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Nonlinear Hyperbolic Problems: Theoretical, Applied, and Computational Aspects

Proceedings of the Fourth International Conference on Hyperbolic Problems, Taormina, Italy, April 3 to 8, 1992

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Regularity of a scalar Riemann problem in two space dimensions

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Abstract

For the one dimensional scalar non-convex conservation law $u_t + f(u)_x = 0$ let $f'''(u)$ have finitely many changes of sign. We show that if the initial data consists of finitely many constant states, the solution will be piecewise smooth with finitely many shock curves. Hence the same holds true for two dimensional Riemann problems for the scalar equation $u_t + f(u)_x + f(u)_y = 0$.

1. Introduction

Consider the scalar conservation law:

$$u_t + f(u)_x + f(u)_y = 0 \quad , f \in C^3 : R \rightarrow R$$

with Riemann initial data: u(0,x,y) is constant in finitely many wedges meeting at the origin.

Rotating by 45°, the solution to this Riemann problem is equivalent to solving one dimensional initial value problems:

$$u_t + f(u)_x = 0$$

with initial data u(0,x) consisting of finitely many constant states separated by jumps. In fact the solutions to two initial value problems like this may be transformed back to the given two dimensional Riemann problem (for more details see [K] or [L]).

We proceed to show that for many flux functions (like polynomials) the solution to the one dimensional problem has finitely many smooth pieces, which then also holds for the two dimensional Riemann problem.

Regularity of solutions to conservation laws in one space dimension has been studied for convex fluxes by Oleinik [O], Schaeffer [S], Dafermos [D1], for one inflection point by Dafermos [D2]. Generally speaking they show that for C^∞ initial data, generically the solution is C^∞ smooth except on a finite set of C^∞ arcs. Lindquist [L] conjectures that for more than three inflection points there is a loss of piecewise smoothness. We show here this need not be so.

We allow any finite number of inflection points. The initial data is restricted to piecewise constant with finitely many jumps. Any compactly supported initial data may be approximated by this. To the author´s knowledge, here is the first explicit construction of the global solution yielding the regularity result.

2. The one-dimensional problem for small times

Consider $u_t + f(u)_x = 0$, $f \in C^3(R)$, with initial data consisting of finitely many constant states. Allow $f'''(u)$ to change sign only finitely often.

Note: This implies that f(u) has finitely many inflection points.

For small times solve the local Riemann problems at the initial jump discontinuities by constructing the convex or concave hulls $CH(u_l,u_r)$, where u_l and u_r are two neighboring constant states (for more details on $CH(u_l,u_r)$, see [K] or [L]). Each $CH(u_l,u_r)$ corresponds to the Oleinik entropy solution of the local Riemann solution. For an example see Fig. 1.

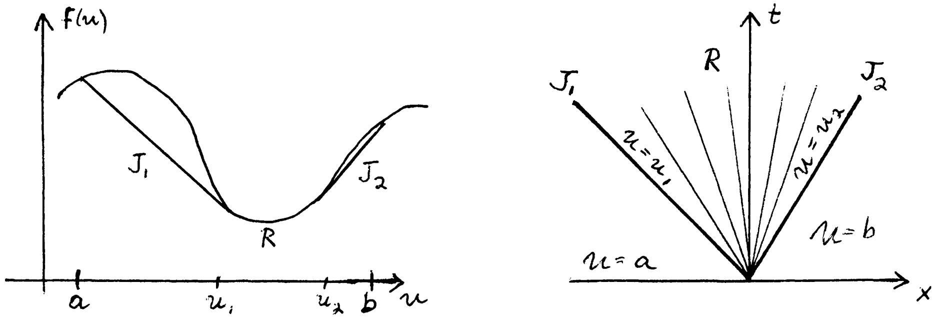


Figure 1: An example of a convex hull corresponding to the one dimensional Riemann problem between a and b.

Definition: Let $k = k(u_l,u_r)$ be the number of inflection points of f in the open interval between u_l and u_r . Let

$$\#CH(u_l,u_r) = \begin{cases} \text{first integer} \geq \frac{k}{2} & \text{if } \{u_l<u_r \text{ and } f''(u) \geq 0\} \text{ or if } \{u_l>u_r \text{ and } f''(u) < 0\} \\ \text{first integer} \geq \frac{1+k}{2} & \text{if } \{u_l<u_r \text{ and } f''(u) < 0\} \text{ or if } \{u_l>u_r \text{ and } f''(u) \geq 0\}. \end{cases}$$

Note: $\#CH(u_l,u_r)$ counts the "valleys" of f traversed between u_l and u_r .

