

CONVEX CONSERVATION LAWS
WITH DISCONTINUOUS COEFFICIENTS.
EXISTENCE, UNIQUENESS
AND ASYMPTOTIC BEHAVIOR

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Abstract. Existence and uniqueness is proved, in the class of functions satisfying a wave entropy condition, of weak solutions to a conservation law with a flux function that may depend discontinuously on the space variable. The large time limit is then studied, and explicit formulas for this limit is given in the case where the initial data as well as the x dependency of the flux vary periodically. Throughout the paper, front tracking is used as a method of analysis. A numerical example which illustrates the results and method of proof is also presented.

0. Introduction. This paper is concerned with scalar conservation laws of the form

$$(0.1) \quad u_t + (k(x)f(u))_x = 0.$$

where $u = u(x, t)$ is the unknown function. This equation expresses that u is conserved with a flux density given by $k(x)f(u)$. Such conservation laws arise in a diversity of contexts, ranging from models of traffic flow [31],

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via models of flow in porous media [20], to hydrodynamic limits of nearest particle processes [21].

Independently of the smoothness of the coefficient $k(x)$, and of the initial data $u(x, 0)$, discontinuities will generally develop in $u(x, t)$. Therefore (0.1) is interpreted in the distributional sense, this means that one has to impose additional conditions in order to ensure uniqueness of a solution. In analogy with gas dynamics, these conditions are usually referred to as *entropy conditions*.

If $k(x)$ is continuous with bounded derivative, one can use the Kružkov entropy condition which says that

$$(0.2) \quad \partial_t |u - c| + \partial_x (\operatorname{sgn}[u - c] (k(x)f(u) - k(x)f(c))) \leq 0,$$

should hold distributionally for every constant c . Kružkov showed in [12] that there is a unique function u of bounded variation which satisfies (0.2), and takes the correct initial data. If $k(x)$ is not continuous, (0.2) does not make sense, and other entropy conditions must be considered.

In the present paper we use a “smallest jump” entropy condition, introduced by Gimse and Risebro in [4], when constructing approximate solutions to (0.1). We show that the approximate solutions lie in a compact set, and that any limit also is a weak solution to (0.1). The “smallest jump” entropy condition is shown to imply that a limit satisfies the wave entropy condition

$$(0.3) \quad \partial_x (k(x)f'(u)) \leq K \left(\frac{1}{t} + 1 \right),$$

for some constant K . And, via estimates for an adjoint problem, we show that weak solutions of (0.1) satisfying (0.3), are unique if their initial data $u(x, 0)$ coincide.

We then proceed to study the large time behavior of solutions to (1.1). This is done by analyzing the behavior of the approximate solutions generated by the front tracking method. We show that for a periodic coefficient $k(x)$, and periodic initial data $u(x, 0)$, the solution does not converge to zero, but to a “standing N -wave”, whose shape resembles a roman ‘N’, in contrast to the standard N -wave which resembles a cyrillic ‘И’.

Existence of solutions to scalar conservation laws of type (0.1) have often been established using difference methods, [19, 1], but a straightforward generalization of these techniques is not possible if $k(x)$ is not continuous. Instead we choose to write (0.1) as a 2×2 system of equations, the first equation being (0.1), the second expressing “conservation” of k , that is

$$(0.4) \quad k_t = 0.$$

The Riemann problem for this system, (0.1) and (0.4), may not be solvable if $k(x)$ is not bounded away from 0, consequently we assume that $k(x)$ is never zero.

In [15] LeFloch and Nedelec present another approach to existence and uniqueness of solutions to (0.1). They study an equivalent equation

$$(r(x)u)_t + (r(x)f(u))_x = 0,$$

and use an explicit representation formula to show existence. This equation is transformed into (0.1) by rescaling the time variable. Again, these techniques depend on the differentiability of $r(x)$, and it would be interesting to see whether the argument in [15] could be modified to cover the case where $r(x)$ is not assumed to be continuous.

The structure of the solution of the Riemann problem for (0.1) and (0.4) is remarkably similar to the solution of the Riemann problem for a system of equation modeling flow of oil, water and polymer in a one dimensional porous medium

$$(0.5) \quad \begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc)_t - (cf(s, c))_x &= 0. \end{aligned}$$

Here s denotes the saturation of water, and c the concentration of dissolved polymer in the water. This system has one linearly degenerate characteristic field, and is not strictly hyperbolic. Both of these properties are shared by (0.1) and (0.4). The system (0.5) was studied by Isaacson [9], and later existence of a solution was proved by Temple [28] using the Glimm scheme. It is interesting to note that both existence, uniqueness and continuous dependence of solutions to (0.5) was proved, using a difference method and a "Kruzkov type" entropy condition, by Tveito and Winther in the case where $c(x, 0)$ is Lipschitz continuous [30]. The estimates used in [30] rely on smoothness estimates on $c(x, t)$ derived in [29], hence a simple adaptation of this difference method to (0.1) and (0.4) can only be expected to work if $k(x)$ is continuous.

Because of these similarities, one may regard (0.1) as a "model 2×2 system" of nonstrictly hyperbolic conservation laws. This model system of conservation laws is an example of a system of resonant conservation laws. Such systems have been studied by Isaacson and Temple [10, 11] in a more general setting. In particular, in [11] the Riemann problem for

$$u_t + f(a(x), u)_x = 0,$$

where u is a vector, was shown to have a unique solution provided the initial states were close.

The large time asymptotics for scalar conservation laws with a flux function $f(u)$ has been studied by many authors, starting with Hopf [7] who studied Burgers' equation, that is $f(u) = u^2/2$, and established that

$$(0.6) \quad |u(x, t)| \leq \text{const} \cdot t^{-1/2}$$

for initial data in $L_1 \cap L_\infty$. Generalizations of this result was then obtained by Lax [14] and many other authors [8, 24, 17, 3] to mention just a few. For a review of the results on asymptotic behavior of solutions to scalar conservation laws, see the article by Kruřkov and Petrosyan [13].

If $k(x)$ is continuous, one can introduce a new variables $y = \int_0^x dz/k(z)$ and $v = u/k(x)$, and write (0.1) as

$$(0.7) \quad v_t + f(k(y)v)_y = -f(k(y)v) \frac{d}{dy} \log |k(y)|.$$

Such equations are commonly called *balance laws*, since v is not conserved, and the deviation from conservation is given by the source term $-f(k(y)v) \frac{d}{dy} \log |k(y)|$. Asymptotic behavior for balance laws was studied by Dafermos [3], Lyberpopoulos [18], and recently by Sinestrari [25, 26]. All these authors considered the case where the source term does not depend on the spatial position. Lyberpopoulos [18] assumed that the source term was equal to u , and found that if the initial data was periodic with mean 0, then the solution tends to a traveling wave whose amplitude does not decay with time. This is to be contrasted with the behavior of solutions to conservation laws where k is constant (0.6), and is similar to the results obtained in this paper.

The rest of this paper is organized as follows: In section 1 we define the front tracking method and construct the approximate solutions. The front tracking scheme is based on the solution of the Riemann problem, and we therefore restate the solution of this from [4]. Then we show that the front tracking method is well defined, and that the functions generated by front tracking lie in a compact set. We then proceed to show that any limit is a weak solution which also satisfies a wave entropy condition. In the manner used in [16], we show that this wave entropy condition implies uniqueness.

Section 2 is concerned with the asymptotic limit for large times. In the case where $k(x)$ and the initial data $u(x, 0)$ both vary periodically, we show explicit formulas for this limit. This is done by examining the corresponding limits for the approximate solutions. The approximate solutions have the property that they are stationary, i.e., $k(x)f(u)$ is constant, after some finite time which depends on the level of approximation. This means that the large time limit of the approximate solutions actually is attained after some finite time.

Since front tracking also is a viable practical numerical method, in section 3 we give a numerical example which illustrates the results from the previous sections.

1. Construction of the weak solution. We here study scalar conservation laws of the following type

$$(1.1) \quad \begin{aligned} u_t + (k(x)f(u))_x &= 0 \\ u(x, 0) &= u_0(x). \end{aligned}$$

Here $k(x)$ is a function of bounded total variation, not necessarily smooth, bounded away from 0, and f is a strictly convex or concave function. We assume that there are constants $a < b$, such that $f(a) = f(b) = 0$, and that the initial function $u_0(x)$ takes values in $[a, b]$. Solutions of (1.1) will in general be regular distributions, and are assumed to satisfy (1.1) distributionally, i.e.,

$$(1.2) \quad \int_{-\infty}^{\infty} \int_0^{\infty} u \phi_t + k(x)f(u)\phi_x \, dt dx + \int_{-\infty}^{\infty} u_0(x)\phi(x, 0) \, dx = 0$$

for all test functions ϕ in $C_0^1(x, t)$.

When stating explicit formulas: (1.7), (1.8), and (2.3), we will in the rest of this paper use

$$(1.3) \quad f(u) = u(1 - u).$$

This expression for $f(u)$ is also used implicitly in the remarks below equation (2.1), where we use the fact that f has a maximum for $u = 1/2$. For simplicity we will also assume that $k(x) > 0$. All results can however quite easily be modified to cover the more general case.

