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The global existence of L^∞ solutions to isentropic Euler equations in general nozzle

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Abstract

In this paper, we study the global L^∞ entropy solutions for the Cauchy problem of the isentropic gas dynamics system in a general nozzle with bounded initial date. First, we apply for the flux-approximation technique coupled with the classical viscosity method to obtain the L^∞ estimates of the viscosity solutions. Second, we prove the pointwise convergence of the approximation solutions by using the compactness framework for any adiabatic exponent $\gamma > 1$.

KEYWORDS

global L^∞ solution, isentropic gas flow, general nozzle, flux approximation, compensated compactness

MSC (2020)

35L45, 35L60, 35L65

1 | INTRODUCTION

We consider the following system of isentropic gas dynamics in a general nozzle:

$$\begin{cases} \rho_t + (\rho u)_x = -\dfrac{a'(x)}{a(x)}\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\dfrac{a'(x)}{a(x)}\rho u^2 \end{cases} \tag{1.1}$$

with bounded initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ is a slowly variable cross-section area at x in the nozzle. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^{\gamma}$, where $\gamma > 1$ is the adiabatic exponent. The nozzle is widely used in some types of steam turbines, rocket engine nozzles, supersonic jet engines, and jet streams in astrophysics.

To study the existence of entropy solutions of the Cauchy problem (1.1) and (1.2), the main difficulty is to establish L^{∞} estimate of solutions because the equations are not in conservative form and the Conley–Chuey–Smoller principle of invariant regions does not apply (see [12, 21] for the details about the physical background of system (1.1) and its difficulty in analysis). For the polytropic gas and the adiabatic exponent $\gamma \in (1, \frac{5}{3}]$, the definition of a finite energy solution (unbounded) is given and its existence is obtained by using the compensated compactness method [16] in [12]. In [8, 18], we used the maximum principle to obtain the L^{∞} estimate of the viscosity solutions of Equation (1.1), and the compensated compactness method to prove the existence of bounded entropy solutions of the Cauchy problem (1.1) and (1.2) for general pressure function $P(\rho)$ under the uniformly bounded condition $|a'(x)| \leq M$ and a monotonic, bounded and discontinuous condition on $a(x)$, respectively. In [22, 23], the author introduced a modified Godunov scheme to construct the approximate solutions of Equation (1.1), and obtained the global existence of weak solutions of the Cauchy problem (1.1) and (1.2) for the Laval nozzle, which is corresponding to $a'(x) \cdot x \geq 0$, in [22] and the general nozzle in [23] for the usual gases $1 < \gamma \leq \frac{5}{3}$ under the smallness assumption on $|a(x)|_{L^1(R)}$.

In [1], the authors introduced the following approximate system, which is different from the viscosity method introduced in [18],

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u + \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 + \varepsilon(\rho u)_{xx} - 2\varepsilon b(x)\rho_x, \end{cases} \tag{1.3}$$

to study the general nozzle for more general gases $1 \leq \gamma \leq 3$.

When $\gamma \geq 3$, the technique introduced in [1] to obtain the a priori L^{∞} estimates of viscosity solutions does not work because the necessary conditions $a_{12} \leq 0$ and $a_{21} \leq 0$, to guarantee the maximum principle (cf. Lemma 3.1 in [1]), are not true.

In this paper, we apply our method introduced in [18] to give a simple proof of the global existence of the entropy solutions for general nozzle and to extend the results of [1] for any adiabatic exponent $\gamma > 1$.

It is worthwhile to point out that, for a general inhomogeneous system of hyperbolic conservation laws, the Riemann problem was resolved by Isaacson and Temple in [10]. More results on inhomogeneous hyperbolic systems can be found in [2, 5–7, 9, 11], and references therein.

In [19], the following system of isentropic gas dynamics in the Laval nozzle with the friction (cf. [20]):

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 - \alpha(x)\rho u|u| \end{cases} \tag{1.4}$$

was studied for the polytropic gas $P(\rho) = \frac{1}{\gamma}\rho^{\gamma}$ and γ is limited in $(3, \infty)$ for a technical difficulty; and the initial-boundary value problem of the compressible Euler equations with friction and heating

$$\begin{cases} (a(x)\rho)_t + (a(x)\rho u)_x = 0, \\ (a(x)\rho u)_t + (a(x)\rho u^2 + a(x)P)_x = a'(x)P - \alpha\sqrt{a(x)}\rho u|u|, \\ (a(x)E)_t + (a(x)u(E+P))_x = \beta a(x)q(x) - \alpha\sqrt{a(x)}\rho u^2|u|, \end{cases} \tag{1.5}$$

was studied in [3] , under suitable conditions among the initial data, $a(x)$ and $\alpha(x)$, by using a new version of a generalized Glimm scheme, where ρ, u, E are, respectively, the density, velocity, total energy and pressure of the gas, α is the coefficient of friction, and $q(x)$ is a given function representing the heating effect from the force outside the nozzle.