Definition: Let there be n initial constant states a_i. Then

$$\text{the graph of } \bigcup_{i=1}^{n} CH(a_i,a_{i+1}) =: \text{c-ch.}$$

is called the graph of the convex-concave hull.

The c-ch. consits of
- straight line segments, representing jumps (J)
- pieces that touch the graph of f(u) in an interval [a,b] of the u-axis, called rarefaction wave (R) if a≠ b, or constant state (C), if a = b.

Note: Left and right refer to the x-t plane at the solution corresponding to the c-ch.

On the c-ch. label the left and right sides of the jumps by integers according to the following

Definition: *labeling rule:*

- the left side of the left most jump gets the number 1.
- the right side u_r of any given jump is assigned an integer which increments the integer on its left side u_l by

$$2\ (\#CH(u_l,u_r)) - 1 .$$

- let $\tilde{J}$ be the next jump to the right of J on c-ch. Let the right side of J have label i. Then the left side of $\tilde{J}$ has label i+1.

Finally add these two labels:

- label the first point of c-ch. by 0, the last point by (maximum label so far) + 1

Note: Using the labeling rule, one point may have two labels.

Definition: Call the set of these integers $\boldsymbol{I}$. Let $P_i = (u_i,f(u_i))$ denote those points on the graph of c-ch. labeled by $i \in \boldsymbol{I}$.

Note: max $\boldsymbol{I}$ is finite for small time since we have finitely many inflection points and finitely many initial jumps.

Definition: A jump on c-ch. is denoted by $J(P_i,P_j)$, a rarefaction wave by $R(P_i,P_{i+1})$ and constant state by $C(P_i)$, $i<j$, $i,j \in \boldsymbol{I}$.

For an example, see Fig. 2.

Note: There are finitely many J, R for small time.

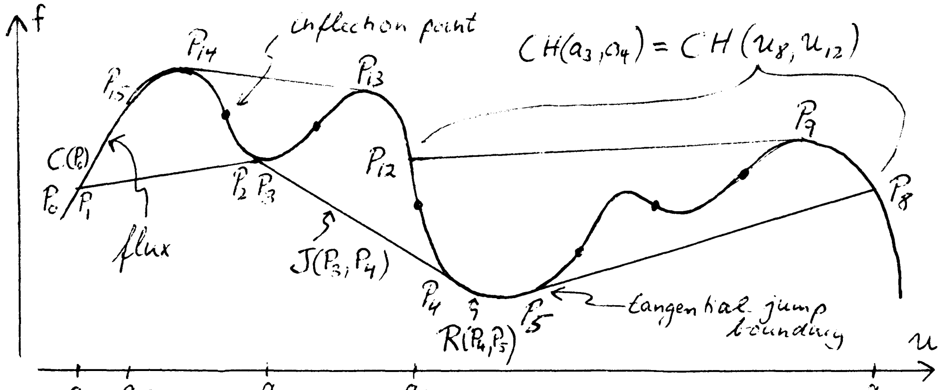
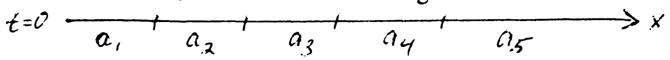


Figure 2: An example of c-ch. with the labels at the jump boundaries. Note that e.g. $P_2 = P_3$. The initial data on the x - axis was arranged like this:



After labeling P_i these initial states have new names. Here max $\boldsymbol{I}$ = 15.

3. The time evolution of the convex-concave hull

We proceed to describe the unique deformation of the c-ch. as time increases. This is done by considering those points P_i of c-ch. where the slope of the corresponding jump $\neq f'(u_i)$. Call these *interactions*.

There are two types of interactions, either J meets J or J meets R.

3.1. *Interaction J J*

Say this interaction takes place at P_i, $i \in \boldsymbol{I}$, with jumps

$$J(P_h,P_i) \text{ and } J(P_{i+1},P_j)\ , \quad h<i<i+1<j\ , \ h,i,j \in \boldsymbol{I}.$$

Possibly P_h and P_j may be interaction points for jumps to the left and right resp. This corresponds to two or more jumps $\{J_k\}_{k=1,\dots,n}$ (each having left and right boundary points u_l^k , u_r^k) meeting at one point in the x-t plane. The solution to this interaction is given by solving the Riemann problem (constructing $CH(u_l^1,u_r^n)$) using the left most state u_l^1 and the right most state u_r^n of the set $\{J_k\}_{k=1,\dots,n}$.