In order to be able to deal with a discontinuous coefficient k , we use the strategy from [5] and introduce an auxiliary system with unknown $v = (u, k)$, and corresponding flux function $G(v) = (kf(u), 0)$, so that (1.1) can be written

$$(1.4) \quad v_t + G(v)_x = 0.$$

The aim of writing a scalar equation as a system of two equations is that the behavior of $u(x, t)$ at discontinuities of $k(x)$ is more easily analyzed using (1.4). More precisely, the Riemann problem for (1.4) was shown in [4] to have a unique solution provided an additional "entropy" condition was assumed to hold. We will now briefly summarize the construction of the Riemann problem solution reported in [4].

The Riemann problem. The Riemann problem for (1.4) is the initial value problem where the initial function $v_0(x)$ is given by

$$(1.5) \quad u_0(x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x \geq 0, \end{cases} \quad k(x) = \begin{cases} k_l & \text{for } x < 0, \\ k_r & \text{for } x \geq 0. \end{cases}$$

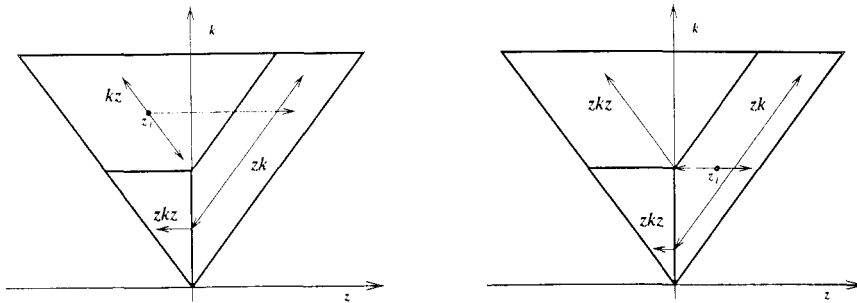


Figure 1. Solution of the Riemann problem

We define v_+ and v_- to be the left and right hand limits of the solution of (1.5) as $x \rightarrow 0^+$ and $x \rightarrow 0^-$ respectively. The quantities u_+ and k_+ are similarly defined. The additional entropy condition which is required in order to obtain uniqueness, says that u_+ and u_- should be chosen such that the jump $|u_+ - u_-|$ is the smallest possible jump satisfying the Rankine-Hugoniot condition

$$(1.6) \quad k_- f(u_+) = k_+ f(u_-).$$

In [4] it is shown that this jump condition is equivalent to a viscous profile entropy condition for the enlarged system (1.4). If equation (1.1) is viewed as the system (1.4), we see that we have two types of waves: a u wave, over which k is constant, and a k wave. The system (1.4) is non-strictly hyperbolic, k waves always have zero speed, and u waves may have both positive and negative speeds. Note that by (1.6), $kf(u)$ is constant along k waves, so that if we picture the solution as a curve in (u, k) space, k waves will be contour lines of $kf(u)$. In order to simplify our calculations and diagrams, we will not use (u, k) coordinates, but rather $(\Psi(u, k), k)$, where Ψ is defined by

$$(1.7) \quad \Psi(u, k) = \operatorname{sgn} \left[u - \frac{1}{2} \right] k (1 - 4u(1 - u)).$$

We see that the contour lines of $kf(u)$ is mapped to straight lines with slope $+1$ if $u > 1/2$ and slope -1 if $u < 1/2$. Note that Ψ is injective, and regular everywhere except on $u = 1/2$. In the following let $z = \Psi(u, k)$, so that a Riemann problem is solved by a combination of z -waves and k -waves. The solution of each Riemann problem is indicated in figure 1. To read how a Riemann problem is solved, follow the arrows from (z_l, k_l) until the desired (z_r, k_r) is reached.

For a further explanation and more detailed description of the solution of such Riemann problems, the reader is referred to [4].

The front tracking scheme. We will use the above solution of the Riemann problem for (1.4) to construct a front tracking scheme for the initial value problem. If $k(x)$ is constant, this scheme coincides with Dafermos' scheme [2], furthermore the present scheme is an adaptation of the scheme used in [9].

The accuracy of the scheme is controlled by some (small) parameter $\delta > 0$. For each fixed δ we define $k_i = i\delta$. Define z_{ij} for $-i \leq j \leq i$ by $z_{ij} = j\delta$. We will now define an approximate flux function $k_i f_\delta(u)$ by making an approximation to $k_i f(u)$ which is linear between the u values $\Psi^{-1}(z_{ij}, k_i)$ and $\Psi^{-1}(z_{i,j-1}, k_i)$. By $\Psi^{-1}(z, k)$ we denote the inverse of (1.7) for a fixed k , i.e.,

$$(1.8) \quad \Psi^{-1}(z, k) = \frac{1}{2} \left(1 + \operatorname{sgn}[z] \sqrt{|z|/k} \right).$$

To be more precise, we let

$$(1.9) \quad k_i f_\delta(u) = \begin{cases} k_i f(u) & \text{if } u = u_{ij} \text{ for some } j \\ k_i \left(f(u_{ij}) + \frac{f(u_{i,j-1}) - f(u_{ij})}{u_{i,j-1} - u_{ij}} (u - u_{ij}) \right) & \text{if } u_{ij} < u < u_{i,j-1}. \end{cases}$$

here $u_{ij} = \Psi^{-1}(z_{ij}, k_i)$. For some fixed i , the Riemann problem with initial states

$$(1.10) \quad u_0(x) = \begin{cases} u_{in} & \text{for } x < 0 \\ u_{im} & \text{for } x \geq 0 \end{cases}$$

for some integers n, m such that $-i \leq n, m \leq i$ can be found by taking envelopes. In particular, the solution will in this case consist of a number ($\max\{n - m, 1\}$) of discontinuities moving apart in (x, t) space, furthermore all intermediate states will also be in the set $\{u_{ik}\}$. This last property, namely that the intermediate states in the solution of the Riemann problem are in some fixed finite discrete set, are also seen to hold for the approximate version of (1.4)

$$(1.11) \quad \begin{aligned} v_{xt} + G_\delta(v_\delta)_x &= 0 \\ v_\delta(x, 0) &= \begin{cases} v_{in} & \text{for } x < 0 \\ v_{jm}, & \text{for } x \geq 0 \end{cases} \end{aligned}$$

where $v_{ij} = (u_{ij}, k_i)$ and $G_\delta = (k f_\delta, 0)$. In this case the intermediate u values will be in the set $\{u_{ij}\}$, $0 \leq i \leq N$, and $-i \leq j \leq i$. For a more

detailed description of Dafermos' method the reader is referred to either [2], or [6], where convergence properties are shown.

Let now $v_0(x)$ be some function taking values in the rectangle $[0, 1] \times (0, K]$. We will approximate v_0 in L_1 with a step function $v_{\delta 0}$ taking values in the finite set $\{v_{ij}\}$, such that

$$(1.12) \quad \lim_{\delta \rightarrow 0} \|v_0 - v_{\delta 0}\|_{L_1^{loc}} = 0.$$

We shall now proceed to construct a weak solution $v_\delta(x, t)$ to the initial value problem

$$(1.13) \quad \begin{aligned} v_{\delta t} + G_\delta(v_\delta)_x &= 0 \\ v_\delta(x, 0) &= v_{\delta 0}(x). \end{aligned}$$

The initial value function defines a series of Riemann problems, these can be solved independently, and the solutions consist of constant states separated by discontinuities which move linearly in (x, t) . We track these discontinuities and thereby propagate the solution forward in time until two discontinuities collide. At the collision point a new Riemann problem is defined by the state to the left of the leftmost colliding discontinuity and the state to the right of the rightmost. Thus the tracking can continue up to the next collision and so on. For more details on this type of front tracking schemes see [6, 22, 23] and the references therein. In analogy with the terminology used in the solution of the Riemann problem, we label the discontinuities in v_δ either z waves or k waves.

Now we have the following lemma which implies that the front tracking procedure is well defined:

Lemma 1.1a. *The number of discontinuities in v_δ is nonincreasing for each collision of discontinuities. Furthermore, the number of discontinuities decreases by at least one if two z waves collide, and is constant if a z wave collides with a k wave.*

This lemma also has parts (b) and (c) which will be needed later when we consider uniqueness and asymptotic behavior, the proof of the lemma is a straightforward study of cases and is therefore contained in an appendix.

Compactness of the approximations. In order to show that the approximations have uniformly bounded total variation with respect to the variables (z, k) , we use the argument in section 4 of [5].

The solution v_δ defines a directed path in (z, k) space. This path consists of z and k waves. The k waves are line segments which have slope ± 1 , and the z waves are horizontal line segments. We will call any finite connected sequence of z and k waves an I curve, and say that an I curve connects

v_L to v_R if the left state of the first segment is v_L and the right state of the last segment is v_R . We will write an I curve consisting of z and k waves as $I = z_1 k_1 z_2 \dots z_N$. This terminology and the subsequent techniques are ultimately borrowed from Temple [28], see also [5]. We now define a functional F on I curves, this functional is defined so that it dominates the total variation of v_δ , and $F(v_\delta(\cdot, t))$ will be shown to be nonincreasing in t .

