u_h(x,t)\,dx, \tag{2.57}$$

and the forward and backward differential operators of the cell average value are denoted by

$$\triangle_+\overline{u}_j := \overline{u}_{j+1} - \overline{u}_j \quad \text{and} \quad \triangle_-\overline{u}_j := \overline{u}_j - \overline{u}_{j-1}. \tag{2.58}$$

In order to rewrite the average value of the ALE-DG solution, we apply the p-point Gauss-Lobatto quadrature rule in the reference cell $[-1,1]$, where we choose p to be the smallest integer satisfying $p-3 \geq k$, if a piecewise P^k polynomial approximation space is used. We denote the quadrature points by

$$-1 = \xi_1 < \xi_2 < \cdots < \xi_p = 1,$$

and the corresponding weights by σ_ν, $\nu = 1, \cdots, p$. Note that $\sum_{\nu=1}^{p} \frac{\sigma_\nu}{2} = 1$. Next, we define

$$u_{h,j-\frac{1}{2}}^{n,+} := u_h\left(\chi_j\left(-1, t_n\right), t_n\right) := u_h^{n,1}, \quad u_{h,j+\frac{1}{2}}^{n,-} := u_h\left(\chi_j\left(1, t_n\right), t_n\right) := u_h^{n,p} \tag{2.59}$$

and for all $\nu = 2, \cdots, p-1$,

$$u_h\left(\chi_j\left(\xi_j^\nu, t_n\right), t_n\right) := u_h^{n,\nu}.$$

Hence, we obtain

$$\overline{u}_j^n = \frac{1}{2}\int_{-1}^{1} u_h\left(\chi_j\left(\xi, t_n\right), t_n\right)\,d\xi = \sum_{\nu=1}^{p}\frac{\sigma_\nu}{2}u_h^{n,\nu}, \tag{2.60}$$

since the parameter p is chosen such that the Gauss-Lobatto quadrature rule is exact for polynomials of degree k.

In the following, we consider the forward Euler time discretization of the weak formulation (2.13). By Proposition 2.9 we get the geometric conservation law for the forward Euler time discretization of the ALE-DG method. Therefore, we obtain

$$\triangle_j^{n+1} - \triangle_j^n = \triangle t \left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right), \tag{2.61}$$

where $\triangle_j^n = \triangle_j(t_n)$. Next, the forward Euler time discretization of the discrete weak formulation (2.13) with the test function $v_h = 1$ and the identities (2.54), (2.55), (2.56) as well as (2.61) provide

$$\begin{aligned}\overline{u}_j^{n+1} = \overline{u}_j^n &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_{h,j-\frac{1}{2}}^{n,+}\right)\right)\\ &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_{h,j+\frac{1}{2}}^{n,-}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_{h,j-\frac{1}{2}}^{n,-}\right)\right)\\ &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\left(\overline{u}_j^n - \frac{1}{2}\left(u_{h,j+\frac{1}{2}}^{n,-} + u_{h,j-\frac{1}{2}}^{n,+}\right)\right).\end{aligned} \tag{2.62}$$

Finally, we are able to state the following lemma.

Lemma 2.10. *Let u_h be the solution of the forward Euler time discretization of the ALE-DG method (2.13) with the Lax-Friedrichs flux (2.52). For all $j = 1, \cdots, N$ all values $u_{h,j-\frac{1}{2}}^{-}(t_n), u_h^{n,1}, \cdots, u_h^{n,p}, u_{h,j+\frac{1}{2}}^{+}(t_n)$ and $\overline{u}_j^n$ are in the interval $[m, M]$ and the grid velocity satisfies the conditions ($\omega1$), ($\omega2$) as well as ($\omega3$). Further, the quantity $h := \max_{t\in[t_n,t_{n+1}]} h(t) \in (0, 1)$ and the CFL condition*

$$\frac{\triangle t}{\rho h} \leq \frac{\min_{1\leq\nu\leq p} \sigma_\nu}{C_{0,1}\left(\min_{1\leq\nu\leq p} \sigma_\nu + 1\right) + 8\lambda} \tag{2.63}$$

is satisfied, where the parameter λ is given by (2.53), the parameter ρ comes from the mesh regularity property (2.5) and the constant $C_{0,1}$ comes from the condition ($\omega3$) of the grid velocity. Then for all $j = 1, \cdots, N$, $\overline{u}_j^{n+1}$ is in the interval $[m, M]$.

Proof. First of all, we define the quantities

$$C_j := \begin{cases} -\left(\dfrac{\widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_h^{n,1}\right)}{u_{h,j+\frac{1}{2}}^{n,+} - u_h^{n,1}}\right), & \text{if } u_{h,j+\frac{1}{2}}^{n,+} \neq u_h^{n,1},\\ 0, & \text{if } u_{h,j+\frac{1}{2}}^{n,+} = u_h^{n,1},\end{cases}$$

and

$$D_j := \begin{cases} \dfrac{\widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_h^{n,p}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_{h,j-\frac{1}{2}}^{n,-}\right)}{u_h^{n,p} - u_{h,j-\frac{1}{2}}^{n,+}}, & \text{if } u_h^{n,p} \neq u_{h,j-\frac{1}{2}}^{n,-},\\ 0, & \text{if } u_h^{n,p} = u_{h,j-\frac{1}{2}}^{n,-}.\end{cases}$$

Note that $C_j \geq 0$ and $D_j \geq 0$, since $\widehat{g}_-\left(\omega_{j+\frac{1}{2}}, \cdot\right)$ is a decreasing function and $\widehat{g}_+\left(\omega_{j-\frac{1}{2}}, \cdot\right)$ is an increasing function. Further, $C_j + D_j \leq 2\lambda$ follows by the mean value theorem. Next, we define for all $a \in \mathbb{R}^{p+2}$ the function

$$\begin{aligned}
&H\left(a_0, \cdots, a_{p+1}\right) \\
&:= \frac{\sigma_1}{2}\left(1 - \frac{\triangle t}{\triangle_j^{n+1}}\left(\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\left(1 - \frac{1}{\sigma_1}\right) + \frac{2}{\sigma_1}C_j\right)\right)a_1 \\
&+ \frac{\sigma_p}{2}\left(1 - \frac{\triangle t}{\triangle_j^{n+1}}\left(\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\left(1 - \frac{1}{\sigma_p}\right) + \frac{2}{\sigma_p}D_j\right)\right)a_p \\
&+ \frac{\triangle t}{\triangle_j^{n+1}}\left(C_j a_{p+1} + D_j a_0\right) \\
&+ \sum_{\nu=2}^{p-1} \frac{\sigma_\nu}{2}\left(1 - \frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\right)a_\nu.
\end{aligned}$$