It is well known that after we have a method to obtain the global existence of solutions for the Cauchy problem of the following homogeneous system:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \end{cases} \tag{1.6}$$

with the bounded initial data (1.2), the main difficulties to study the inhomogeneous system (1.1) are to obtain the a priori L^∞ estimate of the approximation solutions of Equation (1.1), for instance, the a priori L^∞ estimate of the classical viscosity solutions for the Cauchy problem of the parabolic system

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u + \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 + \varepsilon(\rho u)_{xx} \end{cases} \tag{1.7}$$

with the initial data (1.2), and to obtain the positive, lower estimate of ρ^ε since the term $\rho u^2 = \frac{m^2}{\rho}, m = \rho u$ in the second equation in Equation (1.4) is singular when $\rho = 0$.

To obtain these necessary estimates, in this paper, we first use the method given in [17, 18] to construct a sequence of the regular hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x)(\rho - 2\delta)u \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x)(\rho - 2\delta)u^2 \end{cases} \tag{1.8}$$

to approximate system (1.1), where $A(x) = -\frac{a'(x)}{a(x)}, \delta > 0$ denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t)dt. \tag{1.9}$$

As proved in [17], both systems (1.1) and (1.8) have the same Riemann invariants and the entropy equation.

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = u - \sqrt{P'(\rho)}, \quad \lambda_2 = u + \sqrt{P'(\rho)} \tag{1.10}$$

with corresponding Riemann invariants

$$z(\rho, u) = \int_l^\rho \frac{\sqrt{P'(s)}}{s} ds - u, \quad w(\rho, u) = \int_l^\rho \frac{\sqrt{P'(s)}}{s} ds + u, \tag{1.11}$$

where l is a constant, and two eigenvalues of system (1.8) are

$$\lambda_1^\delta = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2^\delta = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)} \tag{1.12}$$

with corresponding two same Riemann invariants (1.11) (both systems (1.1) and (1.8) have the same Riemann invariants as well as the entropy equations [17]).

Second, we consider the Cauchy problem for the following parabolic system:

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x)(\rho - 2\delta)u + \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x)(\rho - 2\delta)u^2 + \varepsilon(\rho u)_{xx} \end{cases} \tag{1.13}$$

with initial data

$$(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) = (\rho_0(x) + 2\delta, u_0(x)), \tag{1.14}$$

where $(\rho_0(x), u_0(x))$ are given in Equation (1.2).

To use the first equation in Equation (1.13), we deduce directly the positive lower bound $\rho^{\varepsilon,\delta} \geq 2\delta > 0$ by the theory of invariant regions [4].

Finally, we made the transformation $z = v + B(x)$, where $B(x)$ is a bounded function to be carefully chosen to control the nonlinear function $A(x)$, so that we might obtain the following inequality on the variable v :

$$v_t + c(x,t)v_x + b(x,t)v \leq \varepsilon v_{xx}, \tag{1.15}$$

which gave us the estimate $v \leq 0$ and so the upper estimate $z(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \leq B(x)$ when the maximum principle was applied to Equation (1.15).

Precisely, we have the following theorems in this paper.

Theorem 1.1. *Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma \geq 3$. If there exist a positive constant M and a nonnegative function $\beta(x)$ such that*

$$\theta M|A(x)| \leq \beta(x), \quad \int_{-\infty}^{\infty} \beta(s)ds \leq \frac{M}{2}, \tag{1.16}$$

then we have

$$z(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{(\rho^{\delta,\varepsilon}(x,t))^\theta}{\theta} - u^{\delta,\varepsilon}(x,t) \leq M - \int_{-\infty}^{x} \beta(s)ds \tag{1.17}$$

and

$$w(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{(\rho^{\delta,\varepsilon}(x,t))^\theta}{\theta} + u^{\delta,\varepsilon}(x,t) \leq M + \int_{-\infty}^{x} \beta(s)ds \tag{1.18}$$

if the initial data

$$z(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) < M - \int_{-\infty}^{x} \beta(s)ds \tag{1.19}$$

and

$$w(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) < M + \int_{-\infty}^{x} \beta(s)ds, \tag{1.20}$$

where $\theta = \frac{\gamma-1}{2}$ and $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ are the solutions of the Cauchy problem (1.13) and (1.14).