We first define F on simple wave segments. Let first $I = z$ be a wave connecting z_l and z_r , let $\Delta z = |z_r - z_l|$, then

$$(1.14) \quad F(z) = |\Delta z|.$$

If $I = k$, the definition of $F(k)$ is more complicated. Let k connect (z_l, k_l) and (z_r, k_r) . We say that k is clockwise if $z_l > z_r$, otherwise we say that k is counterclockwise. The reason for these terms are to be found in the diagrams in figure 1. Then we define

$$(1.15) \quad F(k) = \begin{cases} 2|\Delta k| & \text{if } k \text{ is clockwise} \\ 4|\Delta k| & \text{if } k \text{ is counterclockwise.} \end{cases}$$

For more general I curves, $I = b_1 b_2 b_3 \dots b_n$, where each b_i is a k or a z wave, we define $F(I)$ additively

$$(1.16) \quad F(I) = \sum_{i=1}^n F(b_i).$$

It is now not difficult to prove the following lemma

Lemma 1.2. *Let I be any I curve connecting v_L to v_R . Let $[v_L v_R]$ be the I curve defined by the solution of the Riemann problem with left and right states v_L and v_R . Then*

$$(1.17) \quad F([v_L v_R]) \leq F(I).$$

Proof. The proof of this lemma is similar to the proof of the corresponding lemmas in [28] or [5] (lemma 3.1). \square

This lemma has the immediate and important consequence that

Lemma 1.3. *$F(v_\delta)$ is nonincreasing in time.*

Proof. It is clear that F only changes at collisions of discontinuities. At collisions, a section of the I curve traced by v_δ connecting states v_l and v_r is replaced by the Riemann solution $[v_l v_r]$. Thus F is nonincreasing. \square

Let $\text{Var}_{ab} v$ denote the total variation of v with respect to the variables a and b . By construction of F we have that $F(I) \geq \text{Var}_{z,k} I$ for any I curve.

Therefore we have that $\text{Var}_{zk} v_\delta$ is uniformly bounded if $F(v_{\delta_0})$ is bounded. For any v_l and v_r , we see from figure 1 that $[v_l v_r]$ consists of a finite number of waves (≤ 3) which intersect transversally in the (z, k) plane. Furthermore, we see that in all cases $F([v_l v_r]) \leq 5(|z_l - z_r| + |k_l - k_r|)$. Thus

$$(1.18) \quad \text{Var}_{zk} v_\delta \leq F(v_\delta) \leq F(v_{\delta_0}) \leq 5 \text{Var}_{zk} v_{\delta_0} \leq O(1) \text{Var}_{zk} v_0$$

For periodic initial data, we have that the total variation over a period is uniformly bounded. Now we may use the boundedness of the total variation, and the fact that (1.13) has finite speed of propagation, to show the following Lipschitz continuity of the L_1 norm of the solution

$$(1.19) \quad \int_a^b |v_\delta(x, t_1) - v_\delta(x, t_2)| \leq O(1) |t_2 - t_1| \text{Var}_{zk} v_0.$$

The integration limits a and b are $\mp\infty$ if the initial data are of bounded variation, and the integration is over one period if the initial data are periodic. The proof of this inequality may be found in [5, Lemma 4.1]. We can then use Helly's theorem and standard arguments as in e.g. [27] to show:

Theorem 1.1. *Let $v_0(x) = \Psi((u_0(x), k(x)))$ be such that $\text{Var}_{zk} v_0$ is finite. Then for any sequence $\{\delta\}$ such that $\delta \rightarrow 0$, there exists a subsequence $\{\delta_j\}$ such that for any finite time $t \geq 0$, $u_{\delta_j}(\cdot, t)$ converges uniformly in $L_1^{\text{loc}}(x)$. Furthermore, the limit of u_{δ_j} is a weak solution to (1.1)*

Proof. As mentioned above the compactness of the sequence follows from standard arguments. To show that the limit is a weak solution we exploit the fact that the approximations are weak solutions to the approximate problems (1.13). Let a and b be as in (1.19), for simplicity let u_δ denote the convergent subsequence, and let u denote the limit of u_δ . We have for a suitable test function ϕ

$$(1.21) \quad \begin{aligned} W(u) &= \int_a^b \int_0^\infty u \phi_t + k(x) f(u) \phi_x \, dx dt + \int_a^b u_0(x) \phi(x, 0) \, dx = \\ &\int_a^b \int_0^\infty (u - u_\delta) \phi_t + \{k(x) f(u) - k_\delta(x) f_\delta(u_\delta)\} \phi_x \, dx dt \\ &\quad + \int_a^b (u_0(x) - u_{\delta_0}(x)) \phi(x, 0) \, dx. \end{aligned}$$

Thus we see that

$$|W(u)| \leq M (\|u - u_\delta\|_1 + \|kf(u) - k_\delta f_\delta(u_\delta)\|_1 + \|u_0 - u_{\delta 0}\|_1) \tag{1.22a}$$

$$\leq M \|u - u_\delta\|_1 + M \|kf(u) - k_\delta f_\delta(u)\|_1 + \tag{1.22b}$$

$$M \|k_\delta f_\delta(u) - k_\delta f_\delta(u_\delta)\|_1 + \tag{1.22c}$$

$$M \|k_\delta f_\delta(u) - k_\delta f_\delta(u_\delta)\|_1 + \tag{1.22d}$$

$$M \|u_0 - u_{\delta 0}\|_1. \tag{1.22e}$$

here M is a bound on ϕ_x , ϕ_t and ϕ . The first term (1.22a) can be made arbitrarily small by construction since u_δ converges to u in L_1 . The second and third terms (1.22b and c) can be made small since f_δ converges to f and k_δ to k uniformly. The fourth term (1.22d) may also be made arbitrarily small since f_δ is uniformly Lipschitz continuous, and finally the last term (1.22e) converges as $\delta \rightarrow 0$ by construction of $u_{\delta 0}$. \square

Uniqueness. We now turn to the question of uniqueness. If $k(x)$ is continuous with $k'(x)$ bounded, we can appeal to the fundamental uniqueness result by Kruřkov [12] and conclude that there is a unique weak solution which satisfies the following inequality

$$\frac{\partial}{\partial t} |u - c| + \frac{\partial}{\partial x} [\text{sgn}(u - c) (k(x)f(u) - k(x)f(c))] \leq 0 \tag{1.23}$$

weakly for all constants c . This equation will almost be satisfied for each u_δ with f_δ replacing f in (1.23), the discrepancy is limited by $\text{const} \cdot \|k_\delta - k\|_\infty$. From this it follows that the limit constructed by front tracing also satisfies (1.23), i.e., it belongs to the "right" class. If we have a $k(x)$ which is not continuous, we have seen that we must impose an additional entropy condition at the discontinuity points of $k(x)$ in order to solve the Riemann problem uniquely. This entropy condition states that among all discontinuities in u which satisfies the Rankine-Hugoniot condition (1.6) across the discontinuity in k , the correct discontinuity in u is the smallest such.

We will now show that this entropy condition implies the the limit function satisfies a wave entropy condition (1.33), and then show that this implies uniqueness. This proof is motivated by a recent result of LeFloch and Xin [16], where a wave entropy condition is used to show uniqueness of solutions to a certain class of systems.

In order to define the wave entropy condition and to show that the limit function u satisfies this condition, we first state some preliminary results. We say the the function u_δ has an approximate centered rarefaction wave

at some point (x, t) , if (x, t) is a point of collision between discontinuities in v_δ , or $t = 0$, and the solution of the Riemann problem at (x, t) involves a z wave larger than δ and such that $z_l > z_r$. In other words, the solution of the Riemann problem with f replacing f_δ would involve a rarefaction wave, and this rarefaction wave is approximated with more than one discontinuity. Regarding approximate rarefaction waves, we have:

Lemma 1.1b. *There are no approximate centered rarefaction waves in u_δ for $t > 0$.*

From now on we assume that $k(x)$ is discontinuous at finitely many points: $\{x_j\}$, and has uniformly bounded derivative at all other points. We will also make the approximation $k_\delta(x)$ such that if a discontinuity in k is larger than δ_0 , also k_δ has a discontinuity at the same point for all $\delta \leq \delta_0$.

We will now define an approximate characteristic speed σ_δ , and then show that σ_δ satisfies an entropy inequality. This will then imply that the characteristic speed of the front tracking limit u satisfies a similar inequality.