Then, by applying (2.60) the scheme (2.62) can be written as

$$\overline{u}_j^{n+1} = H\left(u_{h,j-\frac{1}{2}}^{n,-}, u_h^{n,1}, \cdots, u_h^{n,p}, u_{h,j+\frac{1}{2}}^{n,+}\right). \tag{2.64}$$

The mean value theorem and the condition $(\omega 3)$ of the grid velocity provide

$$\left|\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right| \leq \max_{x \in K_j(t_n)} \left|\partial_x\left(\omega\left(x, t_n\right)\right)\right| \triangle_j^n \leq C_{0,1} h. \tag{2.65}$$

Thus, by applying the CFL number (2.63) it follows that

$$\partial_\nu H\left(a_0, \cdots, a_{p+1}\right) \geq 0 \text{ for all } a \in \mathbb{R}^{p+2} \text{ and } \nu = 0, \cdots, p+1.$$

Further, $H(a, \cdots, a) = a$ for all $a \in \mathbb{R}$, since $\sum_{\nu=1}^{p} \frac{\sigma_\nu}{2} = 1$. Therefore, (2.64) is a monotone scheme in conservation form. This completes the proof. □

We have seen that the dGCL (2.61) is an important ingredient to prove the local maximum principle for the ALE-DG method. In fact, Grandmont, Guillard and Farhat [9] have proven that a monotone finite volume ALE method satisfies the local maximum principle if and only if the method satisfies a dGCL. Finally, we apply the maximum-principle-satisfying limiter in [31] to ensure the local maximum principle for the ALE-DG method.

2.4.3. *Total variation stability.* In order to stabilize the Runge-Kutta DG method for static grids, Cockburn and Shu have developed TVD and TVB limiters (cf. [2,5,27] and [6]). The limiters ensure that the cell average values of the DG solution become stable in the sense of the seminorm

$$\left|u_h^n\right|_{\mathrm{TVM}} := \sum_{j=1}^{N} \left|\triangle_+ \overline{u}_j^n\right|.$$

In this section, we prove that the classical TVD and TVB limiters can be applied for our ALE-DG method, too. The main difference in our proof compared to the results in [2,5,27] and [6] is that we have to control an extra term resulting from the time-dependent cells.

As in the section before we consider the ALE-DG method merely for the Lax-Friedrichs flux (2.52). In order to obtain the total variation stability property in the average values, we follow the discussion in [2]. Therefore, we apply for all $v, w \in \mathbb{R}$ the notation

$$\eta(v,w) := \operatorname{sign}(v) - \operatorname{sign}(w).$$

First of all, we subtract the equation (2.62) for j from the equation (2.62) for $j+1$. Afterward, we sum over all j and obtain

$$|u_h^{n+1}|_{\mathrm{TVM}} - |u_h^n|_{\mathrm{TVM}} + \Theta + \Xi_\omega = 0. \tag{2.66}$$

The quantity Θ in (2.66) is given by

$$\begin{aligned}\Theta := &\sum_{j=1}^{N}\left(p\left(\overline{u}_{j+1}^n, u_{h,j+\frac{3}{2}}^{n,-}, u_{h,j+\frac{1}{2}}^{n,+}\right) - p\left(\overline{u}_j^n, u_{h,j+\frac{1}{2}}^{n,-}, u_{h,j-\frac{1}{2}}^{n,+}\right)\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_{h,j+\frac{1}{2}}^{n,-}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}, u_{h,j-\frac{1}{2}}^{n,-}\right)\right)\eta\left(\triangle_-\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &-\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}, u_{h,j-\frac{1}{2}}^{n,+}\right)\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_-\overline{u}_j^{n+1}\right),\end{aligned}$$

where for all piecewise continuous functions $v, w \in \mathrm{L}^2(\Omega)$,

$$p\left(v, w^-, w^+\right) := v - \frac{\triangle t}{\triangle_{j+1}^{n+1}}\widehat{g}_+\left(\omega_{j+\frac{1}{2}}, w^-\right) + \frac{\triangle t}{\triangle_j^{n+1}}\widehat{g}_-\left(\omega_{j+\frac{1}{2}}, w^+\right).$$

The other quantity Ξ_ω in equation (2.66) results from the grid velocity. It is given by

$$\begin{aligned}\Xi_\omega := &\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_{j+1}^{n+1}}a_{j+1,\mp}\left(\omega_{j+\frac{3}{2}} - \omega_{j+\frac{1}{2}}\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}b_{j,\mp}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\sum_{j=1}^{N}\frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right)c_{j,\pm}\\ &+\sum_{j=1}^{N}\frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right)d_{j,\pm},\end{aligned}$$

where

$$\begin{aligned}&a_{j,-} := -\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right), \quad b_{j,-} := -\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right),\\ &c_{j,+} := \eta\left(\triangle_+\overline{u}_j^n, \triangle_-\overline{u}_j^{n+1}\right), \quad d_{j,+} := \eta\left(\triangle_-\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\end{aligned}$$

if $\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right) \geq 0$ and

$$\begin{aligned}&a_{j,+} := u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n, \quad b_{j,+} := \overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+},\\ &c_{j,-} := -\eta\left(\triangle_-\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right), \quad d_{j,-} := -\eta\left(\triangle_+\overline{u}_j^n, \triangle_-\overline{u}_j^{n+1}\right)\end{aligned}$$

if $\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right) \leq 0$. Hence, the ALE-DG method is total variation diminishing stable in the average values if we can ensure that $\Theta + \Xi_\omega \geq 0$. In fact, the sum is positive if the ALE-DG solution satisfies the following conditions:

$$\operatorname{sign}\left(\triangle_+ \overline{u}_j^n\right) = \operatorname{sign}\left(r_{j,\mp} - s_{j,\mp}\right), \tag{2.67}$$

$$\operatorname{sign}\left(\triangle_- \overline{u}_j^n\right) = \operatorname{sign}\left(u_{h,j+\frac{1}{2}}^{n,-} - u_{h,j-\frac{1}{2}}^{n,-}\right), \tag{2.68}$$

$$\operatorname{sign}\left(\triangle_+ \overline{u}_j^n\right) = \operatorname{sign}\left(u_{h,j+\frac{1}{2}}^{n,+} - u_{h,j-\frac{1}{2}}^{n,+}\right), \tag{2.69}$$

where

$$r_{j,\mp} := p\left(\overline{u}_{j+1}^n, u_{h,j+\frac{3}{2}}^{n,-}, u_{h,j+\frac{1}{2}}^{n,+}\right) + \frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left(\omega_{j+\frac{3}{2}} - \omega_{j+\frac{1}{2}}\right) a_{j+1,\mp}$$

and

$$s_{j,\mp} := p\left(\overline{u}_j^n, u_{h,j+\frac{1}{2}}^{n,-}, u_{h,j-\frac{1}{2}}^{n,+}\right) - \frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right) b_{j,\mp}.$$