Theorem 1.2. *Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, 1 < \gamma < 3$. If there exist a positive constant M and a nonnegative function $\beta(x)$ such that*

$$\frac{(\gamma-1)(\gamma+3)}{4(3-\gamma)}M|A(x)| \leq \beta(x), \quad \int_{-\infty}^{\infty} \beta(s)ds \leq \frac{(\gamma-1)M}{4}, \tag{1.21}$$

then we have the same estimates given in Equations (1.17) and (1.18), if the initial data satisfy Equations (1.19) and (1.20).

Remark 1. If we specially choose $\beta(x) = \theta M|A(x)|$ in Theorem 1.1, then the condition (1.17) is equivalent to $\int_{-\infty}^{\infty} \beta(s)ds \leq \frac{1}{2\theta}$, which is large than $\frac{1-\theta}{1+\theta}$ given in Theorem 1.1 in [1]. However, if we choose $\beta(x) = \frac{(\gamma-1)(\gamma+3)}{4(3-\gamma)}M|A(x)|$ in Theorem 1.2, then Equation (1.21) is equivalent to $\int_{-\infty}^{\infty} \beta(s)ds \leq \frac{3-\gamma}{\gamma+3}$, which is same to $\frac{1-\theta}{1+\theta}$.

Theorem 1.3. *For such functions $A(x)$ and the initial data satisfying the conditions in Theorems 1.1 and 1.2, there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as δ, ε tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1) and (1.2).*

Definition 1. A pair of bounded functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.1) and (1.2) if

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho\phi_t + (\rho u)\phi_x - \frac{a'(x)}{a(x)}(\rho u)\phi dxdt + \int_{-\infty}^\infty \rho_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u\phi_t + (\rho u^2 + P(\rho))\phi_x - \frac{a'(x)}{a(x)}\rho u^2\phi dxdt \\ + \int_{-\infty}^\infty \rho_0(x)u_0(x)\phi(x,0)dx = 0 \end{cases} \tag{1.22}$$

holds for all test function $\phi \in C_0^1(R \times R^+)$ and

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \eta(\rho,m)\phi_t + q(\rho,m)\phi_x - \frac{a'(x)}{a(x)}\eta(\rho,m)_\rho \rho u \\ &- \frac{a'(x)}{a(x)}\rho u^2 \eta(\rho,m)_m \phi dxdt \geq 0 \end{aligned} \tag{1.23}$$

holds for any non-negative test function $\phi \in C_0^\infty(R \times R^+ - \{t=0\})$, where $m = \rho u$ and (η, q) is a pair of convex entropy–entropy flux of system (1.1).

2 | PROOF OF THEOREMS 1.1–1.3

In this section, we shall prove Theorems 1.1–1.3.

Proof of Theorem 1.1. We multiply Equation (1.13) by (z_ρ, z_m) and (w_ρ, w_m), respectively, where (z, w) are given in Equation (1.11), to obtain

$$\begin{aligned} &z_t + \lambda_1^\delta z_x \\ &= \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ &\quad + A(x)(\rho - 2\delta)u(\frac{m}{\rho^2} + \frac{\sqrt{P'(\rho)}}{\rho}) - \frac{1}{\rho}A(x)(\rho - 2\delta)u^2 \\ &= \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ &\quad + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &w_t + \lambda_2^\delta w_x \\ &= \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ &\quad + A(x)(\rho - 2\delta)u(-\frac{m}{\rho^2} + \frac{\sqrt{P'(\rho)}}{\rho}) + \frac{1}{\rho}A(x)(\rho - 2\delta)u^2 \\ &= \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ &\quad + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}. \end{aligned} \tag{2.2}$$

Letting $z = B(x) + v$ in Equation (2.1), where $B(x) = M - \int_{-\infty}^{x} \beta(s)ds$, we have

$$\begin{aligned} & v_t + (u - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(v_x + B'(x)) - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & = \varepsilon v_{xx} + \varepsilon B''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_x + \tfrac{2\varepsilon}{\rho}\rho_x B'(x) - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} & v_t + (u - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})v_x - B'(x)(B(x) + v - \textstyle\int_l^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho) \\ & -B'(x)\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & = \varepsilon v_{xx} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \tfrac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x B'(x) + (\tfrac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x))^2] \\ & +\varepsilon B''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_x + \tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 \end{aligned} \tag{2.4}$$

or

$$\begin{aligned} & v_t + a(x,t)v_x + b(x,t)v + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_1 B(x)B'(x)] \leq \varepsilon v_{xx} \\ & - \textstyle\int_l^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) + (1-\varepsilon_1)B(x)B'(x) + B'(x)(\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & +A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}, \end{aligned} \tag{2.5}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a(x,t) = u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b(x,t) = -B'(x)$.