In the following we use the notation that a discontinuity in v_δ separates between a left state (u_l, k_l) or (z_l, k_l) , and a right state (u_r, k_r) or (z_r, k_r) . For each t we let $\sigma_\delta(x, t)$ be a piecewise linear function in x , and $\sigma_\delta(x, t)$ is defined to be linear between the discontinuities of v_δ . We then define the right and left limits of $\sigma_\delta(x, t)$ as x approaches a discontinuity in v_δ from above or below. Assume first that v_δ has a z wave located at y for some time t . Then

$$(1.24) \quad \begin{aligned} \lim_{x \rightarrow y^-} \sigma_\delta(x, t) &= k_\delta(y) \lim_{s \rightarrow 0^+} f'_\delta(u_l + s(u_r - u_l)) \\ \lim_{x \rightarrow y^+} \sigma_\delta(x, t) &= k_\delta(y) \lim_{s \rightarrow 0^+} f'_\delta(u_r + s(u_l - u_r)). \end{aligned}$$

Note that if the z wave is such that $|z_l - z_r| = \delta$, then $\sigma_\delta(x, t)$ is continuous at y . Furthermore, σ_δ is nonincreasing between two consecutive z waves when the left wave has a speed larger or equal to the right, and increasing if the left wave has a smaller speed than the right wave. For k waves the limiting values of σ_δ is defined as follows. Assume that v_δ has a k wave at $x = y$, then

$$(1.25) \quad \begin{aligned} \lim_{x \rightarrow y^-} \sigma_\delta(x, t) &= k_l \min \left\{ \lim_{s \rightarrow 0^+} f'_\delta(u_l + s), \lim_{s \rightarrow 0^-} f'_\delta(u_l + s) \right\} \\ \lim_{x \rightarrow y^+} \sigma_\delta(x, t) &= k_r \max \left\{ \lim_{s \rightarrow 0^+} f'_\delta(u_r + s), \lim_{s \rightarrow 0^-} f'_\delta(u_r + s) \right\} \end{aligned}$$

Since f' is continuous, we have that $|\sigma_\delta(y^+, t) - \sigma_\delta(y^-, t)| \leq O(\sqrt{\delta} + |k_l - k_r|)$. Note that σ_δ is defined so that if σ_δ is increasing on an interval between two discontinuities of v_δ , then these two discontinuities constitute part of an approximate rarefaction wave.

The following lemma justifies the term “approximate characteristic speed”:

Lemma 1.4. For x in any interval $\langle x_j, x_{j-1} \rangle$,

$$(1.26) \quad \lim_{\delta \rightarrow 0} \sigma_\delta(x, t) = k(x) f'(u)$$

in the sense of distributions.

Proof. For some time t , assume that the discontinuities in v_δ in the interval $[x_j, x_{j-1}]$ are located at y_i for $i = 1, \dots, N$. Let $\theta(x)$ be a test function with compact support in $\langle x_j, x_{j-1} \rangle$, we then compute

$$(1.27) \quad \begin{aligned} \int_{x_j}^{x_{j-1}} \sigma_\delta(x, t) \theta(x) dx &= \sum_i^N \int_{y_i}^{y_{i-1}} \sigma_\delta(x, t) \theta(x) dx \\ &= \sum_i^N \int_{y_i}^{y_{i-1}} k(x) f'(u_\delta) \theta(x) + O(\sqrt{\delta}) dx \\ &= \int_{x_j}^{x_{j-1}} k(x) f'(u_\delta) \theta(x) dx + O(\sqrt{\delta}). \end{aligned}$$

The lemma now follows by letting $\delta \rightarrow 0$ in (1.27). \square

The entropy inequality satisfied by σ_δ reads as follows:

Lemma 1.5. The following inequality holds weakly in x in each interval $\langle x_j, x_{j-1} \rangle$, and for all $t > 0$

$$(1.28) \quad \partial_x \sigma_\delta(x, t) \leq \frac{C_1}{t} + C_2 |k'| + O(\sqrt{\delta}),$$

where C_1 and C_2 are constants independent of t and δ .

Proof. Let $\theta(x)$ be a positive test function, and assume that at some time t , the discontinuities of v_δ are located at y_i . Then

$$(1.29) \quad \begin{aligned} \langle \partial_x \sigma_\delta, \theta(x) \rangle &= - \int \sigma_\delta(x, t) \theta'(x) dx \\ &= - \sum_i \int_{y_i}^{y_{i-1}} \sigma_\delta(x, t) \theta'(x) dx \\ &= \sum_i \theta(y_i) [\sigma_\delta(y_i^+) - \sigma_\delta(y_i^-)] + \sum_i \int_{y_i}^{y_{i-1}} \partial_x \sigma_\delta(x, t) \theta(x) dx. \end{aligned}$$

If a z wave is located at y_i then, $\sigma_\delta(y_i^+) \leq \sigma_\delta(y_i^-)$, so that the contribution from z waves in the first sum in (1.29) is nonpositive. Also, if a k wave is

located at y_i , and a z or a k wave is located at $y_{i \pm 1}$, then $\partial_x \sigma_\delta$ is nonpositive between y_i and $y_{i \pm 1}$. Hence

$$(1.30) \quad \langle \partial_x \sigma_\delta, \theta(x) \rangle \leq \sum_{k \text{ waves}} C_2 \theta(y_i) \Delta k + O(\sqrt{\delta}) - \sum_{z-z \text{ segments}} \int_{y_i}^{y_{i+1}} \partial_x \sigma_\delta(x, t) \theta(x) dx.$$

The remark immediately prior to lemma 1.4 shows that any segment over which $\partial_x \sigma_\delta$ is positive, is part of an approximate rarefaction wave, in the sense that the two z waves at its endpoints constitute part of an approximate rarefaction wave. Now we have the following result for approximate rarefaction waves:

Lemma 1.6. *Assume that z_1 and z_2 are two consecutive z waves in the approximate solution v_δ , and that on the segment between them $\partial_x \sigma_\delta(x, t) > 0$, then*

$$(1.31) \quad x(z_2) - x(z_1) \geq t k_{\min} |f''|_{\min} O(u(z_2^-) - u(z_1^-)),$$

where $u(z^\pm)$ denotes the u value of the right or left state of z , and $x(z)$ denotes the position of z , and k_{\min} denotes the minimum of $k(x)$.

Proof (of lemma 1.6). By lemma 1.3, an approximate rarefaction wave can only be centered at $t = 0$. Consider a segment of an approximate rarefaction wave. Over every interval of constant k_δ , the difference in speed of the two endpoints will be at least $k_{\min} |f''|_{\min} O(u(z_2^-) - u(z_1^-))$, which implies (1.31). This concludes the proof of lemma 1.6. \square

Now the difference of σ_δ over a segment of an approximate rarefaction wave is bounded by $k_{\max} |f''|_{\max} O(u(z_2^-) - u(z_1^-))$. Since f'' and k both are bounded and bounded away from 0, on every interval where $\partial_x \sigma_\delta > 0$ we have

$$(1.32) \quad \partial_x \sigma_\delta(x, t) \leq \frac{C_1}{t}.$$

Using this in (1.30) we obtain

$$(1.32) \quad \langle \partial_x \sigma_\delta, \theta(x) \rangle \leq \langle C_2 k', \theta \rangle + \left\langle \frac{C_1}{t}, \theta \right\rangle + O(\sqrt{\delta}),$$

which concludes the proof of lemma 1.5. \square

Combining lemma 1.4 and lemma 1.5, we have that the front tracking limit u satisfies the following entropy inequality weakly in each interval $\langle x_j, x_{j+1} \rangle$

$$(1.33) \quad \partial_x (k(x)f'(u)) \leq K \left(\frac{1}{t} + |k'| \right)$$

for some constant K .

Consider therefore two weak solutions u_1 and u_2 of (1.1) having the same initial data $u_0(x)$, such that $\text{Var}_{zk}\Psi(u_1)$ and $\text{Var}_{zk}\Psi(u_2)$ are bounded independently of t . The solutions u_1 and u_2 are assumed to satisfy (1.33). We then define the potentials $\varphi_i(x, t)$ as

$$(1.34) \quad \varphi_i(x, t) = \int_{(0,0)}^{(x,t)} u_i dx - (k(x)f(u_i)) dt.$$

for $i = 1, 2$. The well definedness of φ_i follows from (1.1), furthermore φ_i are bounded and uniformly Lipschitz continuous functions which satisfy

$$(1.31) \quad \varphi_{ix} = u_i, \quad \varphi_{it} = -k(x)f'(u_i).$$

Thus the difference between the potentials: $\varphi = \varphi_2 - \varphi_1$, satisfies the linear adjoint equation

$$(1.35) \quad \begin{aligned} \varphi_t + a(x, t)\varphi_x &= 0 \\ \varphi(x, 0) &= 0 \end{aligned}$$

for $0 < t$, where the coefficient a is given by

$$(1.36) \quad a(x, t) = \int_0^1 k(x)f'((1-s)u_1 + su_2) ds.$$

Note that a is bounded. Now the entropy condition (1.33) implies that for $x \in \langle x_j, x_{j+1} \rangle$

$$(1.37) \quad \begin{aligned} a_x &= \int_0^1 (k(x)f'((1-s)u_1 + su_2))_x ds \\ &= \int_0^1 (k(x)((1-\xi(s))f'(u_1) + \xi(s)f'(u_2)))_x ds \\ &= \int_0^1 (1-\xi(s))(k(x)f'(u_1))_x + \xi(s)(k(x)f'(u_2))_x ds \end{aligned}$$

$$(1.38) \quad \leq K \left(\frac{1}{t} + |k'| \right).$$

where $\xi(s)$ is some strictly decreasing function taking values in $[0, 1]$, (1.37) holds since f is a strictly concave function.