In addition,

$$\operatorname{sign}\left(\triangle_+ \overline{u}_j^n\right) = \operatorname{sign}\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right), \tag{2.70}$$

$$\operatorname{sign}\left(\triangle_- \overline{u}_j^n\right) = \operatorname{sign}\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right) \tag{2.71}$$

if $\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}} \geq 0$ and

$$\operatorname{sign}\left(\triangle_- \overline{u}_j^n\right) = \operatorname{sign}\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right), \tag{2.72}$$

$$\operatorname{sign}\left(\triangle_+ \overline{u}_j^n\right) = \operatorname{sign}\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right) \tag{2.73}$$

if $\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}} \leq 0$. In general the ALE-DG solution does not satisfy the conditions above. Therefore, the solution has to be revised by a postprocessing procedure. Next, we prove that the TVD limiters of Cockburn and Shu revise the ALE-DG solution such that the conditions (2.67) - (2.73) are satisfied. Let $\widetilde{u}_h$ be the ALE-DG solution revised by a TVD limiter. Then, $\widetilde{u}_h$ satisfies for all $j = 1, \cdots, N$ the conditions

$$\widetilde{u}_{h,j+\frac{1}{2}}^{n,-} = m\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n, \triangle_- \overline{u}_j^n, \triangle_+ \overline{u}_j^n\right) + \overline{u}_j^n \tag{2.74}$$

and

$$\widetilde{u}_{h,j-\frac{1}{2}}^{n,+} = \overline{u}_j^n - m\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}, \triangle_- \overline{u}_j^n, \triangle_+ \overline{u}_j^n\right), \tag{2.75}$$

where the function $m(\cdot)$ is the minmod function for all $a \in \mathbb{R}^s$ given by

$$m(\alpha_1, \cdots, \alpha_s) := \begin{cases} \sigma \min\limits_{1 \leq \tau \leq s} \alpha_\tau, & \text{if } \sigma = \operatorname{sgn}(\alpha_1) = \cdots = \operatorname{sign}(\alpha_s), \\ 0, & \text{else.} \end{cases}$$

The following result indicates that the TVD limiters can be applied for the forward Euler time discretization of the ALE-DG method.

Proposition 2.11. *Let $u_0 \in \mathrm{BV}(\Omega) \cap \mathrm{L}^1(\Omega)$ and $\widetilde{u}_h$ be the solution of the forward Euler time discretization of the ALE-DG method revised by a conservative TVD limiter, such that $\widetilde{u}_h$ satisfies (2.74) and (2.75). The initial data for the method are the L^2 projection of the function u_0, the grid velocity satisfying the conditions*

(ω1), (ω2) as well as (ω3), the quantity $h := \max_{t\in[t_n,t_{n+1}]} h(t) \in (0,1)$ *and the CFL condition*

$$\frac{\triangle t}{\rho h} \leq \frac{1}{C_{0,1}+4\lambda} \tag{2.76}$$

being satisfied. Then, for all $n = 0,\ldots,K$,

$$|u_h^n|_{TVM} \leq |u_0|_{\mathrm{BV}(\Omega)}\,.$$

Proof. We prove that for the revised ALE-DG solution $\widetilde{u}_h$ the sum $\Theta + \Xi_\omega$ in equation (2.66) becomes nonnegative. Therefore, we have to prove that $\widetilde{u}_h$ satisfies the conditions (2.67) - (2.73). Since $\widetilde{u}_h$ satisfies the equations (2.74) and (2.75), the conditions (2.70) - (2.73) are obviously fulfilled. Moreover, we obtain

$$\widetilde{u}^{n,-}_{h,j+\frac12} - \widetilde{u}^{n,-}_{h,j-\frac12} = \widetilde{u}^{n,-}_{h,j+\frac12} - \overline{u}^n_j + \triangle_-\overline{u}^n_j - \left(\widetilde{u}^{n,-}_{h,j-\frac12} - \overline{u}^n_{j-1}\right).$$

Since $\triangle_-\overline{u}^n_j = 0$, the equations (2.74) and (2.75) imply that $\widetilde{u}^{n,-}_{h,j+\frac12} - \widetilde{u}^{n,-}_{h,j-\frac12} = 0$. Next, by the definition of the minmod function and (2.74) as well as (2.75) we obtain

$$0 \leq \left(1 - \frac{\widetilde{u}^{n,-}_{h,j-\frac12} - \overline{u}^n_{j-1}}{\triangle_-\overline{u}^n_j} + \frac{\widetilde{u}^{n,-}_{h,j+\frac12} - \overline{u}^n_j}{\triangle_-\overline{u}^n_j}\right) \leq 2 \tag{2.77}$$

if $\triangle_-\overline{u}^n_j \neq 0$. Hence, (2.68) is satisfied, since

$$\widetilde{u}^{n,-}_{h,j+\frac12} - \widetilde{u}^{n,-}_{h,j-\frac12} = \left(1 - \frac{\widetilde{u}^{n,-}_{h,j-\frac12} - \overline{u}^n_{j-1}}{\triangle_-\overline{u}^n_j} + \frac{\widetilde{u}^{n,-}_{h,j+\frac12} - \overline{u}^n_j}{\triangle_-\overline{u}^n_j}\right)\triangle_-\overline{u}^n_j.$$

Likewise,

$$0 \leq \left(1 - \frac{\overline{u}^n_{j+1} - \widetilde{u}^{n,+}_{h,j+\frac12}}{\triangle_+\overline{u}^n_j} + \frac{\overline{u}^n_j - \widetilde{u}^{n,+}_{h,j-\frac12}}{\triangle_+\overline{u}^n_j}\right) \leq 2 \tag{2.78}$$

if $\triangle_+\overline{u}^n_j \neq 0$. Thus, the condition (2.69) can be proven similarly.