Similarly, letting $w = C(x) + v_1$ in Equation (2.2), where $C(x) = M + \int_{-\infty}^{x} \beta(s)ds$, we have

$$\begin{aligned} & v_{1t} + (u + \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(v_{1x} + C'(x)) - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & = \varepsilon v_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_{1x} + \tfrac{2\varepsilon}{\rho}\rho_x C'(x) - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} & v_{1t} + (u + \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})v_{1x} + C'(x)(C(x) + v_1 - \textstyle\int_l^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho) \\ & +C'(x)\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & = \varepsilon v_{1xx} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \tfrac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x C'(x) + (\tfrac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x))^2] \\ & +\varepsilon C''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_{1x} + \tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2 \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} & v_{1t} + a_1(x,t)v_{1x} + b_1(x,t)v_1 + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x)] \leq \varepsilon v_{1xx} \\ & + \textstyle\int_l^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x) - (1-\varepsilon_1)C(x)C'(x) - C'(x)(\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} \\ & +A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}, \end{aligned} \tag{2.8}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a_1(x,t) = u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b_1(x,t) = C'(x)$.

Using the first equation in Equation (1.13), we have the a priori estimate $\rho \geq 2\delta$. We can choose $\beta(x)$ to be smooth enough, $\varepsilon = o(\delta)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of Equations (2.5) and (2.8)

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_1 B(x)B'(x) \geq 0 \tag{2.9}$$

and

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) \geq 0. \tag{2.10}$$

When $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma \geq 3$, we choose $l = 2\delta$, then by using the following inequality:

$$\frac{1}{\theta}(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \leq \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s}ds \leq (\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \quad \text{for} \quad \gamma \geq 3, \tag{2.11}$$

we have the following estimate on the terms of Equation (2.5)

$$\begin{aligned} L = &-\textstyle\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) + (1-\varepsilon_1)B(x)B'(x) \\ &+B'(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} \\ &\leq (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}. \end{aligned} \tag{2.12}$$

Now, we may analyze the function L point by point. First, at the points (x,t), where $A(x) \geq 0$,

$$\begin{aligned} L &\leq (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} \\ &= (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(w - z) \\ &= A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1 - v) + (1-\varepsilon_1)B(x)B'(x) \\ &\quad +A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(C(x) - B(x)) \\ &= -A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v - v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) \\ &\quad +A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_{-\infty}^x \beta(s)ds \\ &\leq -A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v - v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) \\ &\quad +\theta A(x)\textstyle\int_{-\infty}^x \beta(s)ds \int_{2\delta}^\rho \frac{\sqrt{P'(s)}}{s}ds \\ &= -A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v - v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) \\ &\quad +\frac{1}{2}\theta A(x)\textstyle\int_{-\infty}^x \beta(s)ds(w + z) \\ &= -A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v - v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) \\ &\quad +\frac{1}{2}\theta A(x)\textstyle\int_{-\infty}^x \beta(s)ds(v_1 + v + C(x) + B(x)) \\ &= (\frac{1}{2}\theta A(x)\textstyle\int_{-\infty}^x \beta(s)ds + A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v_1 \\ &\quad +(\frac{1}{2}\theta A(x)\textstyle\int_{-\infty}^x \beta(s)ds - A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v \\ &\quad -((1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) - \theta M A(x)\int_{-\infty}^x \beta(s)ds), \end{aligned} \tag{2.13}$$

where

$$
\begin{aligned}
&(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)-\theta MA(x)\textstyle\int_{-\infty}^{x}\beta(s)ds\\
&>\frac{M}{2}\beta(x)-\theta MA(x)|\beta(x)|_{L^1(R)}\geq 0
\end{aligned}
\tag{2.14}
$$

due to the conditions $|\beta(x)|_{L^1(R)}\leq \frac{M}{2}$ and $\theta M|A(x)|\leq\beta(x)$ in Theorem 1.1. Therefore, we obtain the following inequality from Equations (2.5), (2.9), (2.13), and (2.14)

$$
v_t+a(x,t)v_x+l_1(x,t)v+l_2(x,t)v_1\leq\varepsilon v_{xx}, \tag{2.15}
$$

where $l_1(x,t), l_2(x,t)\leq 0$ are suitable functions.