How does the c-ch. change with such interactions? Note that we distinguish between monotone jumps, where for all k=1,...,n we have $u_l^k<u_r^k$ or for all k=1,...,n we have $u_l^k>u_r^k$, and the non-monotone case, where there exist J_i and J_m in $\{J_k\}_{k=1,\dots,n}$ s.th. $u_l^i<u_r^i$ and $u_l^m>u_r^m$.

Monotone case: two or more jumps will interact to become one single jump, e.g.

$$J(P_1,P_2)\ J(P_3,P_4) \rightarrow J(P_1,P_4)\quad .$$

Here we discard all "middle points", in this example throw out $P_2=P_3$.

Non-monotone case: Since two jumps meeting in this interaction jump in opposite directions, more than one jump may arise. Say

$$u_l^1<u_r^n\quad .$$

Pick all J_k from $\{J_k\}_{k=1,\dots,n}$ with $u_l^k<u_r^k$. Since their intervals satisfy

$$\bigcup_{\forall k \text{ s.th. } u_l^k<u_r^k} [\, u_l^k , u_r^k] \supset [\, u_l^1 , u_r^n]$$

we have

$$\sum_{\forall k \text{ s.th. } u_l^k<u_r^k} \#CH(u_l^k,u_r^k) \geq \#CH(u_l^1,u_r^n)\quad .$$

By non-monotonicity $\exists$ J_m in $\{J_k\}_{k=1,\dots,n}$ with $u_l^m>u_r^m$ and $\#CH(u_l^m,u_r^m) \geq 1$. Thus

$$\sum_{k=1}^{n} \#CH(u_l^k,u_r^k) \geq 1 + \sum_{\forall k \text{ s.th. } u_l^k<u_r^k} \#CH(u_l^k,u_r^k) > \#CH(u_l^1,u_r^n)\quad .$$

After the interaction we need to relabel. First we discard all jump boundary points from $\{J_k\}_{k=1,\dots,n}$ except for those at u_l^1 and u_r^n . Then we label the jump boundary points in $CH(u_l^1,u_r^n)$ using the labeling rule. Because of the strict inequality in the above equation, the new labels will maintain the increasing order of $\boldsymbol{I}$, without changing the remaining points. See

Fig. 3.

Note: In the interaction J J the set I changes but max I does not change.

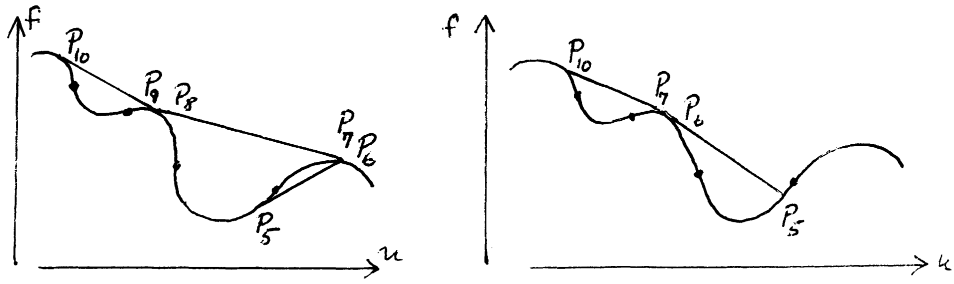


Figure 3: An example of a non-monotone J J interaction

$$J(P_5,P_6)\ J(P_7,P_8)\ J(P_9,P_{10}) \rightarrow J(P_5,P_6)\ R(P_6,P_7)\ J(P_7,P_{10})$$

on the left before interaction, on the right after.

3.2. *The interaction J R*

Say this interaction takes place at P_j with $J(P_i,P_j)$. Then P_i is either a constant state $C(P_i)$, or the boundary of a rarefaction wave $R(P_h,P_i)$, or a boundary of a jump $J(P_h,P_i)$, $h<i<j$, $h,i,j \in I$. We only describe the case of a rarefaction wave $R(P_h,P_i)$. The other cases are evident then. Say the interaction at P_j begins at time t_0. There exists some time interval $[\ t_0\ ,\ T]$ such that for all $\tilde{t} \in [\ t_0\ ,\ T]$ we construct $CH(u_i(\tilde{t}),u_j(\tilde{t}))$, as $u_i(\tilde{t})$ and $u_j(\tilde{t})$ move along their respective rarefaction waves in time.