It remains to prove that condition (2.67) is satisfied. Note that $r_{j,\mp} - s_{j,\mp}=0$ if $\triangle_+\overline{u}^n_j = 0$. Therefore, in the following we assume that $\triangle_+\overline{u}^n_j \neq 0$. Since $\widetilde{u}_h$ satisfies the equations (2.74) as well as (2.75), by the mean value theorem, the mesh regularity property (2.5), (2.77) and (2.78) we obtain

$$\frac{\triangle t}{\triangle^{n+1}_{j+1}}\left|\frac{\widehat{g}_+\left(\omega_{j+\frac12}, \widetilde{u}^{n,-}_{h,j+\frac32}\right) - \widehat{g}_+\left(\omega_{j+\frac12}, \widetilde{u}^{n,-}_{h,j+\frac12}\right)}{\triangle_+\overline{u}^n_j}\right| \leq \frac{2\lambda\triangle t}{\rho h}$$

and

$$\frac{\triangle t}{\triangle^{n+1}_{j}}\left|\frac{\widehat{g}_-\left(\omega_{j+\frac12}, u^{n,+}_{h,j+\frac12}\right) - \widehat{g}_-\left(\omega_{j+\frac12}, u^{n,+}_{h,j-\frac12}\right)}{\triangle_+\overline{u}^n_j}\right| \leq \frac{2\lambda\triangle t}{\rho h}.$$

In addition, the equations (2.74) and (2.75) yield

$$\left|\frac{a_{j+1,\mp}}{\triangle_+\overline{u}^n_j}\right| \leq 1 \quad \text{and} \quad \left|\frac{b_{j,\mp}}{\triangle_+\overline{u}^n_j}\right| \leq 1.$$

Thus, since the grid velocity satisfies the condition ($\omega 3$) and $h \in (0,1)$, by the mesh regularity property (2.5) as well as (2.65) we obtain

$$\frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\left(\omega_{j+\frac{3}{2}}-\omega_{j+\frac{1}{2}}\right)\frac{a_{j+1,\mp}}{\triangle_{+}\overline{u}_{j}^{n}}\right| \leq \frac{C_{0,1}\triangle t}{2\rho h}$$

and

$$\frac{1}{2}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\left(\omega_{j+\frac{1}{2}}-\omega_{j-\frac{1}{2}}\right)\frac{b_{j,\mp}}{\triangle_{+}\overline{u}_{j}^{n}}\right| \leq \frac{C_{0,1}\triangle t}{2\rho h}.$$

Therefore, the CFL condition (2.76) provides

$$\begin{aligned}\left|\triangle_{+}\overline{u}_{j}^{n}\right| \geq{}& \frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\widehat{g}_{+}\left(\omega_{j+\frac{1}{2}},\widetilde{u}_{h,j+\frac{3}{2}}^{n,-}\right)-\widehat{g}_{+}\left(\omega_{j+\frac{1}{2}},\widetilde{u}_{h,j+\frac{1}{2}}^{n,-}\right)\right| \\ &+\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\widehat{g}_{-}\left(\omega_{j+\frac{1}{2}},u_{h,j+\frac{1}{2}}^{n,+}\right)-\widehat{g}_{-}\left(\omega_{j+\frac{1}{2}},u_{h,j-\frac{1}{2}}^{n,+}\right)\right| \\ &+\frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\left(\omega_{j+\frac{3}{2}}-\omega_{j+\frac{1}{2}}\right)a_{j+1,\mp}\right|+\frac{1}{2}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\left(\omega_{j+\frac{1}{2}}-\omega_{j-\frac{1}{2}}\right)b_{j,\mp}\right|. \qquad (2.79)\end{aligned}$$

Hence, the condition (2.67) is satisfied, since for all $a,b\in\mathbb{R}$ with $|a|>|b|$ it follows that $\operatorname{sign}(a)=\operatorname{sign}(a-b)$. Since the TVD limiter is a conservative limiter, the function u_h evaluated in the semi-norm $|\cdot|_{\mathrm{TVM}}$ has the same value as the revised function $\widetilde{u}_h$ evaluated in the same semi-norm. Therefore, by applying successively the inequality resulting from the equation (2.66), we obtain

$$\left|u_h^n\right|_{\mathrm{TVM}} \leq \left|u_h^0\right|_{\mathrm{TVM}}.$$

Since u_h^0 is the L^2 projection of the function $u_0\in \mathrm{BV}(\Omega)$, we obtain the result. □

To maintain the high order accuracy at local extrema, a TVB limiter has been introduced (cf. [2,5,27] and [6]). The TVB limiter based on the modified minmod function

$$\overline{m}\left(\alpha_1,\cdots,\alpha_s\right) := \begin{cases}\alpha_1, & \text{if } |\alpha_1| \leq \widetilde{M}\left(\triangle_j^n\right)^2, \\ m\left(\alpha_1,\cdots,\alpha_s\right), & \text{else.}\end{cases} \qquad (2.80)$$

Selection options for the parameter $\widetilde{M}$ have been discussed in [5]. Further, Cockburn and Shu have proven that the TVB limiter provides TVB stability and does not affect the accuracy of the method. We have a similar result for the forward Euler time discretization of the ALE-DG method with the TVB limiter.

Proposition 2.12. *Let u_h be the solution of the forward Euler time discretization of the ALE-DG method and $\widetilde{u}_h$ be the solution of the method revised by a conservative TVB limiter, such that $\widetilde{u}_h$ satisfies (2.74) and (2.75) for the function (2.80) instead of the minmod function. Suppose for smooth solutions of* (1.1) *u_h is a $(k+1)$-th order accurate approximation. Then under the same assumption as in Proposition 2.11 it holds that for all $n=0,\cdots,K$,*

$$\left|u_h^n\right|_{TVM} \leq |u_0|_{\mathrm{BV}(\Omega)} + \left(4C_{0,1}+16\lambda\right)\widetilde{M}\,|\Omega|\,T,$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω and $t_K=T$. Moreover, for smooth solutions of (1.1) *$\widetilde{u}_h$ is a $(k+1)$-th order accurate approximation.*

This result can be proven by similar arguments as in [27] and the last proof. Therefore, we omit a proof in this paper.

TABLE 3.1. Errors at time $T = 0.1$ for Burgers' equation.