Second, at the points (x,t), where $A(x)\leq 0$, we have

$$
\begin{aligned}
L&\leq(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}\\
&=(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds-z)\\
&\leq(1-\varepsilon_1)B(x)B'(x)-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v+B(x))\\
&\leq(1-\varepsilon_1)B(x)B'(x)-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v-\theta A(x)B(x)\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds\\
&=(1-\varepsilon_1)B(x)B'(x)-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v-\frac{1}{2}\theta A(x)B(x)(w+z)\\
&=(1-\varepsilon_1)B(x)B'(x)-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v-\frac{1}{2}\theta A(x)B(x)(v+v_1+2M)\\
&=-(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+\frac{1}{2}\theta A(x)B(x))v\\
&\quad-\frac{1}{2}\theta A(x)B(x)v_1-((1-\varepsilon_1)\beta(x)+\theta MA(x))B(x)\\
&\leq-(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+\frac{1}{2}\theta A(x)B(x))v\\
&\quad-\frac{1}{2}\theta A(x)B(x)v_1,
\end{aligned}
\tag{2.16}
$$

where $-\frac{1}{2}\theta A(x)B(x)\geq 0$. Thus, we also obtain an inequality

$$
v_t+a(x,t)v_x+l_3(x,t)v+l_4(x,t)v_1\leq\varepsilon v_{xx}, \tag{2.17}
$$

where $l_3(x,t), l_4(x,t)\leq 0$ are suitable functions.

Now, we choose $l=2\delta$ and consider the following terms on the right-hand side of Equation (2.8)

$$
\begin{aligned}
L_1=&\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x)-(1-\varepsilon_1)C(x)C'(x)\\
&-C'(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}.
\end{aligned}
\tag{2.18}
$$

First, at the points (x,t), where $A(x)\leq 0$,

$$
\begin{aligned}
\check{}_1&\leq A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}=A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(w-z)\\
&=A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1-v+C(x)-B(x))\\
&\leq A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1-v),
\end{aligned}
\tag{2.19}
$$

where the coefficient before v, $-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}\geq 0$. So, we have an inequality from Equations (2.8), (2.10), and (2.19) that

$$v_{1t}+a_1(x,t)v_{1x}+l_5(x,t)v+l_6(x,t)v_1\leq \varepsilon v_{1xx}, \tag{2.20}$$

where $l_5(x,t)\leq 0, l_6(x,t)$ are suitable functions. Second, at the points (x,t), where $A(x)\geq 0$,

$$\begin{aligned}
L_1 &\leq -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}\\
&= -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\Big(w-\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho\Big)\\
&\leq -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1+C(x))\\
&\leq -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1+\theta A(x)C(x)\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho\\
&= -(1-\varepsilon_1)C(x)\beta(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1+\frac{1}{2}\theta A(x)C(x)(w+z)\\
&= -(1-\varepsilon_1)C(x)\beta(x)+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1\\
&\quad +\frac{1}{2}\theta A(x)C(x)(v+v_1+C(x)+B(x))\\
&= \Big(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+\frac{1}{2}\theta A(x)C(x)\Big)v_1\\
&\quad +\frac{1}{2}\theta A(x)C(x)v-((1-\varepsilon_1)\beta(x)-\theta MA(x))C(x)\\
&\leq \Big(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}+\frac{1}{2}\theta A(x)C(x)\Big)v_1+\frac{1}{2}\theta A(x)C(x)v
\end{aligned} \tag{2.21}$$

due to $\theta M|A(x)|<\beta(x)$, where the coefficient $\frac{1}{2}\theta A(x)C(x)\geq 0$. So, we also have an inequality

$$v_{1t}+a_1(x,t)v_{1x}+l_7(x,t)v+l_8(x,t)v_1\leq \varepsilon v_{1xx}, \tag{2.22}$$

where $l_7(x,t)\leq 0, l_8(x,t)$ are suitable functions.