We temporarily fix j and define the “stretched” coordinate y_ϵ as (1.39)

$$y_\epsilon(x) = \frac{1}{\bar{d} + \epsilon} (\bar{d}x + \epsilon\bar{x}), \quad \text{where } \bar{d} = \frac{1}{2}(x_{j+1} - x_j), \quad \bar{x} = \frac{1}{2}(x_{j+1} + x_j).$$

Note that $y_\epsilon(x_j - \epsilon) = x_j$ and that $y_\epsilon(x_{j+1} + \epsilon) = x_{j+1}$. For $x \in \langle x_j, x_{j+1} \rangle$ we define the smoothed coefficient a^ϵ as

$$(1.40) \quad a^\epsilon(x, t) = a(y_\epsilon(\cdot), t) * \omega^\epsilon(x),$$

where ω^ϵ is a standard mollifier in the x variable with support in $[-\epsilon, \epsilon]$. The smoothed coefficient satisfies the following entropy inequality:

Lemma 1.7.

$$(1.41) \quad \partial_x a^\epsilon(x, t) \leq K \left(\frac{1}{t} + |k'| \right) \frac{\bar{d} + \epsilon}{\bar{d}}$$

weakly for $x \in \langle x_j, x_{j+1} \rangle$.

Proof. Since $x \mapsto y_\epsilon(x)$ is a linear change of variable, the following inequality holds weakly for $x \in \langle x_j, x_{j+1} \rangle$

$$(1.42) \quad \partial_x a(y(x)) = \partial_y a \frac{dy}{dx} \leq K \left(\frac{1}{t} + |k'| \right) \frac{\bar{d} + \epsilon}{\bar{d}}.$$

Let $\theta(x)$ be a nonnegative test function with support in $\langle x_j, x_{j+1} \rangle$, then

$$\begin{aligned} \langle \partial_x a^\epsilon, \theta \rangle &= - \langle a^\epsilon, \theta' \rangle = - \int_{x_j}^{x_{j+1}} a^\epsilon(x) \theta'(x) dx \\ &= - \int_{x_j}^{x_{j+1}} \int_{x-\epsilon}^{x-\epsilon} a(y_\epsilon(z)) \omega^\epsilon(x-z) \theta'(x) dz dx \\ &= - \int_{x_j-\epsilon}^{x_{j+1}+\epsilon} a(y_\epsilon(z)) \int_{x_j}^{x_{j+1}} \theta'(x) \omega^\epsilon(x-z) dx dz \\ &= - \int_{x_j-\epsilon}^{x_{j+1}+\epsilon} a(y_\epsilon(z)) \left\{ \theta \omega^\epsilon \Big|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \theta(x) \omega^{\epsilon'}(x-z) dx \right\} dz \\ &= \int_{x_j}^{x_{j+1}} \left\{ - \int_{x_j-\epsilon}^{x_{j+1}+\epsilon} a(y_\epsilon(z)) \frac{d}{dz} \omega^\epsilon(x-z) dz \right\} \theta(x) dx \\ &= \int_{x_j}^{x_{j+1}} \langle \partial_z a(y(z)), \omega_{-x}^\epsilon(z) \rangle \theta(x) dx \\ &\leq \int_{x_j}^{x_{j+1}} K \left(\frac{1}{t} + |k'| \right) \frac{\bar{d} + \epsilon}{\bar{d}} \theta(x) dx = \left\langle K \left(\frac{1}{t} + |k'| \right) \frac{\bar{d} + \epsilon}{\bar{d}}, \theta \right\rangle. \end{aligned}$$

This concludes the proof of lemma 1.7. \square

Since a^ϵ is differentiable in $\langle x_j, x_{j+1} \rangle$ this inequality holds strongly. Rewriting (1.35) using the smoothed coefficient a^ϵ we get

$$(1.43) \quad \varphi_t - a^\epsilon \varphi_x = (a^\epsilon - a) \varphi_x = g^\epsilon \varphi_x.$$

If u_1 and u_2 are periodic with the same period, we let $R = [0, P]$, where P is the common period of u_1 and u_2 . If we do not have periodic solutions, we let $R = [-N - Mt, N + Mt]$, where N and M are constants which are chosen so large that $u_1 = u_2$ on ∂R . This is always possible, since (1.1) has finite speed of propagation, and k is continuous with bounded derivative outside a bounded interval. Thus the solution of (1.1) outside R does not depend on k in the interior of R , and therefore $u_1 = u_2$ for almost all x and t outside R .

We multiply (1.43) with $p\varphi^{p-1}$, where p is some even positive integer, and integrate over R to obtain

$$(1.44) \quad \frac{d}{dt} \int_R \varphi^p dx + \int_R a^\epsilon \partial_x \varphi^p dx = \int_R g^\epsilon \partial_x \varphi^p dx.$$

Since a^ϵ is differentiable, we may integrate by parts in each interval $\langle x_j, x_{j+1} \rangle$

$$(1.45) \quad \frac{d}{dt} \int_R \varphi^p dx = \sum_j \int_{x_j}^{x_{j+1}} \partial_x a^\epsilon \varphi^p dx - \sum_j \varphi^p(x_j) A_j + \int_R g^\epsilon \partial_x \varphi^p dx.$$

The constants A_j are given by the jumps in a^ϵ over the discontinuities in k at x_j , and are therefore uniformly bounded since a is uniformly bounded. Since φ^p is Lipschitz continuous, there is a nonnegative constant C such that

$$(1.46) \quad \varphi^p(y) \leq C \int_R \varphi^p(x) dx$$

for any y . Thus,

$$(1.47) \quad \frac{d}{dt} \int_R \varphi^p dx \leq \sum_j \int_{x_j}^{x_{j+1}} K \left(\frac{1}{t} + |k'| \right) \frac{\bar{d}_j + \epsilon}{\bar{d}_j} \varphi^p dx + \tilde{C} \int_R \varphi^p(x) dx + \int_R g^\epsilon \partial_x \varphi^p dx.$$

Since $a^\epsilon \rightarrow a$ in L_p^{loc} as $\epsilon \rightarrow 0$, we have that $\int_R g^\epsilon \partial_x \varphi^p dx \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence, letting $\epsilon \rightarrow 0$, we obtain

$$(1.48) \quad Y'(t) \leq E \left(\frac{1}{t} + 1 \right) Y(t)$$

for some nonnegative constant E , and where $Y(t)$ denotes the p th power of the L_p norm of φ , i.e., $Y(t) = \int_R \varphi^p dx$. Gronwall's inequality now gives

$$(1.49) \quad (Y(t)t^{-E}e^{-Et})' \leq 0.$$

or

$$(1.50) \quad Y(t) \leq t^E e^{Et} e^{-Es} s^{-E} Y(s),$$

for $0 < s < t$. But as $s \rightarrow 0^-$, the Lipschitz continuity of φ implies that $Y(s) = O(s^p)$. Therefore

$$(1.51) \quad Y(t) \leq \tilde{K} t^E e^{Et} e^{-Es} s^{p-E},$$

for some constant \tilde{K} , and by choosing p sufficiently large we get that $Y(t) = 0$, i.e., $u_1 = u_2$ for almost all x and t .

Using the methods and estimates from this section, with some slight generalizations where appropriate, it is now straightforward to prove:

Theorem 1.2. *Assume that $k(x)$ is a bounded, piecewise continuous function with a finite number of discontinuities, such that $k'(x)$ is uniformly bounded at all points of continuity, and $k(x)$ is either strictly positive or negative. Let $f(u)$ be a twice differentiable function such that f' and f'' both are bounded, and $f''(u) \neq 0$. Furthermore assume that there is an interval $[a, b]$ such that $u_0(x) \in [a, b]$ and $f(a) = f(b)$.*

If $u_0(x)$ is such that $\Psi(u_0) \in BV$, then there exists one, and only one, weak solution $u(x, t)$, to the following initial value problem

$$(1.52) \quad \begin{aligned} u_t + (k(x)f(u))_x &= 0 \\ u(x, 0) &= u_0(x). \end{aligned}$$

which also satisfies the entropy condition

$$(1.53) \quad \partial_x (k(x)f'(u)) \leq K \left(\frac{1}{t} + |k'| \right)$$

for some constant K , weakly in all intervals where k is continuous.