	N	$u-u_h^S$ L^∞ norm	order	$u-u_h^S$ L^2 norm	order	$u-u_h^M$ L^∞ norm	order	$u-u_h^M$ L^2 norm	order
P^2	10	4.34E-03	–	9.10E-04	–	4.74E-03	–	9.87E-04	–
	20	7.53E-04	2.53	1.25E-04	2.86	8.10E-04	2.55	1.28E-04	2.95
	40	1.14E-04	2.72	1.70E-05	2.88	1.25E-04	2.70	1.72E-05	2.90
	80	1.60E-05	2.83	2.28E-06	2.90	1.76E-05	2.83	2.32E-06	2.89
	160	2.13E-06	2.91	3.00E-07	2.93	2.36E-06	2.90	3.08E-08	2.91
P^3	10	5.55E-04	–	7.46E-05	–	5.10E-04	–	7.47E-05	–
	20	4.16E-05	3.74	5.21E-06	3.84	3.58E-05	3.83	5.09E-06	3.88
	40	3.12E-06	3.74	3.66E-07	3.83	2.71E-06	3.72	3.51E-07	3.86
	80	2.11E-07	3.89	2.49E-08	3.88	1.83E-07	3.89	2.43E-08	3.85
	160	1.37E-08	3.94	1.66E-09	3.91	1.19E-08	3.94	1.64E-09	3.89

3. NUMERICAL EXPERIMENTS

In this section, we display the performance of the ALE-DG method. We adopt TVD Runge-Kutta methods (cf. Gottlieb and Shu [11]) for the time discretization, which are convex combinations of the forward Euler method. Thus, by an adequate adjustment of the CFL condition, the results for the forward Euler discretization can be extended to TVD Runge-Kutta methods.

Example 3.1 (Burgers' equation)**.**

We solve Burgers' equation with periodic boundary condition:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \quad x \in [0,1], \\ u(x,0) = \frac{1}{4} + \frac{1}{2}\sin(\pi(2x-1)). \end{cases}$$

The exact solution is smooth at $T = 0.1$ and has a well developed shock at $T = 0.4$. Here we choose the time step small enough to demonstrate the spatial error only. To maintain the stability, the TVB limiter is used with the parameter $\widetilde{M} = 20$.

In Table 3.1 we compare the convergence history of the ALE-DG method by using piecewise P^2 and P^3 polynomial elements with different cell numbers N at $T = 0.1$ on the static uniform grid and the moving grid $x_{j+\frac{1}{2}}(t_n) = x_{j+\frac{1}{2}}(0) + 0.4\sin(t_n)(x_{j+\frac{1}{2}}(0)-1)x_{j+\frac{1}{2}}(0)$ respectively. The moving grid starts from a uniform grid initially, and we use u_h^S and u_h^M to denote the numerical solutions on the static and moving grid respectively. It can be seen that numerically the optimal convergence order can be obtained for both grids. Note that the ALE-DG method on a static grid is the original DG method in [2, 5]. In Table 3.2 we show the convergence of the ALE-DG scheme for both grids when the shock is developed. With the help of the TVB limiter, the ALE-DG scheme is uniformly high order in regions of smoothness. Moreover, in Figure 3.1 we compare the exact and the ALE-DG solutions with $N = 80$ and $k = 4$ at time $T = 0.4$. It is shown that shocks are captured in a few elements without production of spurious oscillations.

In Table 3.3 the convergence history of the ALE-DG method with different polynomial degree k is displayed on the same static and moving grids with the cell number $N = 40$ at time $T = 0.1$ and $T = 0.4$. We can see that the ALE-DG method maintains the spectral convergence property of the DG method. This indicates the efficiency of the ALE-DG method using polynomials of higher degree.

TABLE 3.2. Errors in smooth regions $\Omega = \{x : |x - \text{shock}| \geq 0.1\}$ at time $T = 0.4$ for Burgers' equation.

		$u - u_h^S$		$u - u_h^S$		$u - u_h^M$		$u - u_h^M$	
	N	L^∞ error	order	L^2 error	order	L^∞ error	order	L^2 error	order
P^2	10	5.81E-03	–	1.11E-03	–	1.72E-02	–	2.45E-03	–
	20	1.75E-04	5.05	3.81E-05	4.86	8.61E-04	4.32	8.47E-05	4.85
	40	2.41E-05	2.86	4.05E-06	3.23	3.26E-05	4.72	4.12E-06	4.36
	80	3.33E-06	2.86	4.59E-07	3.14	4.58E-06	2.83	4.90E-07	3.07
	160	4.37E-07	2.93	5.41E-08	3.08	6.09E-07	2.91	5.57E-08	3.14
P^3	10	2.26E-03	–	4.27E-04	–	5.39E-03	–	8.85E-04	–
	20	9.99E-06	7.82	1.51E-06	8.14	1.83E-04	4.88	1.81E-05	5.61
	40	8.40E-07	3.57	8.83E-08	4.10	1.25E-06	7.19	9.66E-08	7.55
	80	6.19E-08	3.76	5.37E-09	4.04	9.52E-08	3.71	5.95E-09	4.02
	160	4.17E-09	3.89	3.29E-10	4.03	6.49E-09	3.87	3.63E-10	4.03

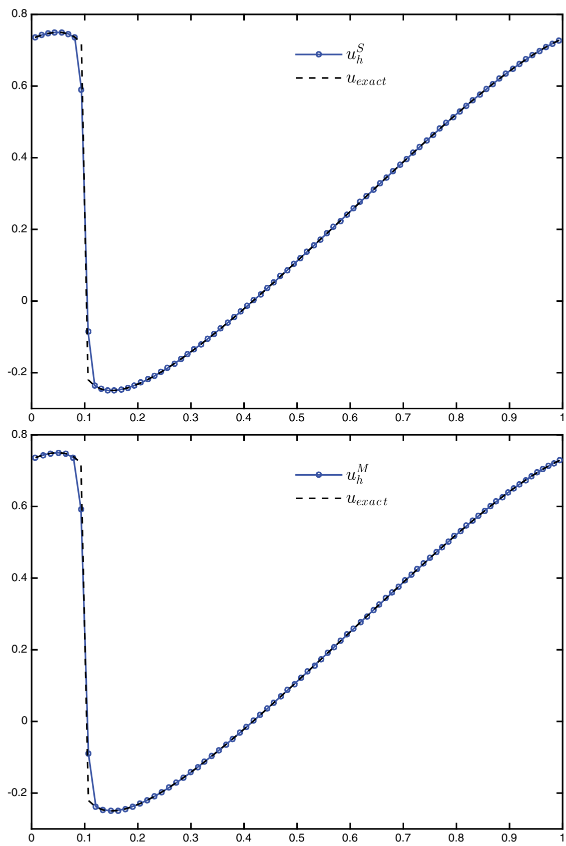

FIGURE 3.1. Comparison of the exact and the ALE-DG solutions u_h^S (top, on the static grid) and u_h^M (bottom, on the moving grid) with $N = 80$, $k = 4$ at time $T = 0.4$.