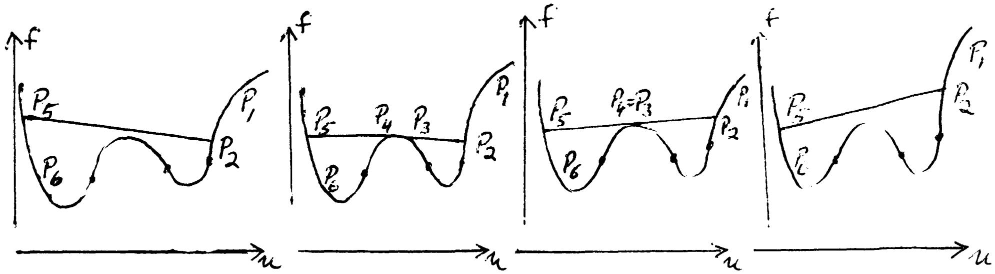
How does c-ch. change in these interactions?

As long as $J(P_i,P_j)$ continuously moves the locations P_i and P_j may change in time. Since $f''(u) \neq 0$ in the rarefaction interval u_j, u_{j+1} , the label P_j does not change. Similarily with P_i.

The jump $J(P_i,P_j)$ may bifurcate, see Fig. 4. If $J(P_i,P_j)$ bifurcates, there may be at most $\#CH(u_i,u_j)$ new jumps arising. We need to label these new jumps. Since

$$j = i + 2\ (\#CH(u_i,u_j) - 1$$

we may label the new jump boundary points using the *labeling rule* without changing j and keeping the increasing order of the jump boundary points from left to right.



$J(P_2,P_5) \rightarrow J(P_2,P_3)\, R(P_3,P_4)\, J(P_4,P_5) \rightarrow J(P_2,P_3)\, C(P_3)\, J(P_4,P_5) \rightarrow J(P_2,P_5)$

Figure 4: The first arrow describes a bifurcation, the last one a monotone jump interaction.

Note: max $\boldsymbol{I}$ remains unchanged in the interaction J R .

Note: A point P_i may change from a tangential jump boundary point to an interaction point and vice versa as time evolves.

Note: This c-ch. construction has the property that if we found the solution up to a certain time t_0 , we may continue on for a certain time interval. Suppose we have constructed the solution using the c-ch as far in time as possible.

Note: Since max $\boldsymbol{I}$ remains unchanged in time, at any given time the number of jumps in the c-ch. is uniformly bounded.

4. Counting the number of interactions possible

Lemma: The solution has at most finitely many of the following interactions:
- non - monotone jump interactions

$J(P_i,P_j)\, J(P_{j+1},P_k) \rightarrow CH(u_i,u_k)$ with $u_i<u_j$ and $u_{j+1}>u_k$ or with $u_i>u_j$ and $u_{j+1}<u_k$
- jump decreasing to zero strength

$$J(P_i,P_j)\, R(P_j,P_{j+1}) \rightarrow C(P_i)\, R(P_i,P_{j+1}) = CH(u_i,u_{j+1}) \quad i<j \quad .$$

Proof: We have collected those interactions, where

$$\sum_{\text{waves before interaction}} \#CH(u_l,u_r) > \sum_{\text{waves after interaction}} \#CH(u_l,u_r)$$

Thus the number of distinct P_i, $i\in \boldsymbol{I}$ that may still possibly arise for t > (time of interaction) has decreased. Since max $\boldsymbol{I}$ is uniformly bounded in time, such interactions can only arise finitely often.

Note: Both in the monotone jump interaction and bifurcation we have

$$\sum_{\text{waves before interaction}} \#CH(u_l,u_r) = \sum_{\text{waves after interaction}} \#CH(u_l,u_r) \quad .$$

Thus we need a different argument if we want to rule out infinitely many of these interactions.

Lemma: If the solution has infinitely many bifurcations, then $f'''(u)$ changes sign infinitely often.

Proof: Suppose we had infinitely many bifurcations. Here is an example of one bifurcation:

$$J(P_1,P_4) \rightarrow J(P_1,P_2)\ R(P_2,P_3)\ J(P_3,P_4)\ .$$

The c-ch. allows only at most a fixed finite number of jumps. Thus jumps need to interact infinitely often and cause two jumps to become one. Only a finite number of non-monotone jump interactions are allowed. Thus there have to be infinitely many monotone jump interactions, called recombinations, e.g.

$$J(P_1,P_2)\ J(P_3,P_4) \rightarrow J(P_1,P_4)\ .$$

Since we have only finitely many points in P_i, $i \in I$, there exist two points, say P_1 and P_4, with infinitely many bifurcations and recombinations between these points.