Summing up the analysis above, we have the following two inequalities on v and v_1:

$$\begin{cases} v_t+a(x,t)v_x+b(x,t)v+c(x,t)v_1\leq \varepsilon v_{xx},\\ v_{1t}+a_1(x,t)v_{1x}+b_1(x,t)v_1+c_1(x,t)v\leq \varepsilon v_{1xx},\end{cases} \tag{2.23}$$

where the coefficient functions $c(x,t)\leq 0, c_1(x,t)\leq 0$, so the maximum principle [15] on nonlinear-coupled parabolic equations gives us the estimates $v(x,t)\leq 0, v_1(x,t)\leq 0$ and the upper bounds of z and w. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. To prove Theorem 1.2, when $P(\rho)=\frac{1}{\gamma}\rho^\gamma, 1<\gamma<3$, we let $l=0$ and rewrite Equations (2.5) and (2.8) as follows:

$$\begin{aligned}
& v_t+a(x,t)v_x+b(x,t)v\\
&\quad +\Big[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2-\varepsilon B''(x)-\varepsilon_1 B(x)B'(x)+2\delta\frac{\sqrt{P'(\rho)}}{\rho}B'(x)\Big]\\
&\leq \varepsilon v_{xx}-\int_0^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x)+(1-\varepsilon_1)B(x)B'(x)\\
&\quad +B'(x)\sqrt{P'(\rho)}+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}\\
&= \varepsilon v_{xx}+\frac{\gamma-3}{\gamma-1}\rho^\theta B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}
\end{aligned} \tag{2.24}$$

and

$$
\begin{aligned}
& v_{1t}+a_1(x,t)v_{1x}+b_1(x,t)v_1 \\
& \quad +[-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2-\varepsilon C''(x)+\varepsilon_1 C(x)C'(x)-2\delta\tfrac{\sqrt{P'(\rho)}}{\rho}C'(x)] \\
& \le \varepsilon v_{1xx}+\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x)-(1-\varepsilon_1)C(x)C'(x) \\
& \quad -C'(x)\sqrt{P'(\rho)}+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\
& =\varepsilon v_{1xx}-\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}.
\end{aligned}
\tag{2.25}
$$

Since

$$
2\delta\frac{\sqrt{P'(\rho)}}{\rho}=2\delta\rho^{\frac{\gamma-3}{2}}\le(2\delta)^{\frac{\gamma-1}{2}},
\tag{2.26}
$$

we may choose $\beta(x)$ to be sufficiently smooth, $\varepsilon=o(\delta)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of Equations (2.24) and (2.25)

$$
-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2-\varepsilon B''(x)-\varepsilon_1 B(x)B'(x)+2\delta\frac{\sqrt{P'(\rho)}}{\rho}B'(x)\ge 0,
\tag{2.27}
$$

$$
-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2-\varepsilon C''(x)+\varepsilon_1 C(x)C'(x)-2\delta\frac{\sqrt{P'(\rho)}}{\rho}C'(x)\ge 0.
\tag{2.28}
$$

Furthermore, we consider the terms on the right-hand side of Equations (2.24) and (2.25)

$$
K=\tfrac{\gamma-3}{\gamma-1}\rho^\theta B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}
\tag{2.29}
$$

and

$$
K_1=-\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}.
\tag{2.30}
$$

First, at the points (x,t), where $A(x)\ge 0$, we have that

$$
\begin{aligned}
K &= \tfrac{\gamma-3}{\gamma-1}\rho^\theta B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} \\
&= \tfrac{\gamma-3}{4}(w+z)B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)\tfrac{\rho-2\delta}{2\rho}(w-z)\rho^\theta \\
&= \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds) \\
&\quad +A(x)\tfrac{\rho-2\delta}{2\rho}(v_1-v+2\int_{-\infty}^x\beta(s)ds)\rho^\theta \\
&= (\tfrac{3-\gamma}{4}\beta(x)-A(x)\tfrac{\rho-2\delta}{2\rho})v+(\tfrac{3-\gamma}{4}\beta(x)+A(x)\tfrac{\rho-2\delta}{2\rho})v_1 \\
&\quad +\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds)+A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^x\beta(s)ds\rho^\theta,
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^x\beta(s)ds\rho^\theta &= A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^x\beta(s)ds\tfrac{\theta}{2}(w+z) \\
&= A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^x\beta(s)ds\tfrac{\theta}{2}(v_1+v+2M)
\end{aligned}
\tag{2.32}
$$

and

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)+A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds\theta M\\
&\leq(\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x)+(1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds+\int_{-\infty}^{x}\beta(s)ds\beta(x)\\
&=\beta(x)((\tfrac{1-\gamma}{2}+\varepsilon_1)M+(2-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds)\leq 0
\end{aligned}\tag{2.33}$$

because $|\theta MA(x)|\leq\frac{2(3-\gamma)}{\gamma+3}\beta(x)\leq\beta(x)$ and $2\int_{-\infty}^{\infty}\beta(s)ds\leq\frac{\gamma-1}{2}M$ as given in Theorem 1.2.