TABLE 3.3. L^∞ errors at time $T = 0.1$ and $T = 0.4$ in a smooth region for Burgers' equation with $N = 40$.

	T =0.1		T =0.4	
k	$u - u_h^S$	$u - u_h^M$	$u - u_h^S$	$u - u_h^M$
1	1.89E-03	1.77E-03	7.21E-04	7.25E-04
2	1.14E-04	1.25E-04	2.41E-05	3.25E-05
3	3.12E-06	2.71E-06	8.40E-07	1.25E-06
4	1.21E-07	1.44E-07	3.33E-08	5.47E-08
5	4.25E-09	3.40E-09	1.30E-09	2.40E-09
6	1.53E-10	1.97E-10	5.36E-11	1.09E-10
7	3.98E-12	3.55E-12	2.42E-12	5.07E-12
8	1.38E-13	1.52E-13	1.45E-13	3.50E-13
9	7.88E-15	4.14E-14	1.61E-14	7.70E-14

TABLE 3.4. Errors of the ALE-DG solutions $u_h^{\tilde{M}}$ in smooth regions at time $T = 0.1$ and $T = 0.4$ for Burgers' equation.

		$T = 0.1$		$T = 0.1$		$T = 0.4$		$T = 0.4$	
	N	L^∞ error	order	L^2 error	order	L^∞ error	order	L^2 error	order
	10	4.46E-03	–	9.28E-04	–	5.76E-03	–	1.10E-03	–
P^2	20	7.61E-04	2.55	1.25E-04	2.89	1.76E-04	5.03	3.24E-05	5.09
	40	1.14E-04	2.74	1.67E-05	2.90	2.41E-05	2.87	3.51E-06	3.21
	80	1.60E-05	2.83	2.26E-06	2.89	3.33E-06	2.86	4.00E-07	3.13
	160	2.43E-06	2.72	3.05E-07	2.89	4.37E-07	2.93	5.05E-08	2.99
P^3	10	5.10E-04	–	7.08E-05	–	2.02E-03	–	3.40E-04	–
	20	3.58E-05	3.83	4.82E-06	3.88	9.78E-06	7.69	1.19E-06	8.16
	40	2.71E-06	3.72	3.40E-07	3.83	8.15E-07	3.58	7.30E-08	4.03
	80	1.83E-07	3.89	2.31E-08	3.88	6.19E-08	3.72	4.42E-09	4.05
	160	1.18E-08	3.95	1.55E-09	3.90	4.17E-09	3.89	2.70E-10	4.03

In Table 3.4, we test the accuracy of the ALE-DG solutions $u_h^{\tilde{M}}$ on another moving grid which does not satisfy the grid assumption (ω3). The grid is defined as $x_{j+\frac{1}{2}}(t_n) = x_{j+\frac{1}{2}}(0) + 0.4\sin(t_n)H(x_{j+\frac{1}{2}}(0) - 0.5)(x_{j+\frac{1}{2}}(0) - 1))x_{j+\frac{1}{2}}(0)$, where $H(x)$ is the Heaviside step function. The table shows the convergence history before and after shock formulated at time $T = 0.1$ and $T = 0.4$ respectively. On this grid the method can still maintain the accuracy in smooth regions.

Example 3.2 (Euler's equations)**.**

We consider Euler's equations of gas dynamics for a polytropic gas:

$$\partial_t \mathbf{u} + \partial_x \Big(\mathbf{f}(\mathbf{u})\Big) = 0, \quad x \in [0,1],$$
$$\mathbf{u} = (\rho, m, E)^T, \quad \mathbf{f}(\mathbf{u}) = v\mathbf{u} + (0, p, pv)^T,$$

with

$$p = (\gamma - 1)(E - \frac{1}{2}\rho v^2), \quad m = \rho v,$$

TABLE 3.5. Errors of the density at time $T = 1.2$ for Euler's equations.

		$\rho - \rho_h^S$		$\rho - \rho_h^S$		$\rho - \rho_h^M$		$\rho - \rho_h^M$	
	N	L^∞ norm	order	L^2 norm	order	L^∞ norm	order	L^2 norm	order
P^2	10	2.63E-03	–	9.95E-04	–	5.14E-03	–	1.48E-03	–
	20	3.87E-04	2.77	1.42E-04	2.78	7.88E-04	2.70	2.20E-04	2.75
	40	5.10E-05	2.92	1.87E-05	2.93	1.06E-04	2.89	2.94E-05	2.91
	80	6.46E-06	2.98	2.38E-06	2.98	1.36E-05	2.96	3.75E-06	2.97
	160	8.08E-07	3.00	2.98E-07	2.99	1.71E-06	2.99	4.71E-07	2.99
P^3	10	7.23E-05	–	1.92E-05	–	1.91E-04	–	3.60E-05	–
	20	4.40E-06	4.04	1.07E-06	4.16	1.27E-05	3.90	1.97E-06	4.19
	40	2.74E-07	4.01	6.65E-08	4.01	8.07E-07	3.98	1.15E-07	4.10
	80	1.71E-08	4.00	4.14E-09	4.00	5.10E-08	3.98	6.99E-09	4.04
	160	1.07E-09	4.00	2.59E-10	4.00	3.20E-09	3.99	4.30E-10	4.02

TABLE 3.6. L^∞ errors at time $T = 1.2$ for Burgers' equation and Euler's equations with constant solutions $u = 1$ and $(\rho, v, p) = (1, 1, 1)$.