2. Asymptotic behavior. In this section we study the behavior of the initial value problem for large times. Lemma 1.1a implies that the number of discontinuities in the front tracking approximants v_δ is constant after some time t'_δ . Also, since the "Glimm functional" F is nonnegative, and at each collision, F is either constant or changes by at least δ , after some time t''_δ , $F(v_\delta)$ is constant. Let $t_\delta = \max\{t'_\delta, t''_\delta\}$. We refer to a collision of discontinuities in v_δ as a zk collision if a z wave is colliding with a k wave from the left, similarly as a kz collision if the z wave collides from the right. The third part of lemma 1.1 describes the behavior of v_δ after t_δ :

Lemma 1.1c. *If $t > t_\delta$, then a zk collision will give a kz solution, and a kz collision will give a zk solution.*

If we have nonperiodic initial data, then from lemma 1.1c it follows that after some finite time the I curve traced out by v_δ will be of the form

$$z_1^l z_2^l \dots z_n^l k_1 z_1^0 \dots k_\ell z_1^r \dots z_m^r.$$

since eventually, all z waves with negative (z_i^l) or positive (z_i^r) speed will have “moved through” all k waves. The waves z_i^0 have zero speed. Since the number of z waves is constant, the I curve traced out by v_δ does not change further, so the asymptotic state is actually attained after some finite number of interactions. It is interesting to note that in contrast to the case with a strictly hyperbolic and genuinely nonlinear system of equations, the I curve given by this asymptotic state is *not* necessarily the solution of the Riemann problem defined by the states at the endpoints of the curve.

We will not pursue the case of nonperiodic initial data further, but instead derive explicit formulas for the asymptotic solution in the case where both the coefficient $k(x)$ and $u_0(x)$ are periodic with the same period.

In this case we also make the approximate initial data $v_{\delta 0}$ periodic, therefore v_δ will form a closed I curve. Lemma 1.1c now implies that after some time, either all z waves will have zero speed, positive speed, or negative speed, since no zz collisions occur after t_δ .

Assume first that for all t larger than some time t_- , all z waves of v_δ have positive speed. Since all z waves have positive speed, there must be z waves over which the flux function $k_\delta f_\delta$ changes. Since I is a closed curve, there must be at least two of them. If there are only two, then the change in z over each wave is $\pm\delta$, and they will have the same speed on each interval where $k_\delta(x)$ is constant. Thus there will be no collisions. In general, there will be no collisions if the maximum value of the flux function ($k f(u)$) minus its minimum is less than δ . We call such a sequence of z waves, each of which carries a z difference of δ , with the sign of the difference alternating, for *ripples*. These waves are moving from left to right, but never colliding, since they will have the same speed on each interval of constant k_δ . If the maximum value of the flux function ($k f(u)$) minus its minimum is larger than δ , since I is a closed curve, there will be some pair of neighboring waves which have the property that the left member moves faster than the right through *every* interval of constant k_δ . This pair will of course eventually collide. Since we have no z collisions after t_+ , the largest difference in flux values is δ .

A similar discussion shows that this is also true if for all t larger than some t_- , we only have waves of negative speed. In particular, this means that after some time *either*, (a), the solution consists of ripples moving to

the left or right. In this case the I curve determined by v_δ does not cross the k axis. Or, (b), the solution is stationary, and the flux $kf(u)$ is everywhere constant.

From now on we assume that $k(x)$ has only one minimum in each period, that is: In each period there is a closed interval $[x_1, x_2]$, possibly consisting of one point, such that $k(x) = k_{\min}$ for $x \in [x_1, x_2]$, and $k(x) > k_{\min}$ for $x \notin [x_1, x_2]$. The reason for this simplifying assumption is that the asymptotic I curve can only cross the k axis where $k(x)$ has a minimum, therefore it follows that this can happen at most once.

Consequently, after some finite time T_δ , the approximate solutions v_δ consist either of

- (a) k waves and small z waves of size at most δ .
- or
- (b) k waves and at most one stationary z wave.

In case (a), v_δ will form an I curve with only nonnegative or nonpositive z values, and in case (b), v_δ will form a triangle-like curve which is traversed in the clockwise direction. In particular, I can only cross the k axis at global minimum values of k , so since k has a minimum which is attained once in each period, the k axis can only be crossed once per period. See the illustration below.

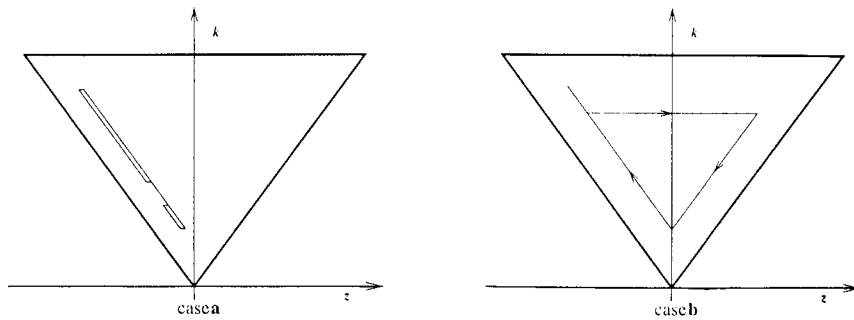


Figure 2. Asymptotic I curves.

If $\{s_\delta\}$ is some sequence such that $s_\delta > T_\delta$, and if u is the weak solution of (1.1) with periodic initial data and periodic k , we have

$$(2.1) \quad \lim_{\delta \rightarrow 0} \|u_\delta(\cdot, s_\delta) - u(\cdot, s_\delta)\|_1 = 0.$$

and, since convergence in L_1 implies pointwise convergence almost everywhere, $u_\infty(x) = \lim_{t \rightarrow \infty} u(x, t)$ will either be:

Theorem 2.1.

(a): A function which is either smaller or equal to $1/2$, or larger or equal to $1/2$. If k is continuous at x , then so is $u_\infty(x)$.

or,

(b): an “N-wave”, which has one stationary shock at some point x_s between two succeeding minima of $k(x)$. Furthermore, for each such stationary shock, $u_\infty^-(x) < 1/2 < u_\infty^+(x)$, where by $u_\infty^-(x)$ (u_∞^+) we denote $u_\infty(x)$ for x between the minimum of k and x_s (x between x_s and the minimum). The functions $u_\infty^-(x)$ and $u_\infty^+(x)$ are continuous where $k(x)$ is continuous.

We can actually compute u_∞ directly from u_0 as follows. From figure 2 we see that u_∞ will be of the form

$$(2.2) \quad z(x) = \mp (k(x) - k)$$

where k is some constant $k \leq k_{\min}$, and the negative sign is chosen for the part to the left of the k axis, and the positive sign for the part on the right. If we are in case **b** there will of course be a horizontal part of the I curve representing the shock from negative z values to positive. We choose x so that the $k(x)$ achieves its minimum for $x = 0$. Assuming $f(u) = u(1 - u)$, and writing (2.2) in u coordinates gives

$$(2.3) \quad u(x, k, x_s) = \begin{cases} u_\infty^-(x, k) & \text{for } x < x_s \\ u_\infty^+(x, k) & \text{for } x \geq x_s \end{cases} = \begin{cases} \frac{1}{2} \left(1 - \sqrt{1 - \frac{k}{k(x)}} \right) & \text{for } x < x_s \\ \frac{1}{2} \left(1 + \sqrt{1 - \frac{k}{k(x)}} \right) & \text{for } x \geq x_s. \end{cases}$$

The mean value of u is conserved, so

$$(2.4) \quad m = \frac{1}{b-a} \int_a^b u_0(x) dx = \frac{1}{b-a} \int_a^b u_\infty(x) dx$$

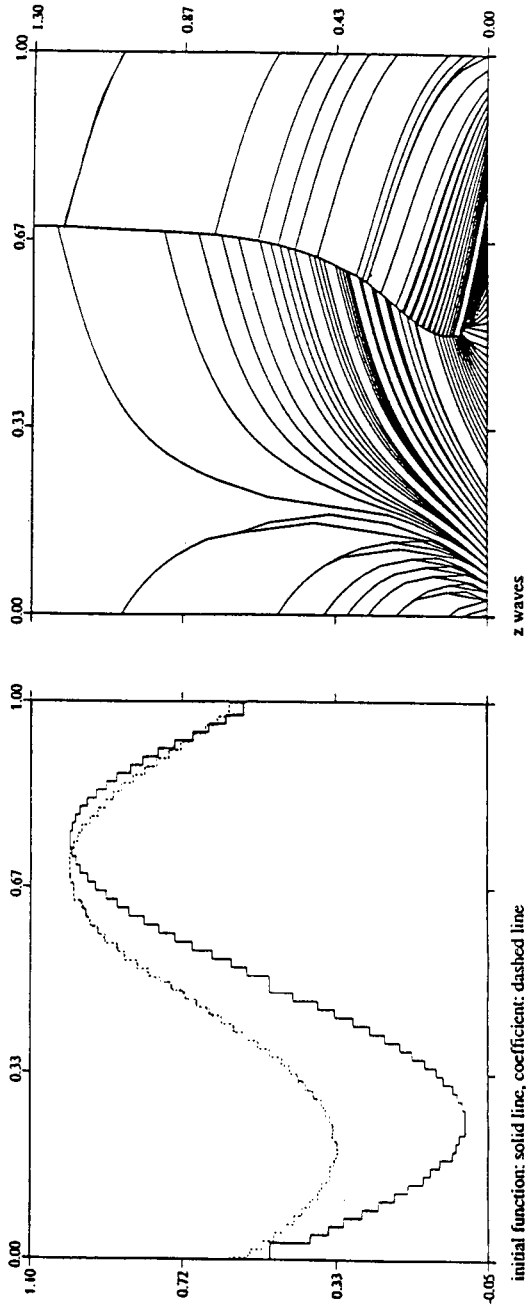
For simplicity setting $a = 0$, $b = 1$, we have the same two cases as before.