Thus, we have from Equations (2.31)–(2.33) that

$$\begin{aligned}
K&\leq(\tfrac{3-\gamma}{4}\beta(x)-A(x)\tfrac{\rho-2\delta}{2\rho})v+(\tfrac{3-\gamma}{4}\beta(x)+A(x)\tfrac{\rho-2\delta}{2\rho})v_1\\
&\quad+A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds\tfrac{\theta}{2}(v_1+v)=l_1(x,t)v+l_2(x,t)v_1,
\end{aligned}\tag{2.34}$$

where $l_2(x,t)\geq 0$.

Second, at the points (x,t), where $A(x)\leq 0$, we have that

$$\begin{aligned}
K&=\tfrac{\gamma-3}{\gamma-1}\rho^{\theta}B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}\\
&=\tfrac{\gamma-3}{4}(w+z)B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)\tfrac{\rho-2\delta}{\rho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2}\\
&=\tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&\quad+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1-v+2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v+v_1+2M)\\
&=(\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds-v-2M))v\\
&\quad+(\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds+2M))v_1+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2\\
&\quad+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds,
\end{aligned}\tag{2.35}$$

where the coefficient before v_1

$$\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds+2M)\\
&\geq\tfrac{3-\gamma}{4}\beta(x)-\tfrac{\gamma-1}{8}|A(x)|\tfrac{\rho-2\delta}{\rho}(\tfrac{\gamma-1}{2}M+2M)\\
&=\tfrac{3-\gamma}{4}\beta(x)-\tfrac{\gamma+3}{8}\theta MA(x)\geq 0
\end{aligned}\tag{2.36}$$

because $|\theta MA(x)|\leq\frac{2(3-\gamma)}{\gamma+3}\beta(x)$;

$$\frac{\gamma-1}{8}A(x)\frac{\rho-2\delta}{\rho}v_1^2\leq 0\tag{2.37}$$

and

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds\leq 0
\end{aligned}\tag{2.38}$$

because the proof of Equation (2.33). Thus, we have from Equations (2.35)–(2.38) that

$$K\leq l_3(x,t)v+l_4(x,t)v_1,\tag{2.39}$$

where $l_3(x,t),l_4(x,t)\geq 0$ are two suitable functions.

Summing up the analysis above, for any $A(x)$, we have the following inequality:

$$v_t+a(x,t)v_x+b(x,t)v+c(x,t)v_1\leq\varepsilon v_{xx},\tag{2.40}$$

where the coefficient function $c(x,t)\leq 0$.

Similarly, we consider K_1 given in Equation (2.30). First, at the points (x,t), where $A(x)\le 0$, we have that

$$\begin{aligned} K_1 &= -\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}\\ &= -\tfrac{\gamma-3}{4}(w+z)C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)\tfrac{\rho-2\delta}{2\rho}(w-z)\rho^\theta\\ &= \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)\\ &\quad +A(x)\tfrac{\rho-2\delta}{2\rho}(v_1-v+2\textstyle\int_{-\infty}^x\beta(s)ds)\rho^\theta\\ &= (\tfrac{3-\gamma}{4}\beta(x)-A(x)\tfrac{\rho-2\delta}{2\rho})v+(\tfrac{3-\gamma}{4}\beta(x)+A(x)\tfrac{\rho-2\delta}{2\rho})v_1\\ &\quad +\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)+A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^x\beta(s)ds\rho^\theta, \end{aligned} \tag{2.41}$$

where

$$A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^x\beta(s)ds\rho^\theta\le 0, \tag{2.42}$$

$$\begin{aligned} &\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)\\ &\quad\le(\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x)-(1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^x\beta(s)ds\le 0 \end{aligned} \tag{2.43}$$

and

$$\frac{3-\gamma}{4}\beta(x)-A(x)\frac{\rho-2\delta}{2\rho}\ge 0. \tag{2.44}$$

Thus, we have from Equation (2.41)–(2.44) that

$$K_1\le l_5(x,t)v+l_6(x,t)v_1, \tag{2.45}$$

where $l_5(x,t)\ge 0, l_6(x,t)$ are two suitable functions.