	$u - u_h^M$		$\rho - \rho_h^M$	
N	P^2	P^3	P^2	P^3
10	4.44E-15	9.77E-15	4.44E-15	5.77E-15
20	9.99E-15	1.24E-14	5.77E-15	9.66E-15
40	1.24E-14	2.51E-14	9.55E-15	1.78E-14
80	2.22E-14	1.89E-14	1.77E-14	2.45E-14
160	2.80E-14	3.62E-14	3.24E-14	3.30E-14

where $\gamma = 1.4$ is used in the following computation. Two sets of initial conditions are considered. One is a smooth function (plain wave)

$$(\rho, v, p) = (1 + 0.5\sin(2\pi(x - t)), 1, 1),$$

with periodic boundary condition. The other is a modified Sod shock tube problem (Riemann problem) with left and right state:

$$(\rho_L, v_L, p_L) = (1, 0.75, 1), \quad (\rho_R, v_R, p_R) = (0.125, 0, 1).$$

In Table 3.5 the convergence history of the density given by the ALE-DG method with piecewise P^2 and P^3 polynomial elements is displayed at time $T = 1.2$, where ρ_h^S is the ALE-DG solution on the static uniform grid and ρ_h^M is the ALE-DG solution on the same moving grid as in the last test. We can see that the optimal convergence order can be obtained numerically for both grids. In Figure 3.2 we compare the exact and the ALE-DG solutions $\mathbf{u}_h^S$ and $\mathbf{u}_h^M$ on static and moving grids with $N = 200$ and $k = 4$ at time $T = 0.2$. The TVB limiter is used with $\widetilde{M} = 20$. It is shown that both solutions converge to the entropy solution and the performance is similar.

In these numerical experiments we do not consider the methodology of how to move the grid, but the scenario when the grid is chosen at two adjacent time levels. These tests show that the ALE-DG method maintains the properties of the DG method for static grids, such as uniformly high order accuracy and shock capturing. Furthermore, Table 3.6 shows that the ALE-DG method satisfies the geometric conservation law numerically as we proved.

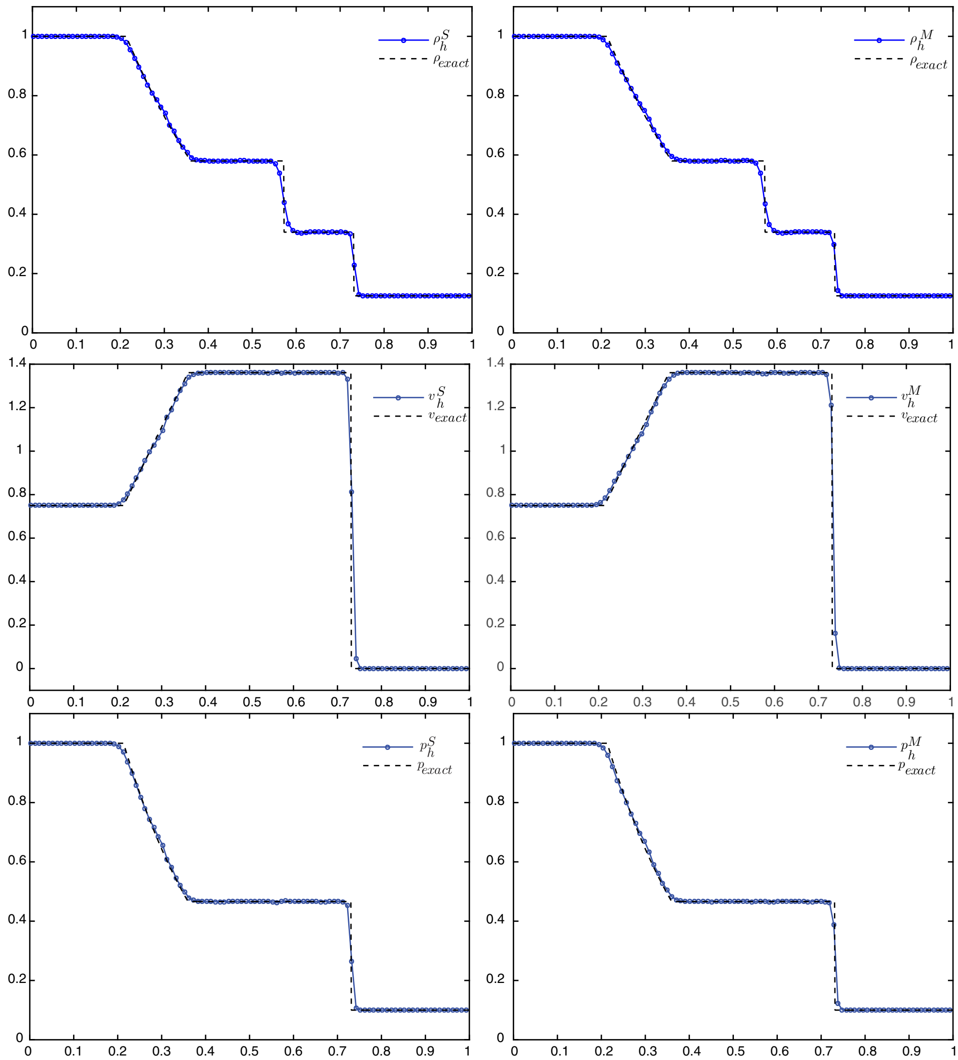

FIGURE 3.2. Comparison of the exact and the ALE-DG solutions $\mathbf{u}_h^S$ (left column, on the static grid) and $\mathbf{u}_h^M$ (right column, on the moving grid) for the modified Sod shock tube problem with $N = 200$, $k = 4$ at time $T = 0.2$ and $\widetilde{M} = 20$.

4. Conclusions

In this paper, we developed an ALE-DG method that satisfies the geometric conservation law on moving grids using a time-dependent approximation space. We began the paper with theoretical results by proving a cell entropy inequality and L^2 stability. We also gave error estimates for the ALE-DG method with monotone numerical fluxes and an upwind flux separately. For the fully discrete scheme, the geometric conservation law and the local maximum principle have been proven. Moreover, for shock capturing, conditions for TVD/TVB limiter have been established. Numerically, it has been shown that our ALE-DG method is uniformly high order accurate and shock capturing. In this paper, we have merely considered how to develop the ALE-DG scheme after the grids are chosen at two adjacent time levels. In future work, we will consider the methodology of how to move the grid efficiently and combine it with our ALE-DG method. Also, the generalization of the method to multidimensional problems is in preparation.