(a):

$$(2.5) \quad m < \int_0^1 u_\infty^-(x, k_{\min}) dx.$$

Since we have that

$$(2.6) \quad \frac{\partial \int_0^1 u_\infty^-(x, k) dx}{\partial k} > 0$$



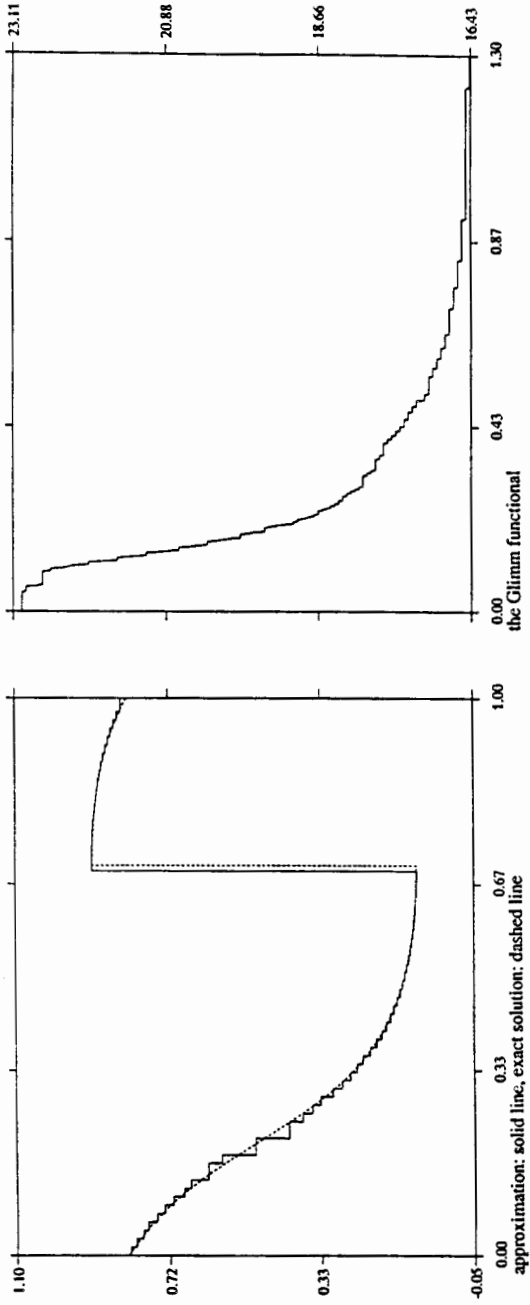


Figure 3

for $k < k_{\min}$, and $\int_0^1 u_{\infty}^{-}(x, 0) dx = 0$, there is a unique $0 \leq \tilde{k} < k_{\min}$ such that

$$(2.7) \quad \int_0^1 u_{\infty}^{-}(x, \tilde{k}) dx = m.$$

Similarly if $m > \int_0^1 u_{\infty}^{+}(x, k_{\min}) dx$ we can find a unique constant k , $0 \leq \tilde{k} < k_{\min}$, such that

$$(2.8) \quad \int_0^1 u_{\infty}^{+}(x, \tilde{k}) dx = m.$$

In this case $u_{\infty}(x) = u_{\infty}^{\pm}(x, \tilde{k})$.

(b):

$$(2.9) \quad \int_0^1 u_{\infty}^{-}(x, k_{\min}) dx \leq m \leq \int_0^1 u_{\infty}^{+}(x, k_{\min}) dx.$$

We have that

$$(2.10) \quad \frac{\partial \int_0^1 u(x, k_{\min}, x_s) dx}{\partial x_s} < 0,$$

comparing this with (2.9), we see that this means that there is a unique x_s , $0 \leq x_s \leq 1$ such that

$$(2.11) \quad \int_0^1 u(x, k_{\min}, x_s) dx = m.$$

In this case $u_{\infty}(x) = u(x, k_{\min}, x_s)$.

3. A numerical example. In this section we present an example where the front tracking construction is used to compute an approximate solution, and we shall see that the stationary solution is obtained after a finite number of interactions (in this case 4982).

As before, let $f(u) = u(1 - u)$. The initial function $u_0(x)$ and the coefficient $k(x)$ are given by

$$(3.1) \quad u_0(x) = \frac{1}{2} (1 - \sin(2\pi x))$$

$$(3.2) \quad k(x) = 1 + 2 \cos^2(\pi(x + 0.3)).$$

The period here is 1, and we have computed the front tracking solution for x in the interval $[0, 1]$, and for $t \leq 1.3$, when we see that the stationary

asymptotic solution is reached. The parameter $\delta = 0.03$. In figure 3 we show the step function approximation to the initial function u_0 and the coefficient $k(x)/3$ in the upper left corner. In the upper right corner we see all z waves in the (x, t) plane. In the lower left corner we see the approximate asymptotic limit $u_{\delta\infty}$ and the true asymptotic solution calculated from (2.11) and (2.3). Finally in the lower right corner we show the Glimm functional $F(I(t))$ plotted against time.

Appendix. In this appendix we prove lemma 1.1, that is

Lemma 1.1.

- a *The number of discontinuities in v_δ is nonincreasing for each collision of discontinuities. Furthermore, the number of discontinuities decreases by at least one if two z waves collide, and is constant if a z wave collides with a k wave.*
- b *There are no approximate centered rarefaction waves in u_δ for $t > 0$.*
- c *If $t > t_\delta$, then a zk collision will give a kz solution, and a kz collision will give a zk solution.*

Proof. The proof is a study of cases. Recall that an approximate centered rarefaction wave denotes a z wave in the solution of a Riemann problem that has magnitude larger than δ , and whose left state z_l is larger than its right state z_r . Any approximate centered rarefaction wave must arise at a collision between two z waves or a z wave and a k wave. Since f is concave, a collision of two z waves will result in one single z wave, and no centered rarefaction waves can arise at such a collision. This shows part (a) and (b) for zz collisions.

It remains to study the collision of a z wave with a k wave, here we have a number of cases:

- A:** The z wave collides with the k discontinuity from the left.
 - Aa:** $k_L > k_R$.
 - Ab:** $k_L < k_R$.
- B:** The z wave collides with the k discontinuity from the right.
 - Ba:** $k_L > k_R$.
 - Bb:** $k_L < k_R$.

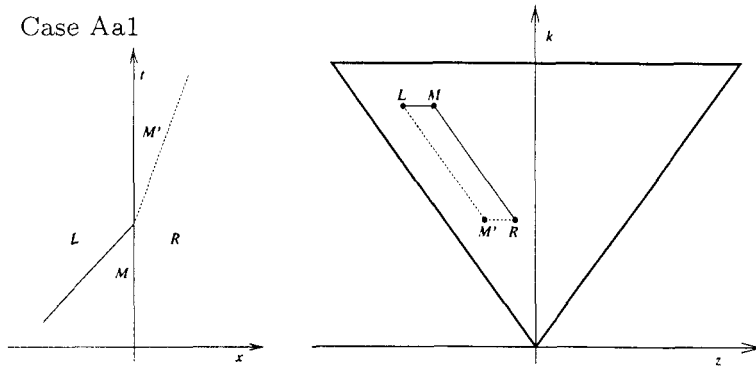
Furthermore, each subcase, **Aa** etc., is divided into two cases depending on the sign of the z value of the middle state.

Case **Aa**. We label the states involved in the collision L , M and R respectively. Thus immediately before the collision we have a discontinuity with positive speed separating states v_L and v_M , and a k wave separating states v_M and v_R .

First we consider the case **Aa1**, where $z_M < 0$. This implies that $z_R \leq 0$. Since the wave separating v_L and v_M has positive speed, $z_L \leq z_M + \delta$. The

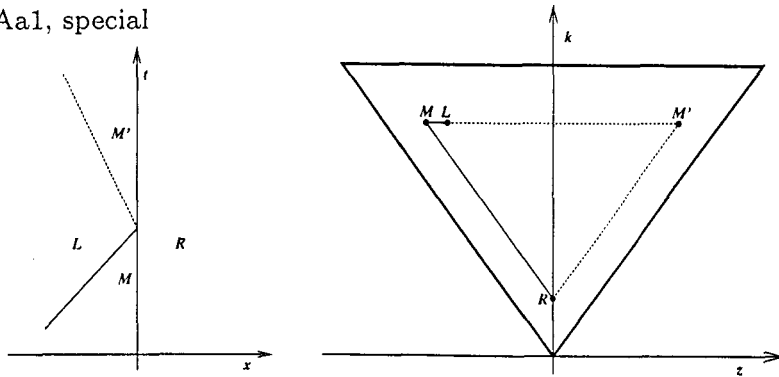
result of such a collision is either a transmitted wave, or if $z_R = 0$, a reflected wave. In this case $\Delta F = 0$. See the figure below.