Now we ask what waves interact at P_1 and P_4. We *claim* that after some finite number of interactions only $R(P_0,P_1)$ and $R(P_4,P_5)$ interact there.

Proof of claim: Suppose infinitely many jumps interact at P_4. Only finitely many of them may be jumps in the opposite direction as $J(P_3,P_4)$ or $J(P_1,P_4)$, Thus infinitely many are in the same direction. Say one of them is $J(P_5,P_8)$. It interacts at P_4, e.g.

$$J(P_3,P_4)\ J(P_5,P_8) \rightarrow J(P_3,P_8)\ .$$

For this to happen infinitely often, $J(P_3,P_8)$ has to bifurcate infintely often, e.g.

$$J(P_3,P_8) \rightarrow J(P_3,P_4)\ R(P_4,P_5)\ J(P_5,P_8)\ .$$

Thus to the right of the interval between u_1 and u_4 there are again infinitely many bifurcations and recombinations in an interval between u_3 and u_8. To the right of this there may be again such intervals, but at most finitely many. Suppose there are no more. Then to the right of P_8 there must be a rarefaction wave

$$R(P_8,P_9)\ .$$

To the left of the bifurcation - recombination interval between u_1 and u_4 there may be only finitely many other such intervals bounded to the left by a rarefaction wave, in our example say by

$$R(P_0,P_1)\ .$$

How do P_1 and P_8 move in time?

a) Suppose both P_1 and P_8 are fixed, $u_0 = u_1$ and $u_8 = u_9$. Then after finitely many interactions the $CH(u_1,u_8)$ would be reached. This is not possible.

b) Suppose P_1 is fixed, $u_0 = u_1$, but P_8 moves along $R(P_8,P_9)$, $u_8 \neq u_9$. Since u_8 moves monotonically towards u_9, only bifurcations or only recombinations are possible. But this would allow only finitely many interactions, thus not possiblke.

c) Suppose both P_1 and P_8 move along their resp. R s, $u_0 \neq u_1$, $u_8 \neq u_9$ and move away from each other, i.e. u_1 moves to the left and u_8 moves to the right. Since all jumps inbetween are monotone, this again implies only bifurcations or recombinations. Similarily if u_1 and u_8 move towards each other.

d) Suppose P_1 and P_8 both move in the same direction. Since we assume infinitely many bifurcation - recombinations between P_1 and P_4, one sees by inspection that in this case it is not possible to also have infinitely many bifurcation - recombinations between P_3 and P_8. Thus only finitely many jumps interact at P_1 to with $J(P_1,P_2)$, similarily at P_4. Hence eventually only rarefaction waves bound P_1 and P_4 and u_1 and u_4 move in the same direction. *End of proof of claim.*

To recapitulate, we have shown that only the example given in Lindquist's paper [L] (that is exactly case d) above may possibly give rise to infinitely many bifurcations, see Fig. 4. Following the explanation in sec. 2.2 in [L], $\frac{du_1}{dt}$ (= h'_1 in sec. 2.2 in [L]) has to become larger and smaller infinitely often. Take the time derivative of the equation defining the value on the characteristic coming from P_1 in $R(P_0,P_1)$, i.e. take $\frac{d}{dt}$ of $f'(u_1) = \frac{x - const.}{t - const.}$ to obtain

$$\frac{du_1}{dt} = \frac{\frac{dx}{dt} - f(u_1)}{(t - const) f''(u_1)} .$$

Thus f'' has to become larger and smaller infintely often. Thus this example requires infinitely many changes of sign of $f'''(u)$. End of proof of lemma.

Corollary: Only finitely many monotone jump interactions may take place.

Proof. Since we have only finitely many bifurcations for the class of flux functions considered here, there may only be finitely many monotone jump interactions.

Lemma: Only finitely many interactions are possible.

Note: This implies that we may extend the solution for all time, since the time interval between two interactions may not decrease to zero

5. Piecewise smoothness of the solution

Definition: A smooth shock curve in the x-t plane for $t \geq t_0 \geq 0$ corresponds
- for $t = t_0$ to a jump $J(P_i,P_j)$ in the c-ch.
- for $t > t_0$ follow $J(P_i,P_j)$ on the c-ch. up to either its interaction with another J, or its bifurcation, or its decay to zero, or up to $t = \infty$, whichever happens first.

Theorem: The solution to

$$u_t + f(u)_x = 0$$

with finitely many constants as initial data and $f'''(u)$ changing sign finitely often has finitely many smooth shock curves.

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