References

[1] P. G. Ciarlet, *The finite element method for elliptic problems*, Classics in Applied Mathematics, vol. 40, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)]. MR1930132

[2] B. Cockburn, *Discontinuous Galerkin methods for convection-dominated problems*, High-order methods for computational physics, Lect. Notes Comput. Sci. Eng., vol. 9, Springer, Berlin, 1999, pp. 69–224, DOI 10.1007/978-3-662-03882-6_2. MR1712278

[3] B. Cockburn, S. Hou, and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. IV. The multidimensional case*, Math. Comp. **54** (1990), no. 190, 545–581, DOI 10.2307/2008501. MR1010597 (90k:65162)

[4] B. Cockburn, S. Y. Lin, and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. III. One-dimensional systems*, J. Comput. Phys. **84** (1989), no. 1, 90–113, DOI 10.1016/0021-9991(89)90183-6. MR1015355 (90k:65161)

[5] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework*, Math. Comp. **52** (1989), no. 186, 411–435, DOI 10.2307/2008474. MR983311 (90k:65160)

[6] B. Cockburn and C.-W. Shu, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput. **16** (2001), no. 3, 173–261, DOI 10.1023/A:1012873910884. MR1873283 (2002i:65099)

[7] J. Donea, A. Huerta, J.-P. Ponthot, and A. Rodríguez-Ferran, *Arbitrary Lagrangian-Eulerian methods*, in Encyclopedia of Computational Mechanics, Volume 1: Fundamentals, Erwin Stein, Rene De Borst and Thomas J.R. Hughes (Eds.), Wiley, 2004.

[8] R. J. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal. **82** (1983), no. 1, 27–70, DOI 10.1007/BF00251724. MR684413 (84k:35091)

[9] C. Farhat, P. Geuzaine, and C. Grandmont, *The discrete geometric conservation law and the nonlinear stability of ALE schemes for the solution of flow problems on moving grids*, J. Comput. Phys. **174** (2001), no. 2, 669–694, DOI 10.1006/jcph.2001.6932. MR1868101 (2002h:76104)

[10] R. Fazio and R. J. LeVeque, *Moving-mesh methods for one-dimensional hyperbolic problems using CLAWPACK*, Comput. Math. Appl. **45** (2003), no. 1-3, 273–298, DOI 10.1016/S0898-1221(03)80019-6. Numerical methods in physics, chemistry, and engineering. MR1991370 (2004e:76082)

[11] S. Gottlieb and C.-W. Shu, *Total variation diminishing Runge-Kutta schemes*, Math. Comp. **67** (1998), no. 221, 73–85, DOI 10.1090/S0025-5718-98-00913-2. MR1443118 (98c:65122)

[12] H. Guillard and C. Farhat, *On the significance of the geometric conservation law for flow computations on moving meshes*, Comput. Methods Appl. Mech. Engrg. **190** (2000), no. 11-12, 1467–1482, DOI 10.1016/S0045-7825(00)00173-0. MR1807009 (2001j:76081)

[13] A. Harten, *High resolution schemes for hyperbolic conservation laws*, J. Comput. Phys. **49** (1983), no. 3, 357–393, DOI 10.1016/0021-9991(83)90136-5. MR701178 (84g:65115)

[14] U. W. Hochstrasser, *Orthogonal polynomials*, in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (M. Abramowitz and I. Stegun, eds.), Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, D.C., 1964, pp. 771–792.

[15] G. S. Jiang and C.-W. Shu, *On a cell entropy inequality for discontinuous Galerkin methods*, Math. Comp. **62** (1994), no. 206, 531–538, DOI 10.2307/2153521. MR1223232 (94h:65099)

[16] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp. **46** (1986), no. 173, 1–26, DOI 10.2307/2008211. MR815828 (88b:65109)

[17] D. Kereš, M. Vogelsberger, D. Sijacki, V. Springel, and L. Hernquist, *Moving-mesh cosmology: Characteristics of galaxies and haloes*, Oxford J. Sci. & Math. MNRAS **425** (2012), 2027–2048.

[18] S. N. Kružkov, *First order quasilinear equations with several independent variables.* (Russian), Mat. Sb. (N.S.) **81 (123)** (1970), 228–255; English transl., Math. USSR Sb. **10** (1970), 217–243. MR0267257

[19] P. Lasaint and P.-A. Raviart, *On a finite element method for solving the neutron transport equation*, Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974), Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 89–123. Publication No. 33. MR0658142 (58 #31918)

[20] M. Lesoinne and C. Farhat, *Geometric conservation laws for flow problems with moving boundaries and deformable meshes, and their impact on aeroelastic computations*, Comput. Methods Appl. Mech. Engrg. **134** (1996), 71–90.

[21] P. D. Thomas and C. K. Lombard, *Geometric conservation law and its application to flow computations on moving grids*, AIAA J. **17** (1979), no. 10, 1030–1037, DOI 10.2514/3.61273. MR544850 (80h:65104)

[22] I. Lomtev, R. M. Kirby, and G. E. Karniadakis, *A discontinuous Galerkin ALE method for compressible viscous flows in moving domains*, J. Comput. Phys. **155** (1999), no. 1, 128–159, DOI 10.1006/jcph.1999.6331. MR1716493 (2000f:76076)

[23] V.-T. Nguyen, *An arbitrary Lagrangian-Eulerian discontinuous Galerkin method for simulations of flows over variable geometries*, J. Fluids and Struc. **26** (2010), 312–329.

[24] P.-O. Persson, J. Bonet, and J. Peraire, *Discontinuous Galerkin solution of the Navier-Stokes equations on deformable domains*, Comput. Methods Appl. Mech. Engrg. **198** (2009), 1585–1595.

[25] T. E. Peterson, *A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation*, SIAM J. Numer. Anal. **28** (1991), no. 1, 133–140, DOI 10.1137/0728006. MR1083327 (91m:65250)

[26] B. A. Robinson, H. T. Y. Yang, and J. T. Batina, *Aeroelastic analysis of wings using the Euler equations with a deforming mesh*, J. Aircraft **28** (1991), 781–788.

[27] C.-W. Shu, *TVB uniformly high-order schemes for conservation laws*, Math. Comp. **49** (1987), no. 179, 105–121, DOI 10.2307/2008252. MR890256 (89b:65208)

[28] Q. Zhang and C.-W. Shu, *Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws*, SIAM J. Numer. Anal. **42** (2004), no. 2, 641–666 (electronic), DOI 10.1137/S0036142902404182. MR2084230 (2005h:65149)

[29] Q. Zhang and C.-W. Shu, *Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws*, SIAM J. Numer. Anal. **44** (2006), no. 4, 1703–1720 (electronic), DOI 10.1137/040620382. MR2257123 (2008f:65162)

[30] Q. Zhang and C.-W. Shu, *Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws*, SIAM J. Numer. Anal. **48** (2010), no. 3, 1038–1063, DOI 10.1137/090771363. MR2669400 (2011g:65205)

[31] X. Zhang and C.-W. Shu, *On maximum-principle-satisfying high order schemes for scalar conservation laws*, J. Comput. Phys. **229** (2010), no. 9, 3091–3120, DOI 10.1016/j.jcp.2009.12.030. MR2601091 (2010k:65181)

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