Second, at the points (x,t), where $A(x)\ge 0$, we have that

$$\begin{aligned} K_1 &= -\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}\\ &= -\tfrac{\gamma-3}{4}(w+z)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)+A(x)\tfrac{\rho-2\delta}{\rho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2}\\ &= \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)\\ &\quad +\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1-v+2\textstyle\int_{-\infty}^x\beta(s)ds)(v+v_1+2M)\\ &= (\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1+2\textstyle\int_{-\infty}^x\beta(s)ds+2M))v_1\\ &\quad +(\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^x\beta(s)ds-2M))v-\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v^2\\ &\quad +\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^x\beta(s)ds, \end{aligned} \tag{2.46}$$

where

$$-\frac{\gamma-1}{8}A(x)\frac{\rho-2\delta}{\rho}v^2\le 0, \tag{2.47}$$

$$\begin{aligned} &\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds)+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^x\beta(s)ds\\ &\quad\le(\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x)+\textstyle\int_{-\infty}^x\beta(s)ds\beta(x)\le 0 \end{aligned} \tag{2.48}$$

and the coefficient before v

$$\begin{aligned}&\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\int_{-\infty}^{x}\beta(s)ds-2M)\\&\geq\tfrac{3-\gamma}{4}\beta(x)-\tfrac{\gamma-1}{8}|A(x)|\tfrac{\rho-2\delta}{\rho}(\tfrac{\gamma-1}{2}M+2M)\\&=\tfrac{3-\gamma}{4}\beta(x)-\tfrac{\gamma+3}{8}\theta MA(x)\geq 0\end{aligned}\tag{2.49}$$

because the proof of Equation (2.36). Thus, we have from Equations (2.46)–(2.49) that

$$K_1 \leq l_7(x,t)v + l_8(x,t)v_1, \tag{2.50}$$

where $l_7(x,t)\geq 0, l_8(x,t)$ are two suitable functions.

Summing up the analysis above, for any $A(x)$, we have the following inequality:

$$v_{1t}+a_1(x,t)v_{1x}+b_1(x,t)v_1+c_1(x,t)v\leq \varepsilon v_{1xx}, \tag{2.51}$$

where the coefficient function $c_1(x,t)\leq 0$.

Therefore, we may apply the maximum principle to the coupled inequalities (2.40) and (2.51) to obtain the estimates $v(x,t)\leq 0, v_1(x,t)\leq 0$ and so the upper bounds of z and w (see [19] for the details). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since the original system (1.1) and the approximated system (1.8) have the same entropy equation or the same entropies [17], also for any weak entropy-entropy flux pair $(\eta(\rho,u),q(\rho,u))$ of system (1.1), it was proved in [17] that

$$\eta_t(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+q_x(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t)) \tag{2.52}$$

are compact in $H^{-1}_{loc}(R\times R^+)$, then there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t),u(x,t))$ as δ,ε tend to zero by using the compactness framework given in [14] for $1<\gamma<3$ and in [13] for $\gamma\geq 3$. It is easy to prove that the limit $(\rho(x,t),u(x,t))$ satisfies Equation (1.22). Moreover, for any weak convex entropy–entropy flux pair $(\eta(\rho,u),q(\rho,u))$ of system (1.1), we multiply Equation (1.13) by (η_ρ,η_m) to obtain that

$$\begin{aligned}&\eta_t(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+q_x(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+\delta q_{1x}(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))\\&=\varepsilon\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})_{xx}-\varepsilon(\rho^{\delta,\varepsilon}_x,m^{\delta,\varepsilon}_x)\cdot\nabla^2\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\cdot(\rho^{\delta,\varepsilon}_x,m^{\delta,\varepsilon}_x)^T\\&\quad+A(x)(\rho^{\delta,\varepsilon}-2\delta)m^{\delta,\varepsilon}\eta_\rho(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})+A(x)(\rho^{\delta,\varepsilon}-2\delta)(u^{\delta,\varepsilon})^2\eta_m(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\\&\leq\varepsilon\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})_{xx}+A(x)(\rho^{\delta,\varepsilon}-2\delta)m^{\delta,\varepsilon}\eta_\rho(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\\&\quad+A(x)(\rho^{\delta,\varepsilon}-2\delta)(u^{\delta,\varepsilon})^2\eta_m(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon}),\end{aligned}\tag{2.53}$$

where $q+\delta q_1$ is the entropy flux of system (1.5) corresponding to the entropy η. Thus, the entropy inequality (1.23) is proved if we multiply a test function to Equation (2.53) and let ε,δ go to zero. Theorem 1.3 is proved. □

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interests.

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