Case Aa1

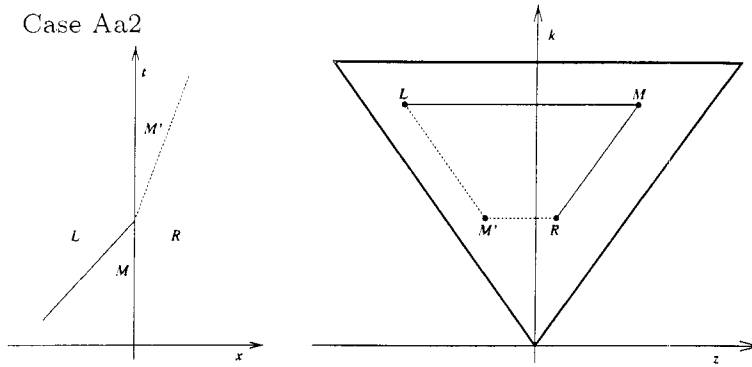


This case has the following special subcase, if $z_r = 0$ and $z_l = z_m + \delta$, then the result of the collision will be a reflected z wave, and F will decrease by δ , see the illustration below

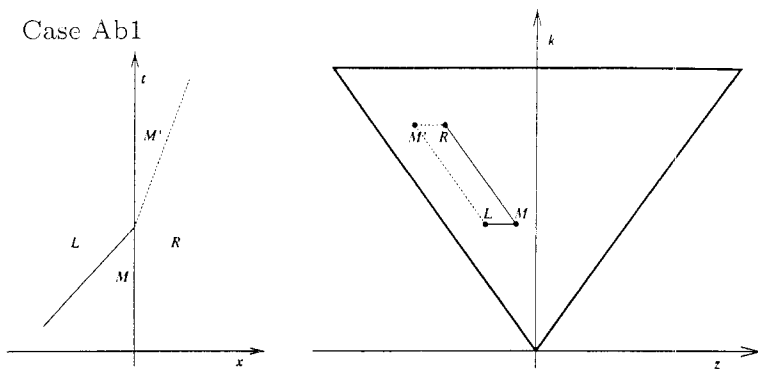
Case Aa1, special



Aa2. now $z_M > 0$ which implies $z_R \geq 0$. Since the colliding z wave has positive speed z_L must be such that $z_L \leq -z_M - \delta$, and the solution of the Riemann problem given at the collision is a transmitted shock, and $\Delta F = 0$. See the figure on the next page.

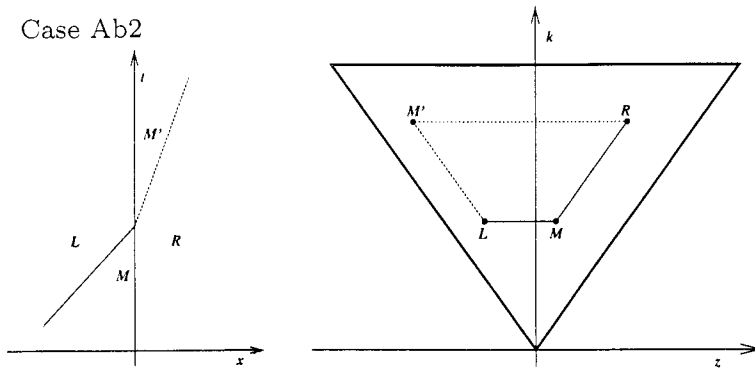


Ab1. now $z_R < 0$ which implies $z_M \leq 0$. In this case $z_L \leq \min[z_M + \delta, 0]$, and the result of the collision will be a transmitted shock, and $\Delta F = 0$. See the figure below.



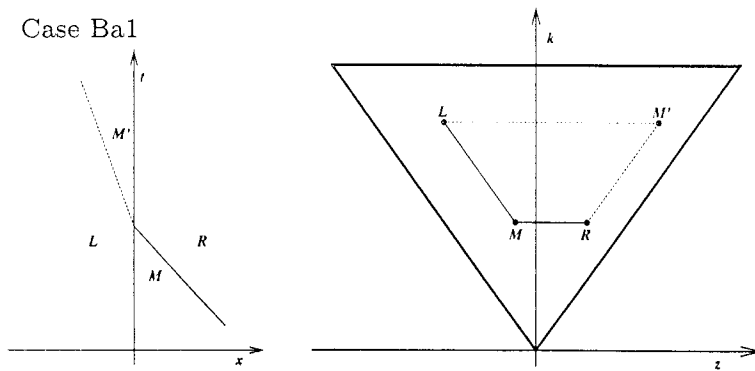
Ab2. now $z_R > 0$ which implies $z_M \geq 0$. In this case $z_L \leq -z_M - \delta$, and the result of the collision will be a transmitted shock, albeit of a larger magnitude than the incoming one, also here $\Delta F = 0$. See the figure on the next page.

Case Ab2



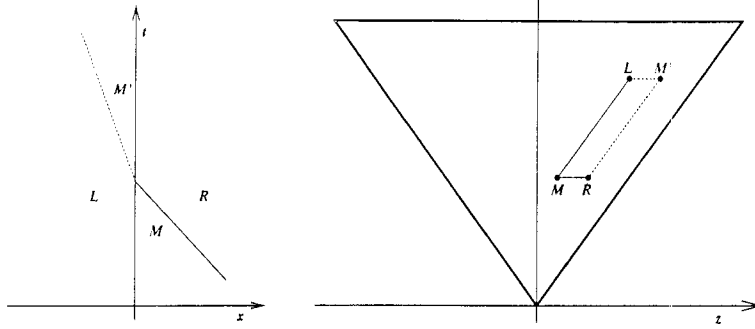
Ba1. now the z wave is colliding from the right. We have that $z_L < 0$ and $k_L > k_R$, this implies that $z_M \leq 0$. Since the speed of the z wave is negative, $z_R \geq -z_M + \delta$. The result of this collision is again a transmitted shock, and $\Delta F = 0$. See the figure below.

Case Ba1



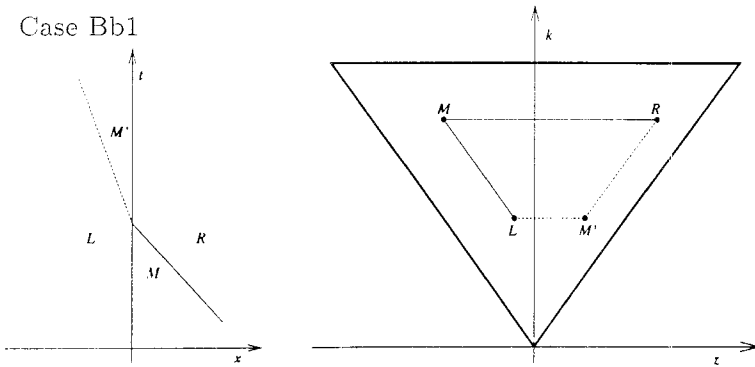
Ba2. now $z_L > 0$ which implies $z_M \geq 0$. Now $z_R \geq \max[z_M - \delta, 0]$, and the result of the collision is a transmitted shock, and $\Delta F = 0$. See the figure on the next page.

Case Ba2

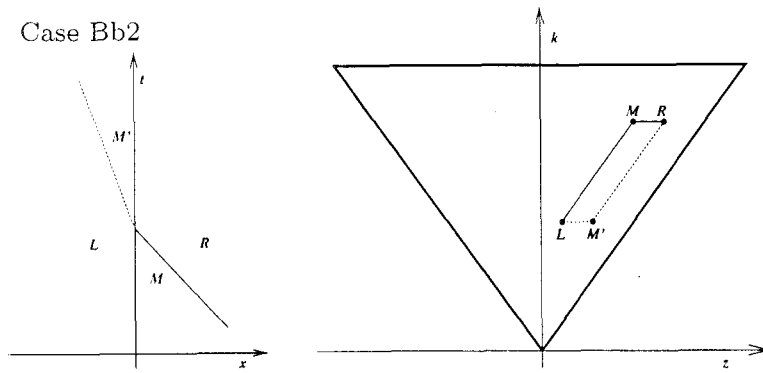


Bb1. now $k_L < k_R$ and $z_M < 0$ which implies $z_L \leq 0$. We must have $z_R \geq -z_M - \delta$, and the result of the collision is a transmitted shock, and $\Delta F = 0$. See the figure below.

Case Bb1



Bb2. now $z_M > 0$ which implies $z_L \geq 0$. In this case, $z_R \geq \max[z_M - \delta, 0]$, and the result is a transmitted shock, and $\Delta F = 0$. See the figure on the next page.



In this case there is a special subcase where $z_l = 0$ and $z_r = z_m - \delta$. This is similar to the special case in **Aa1**, and the result here is also a reflected z wave. In this case $\Delta F = \delta$.

This exhausts the types of zk or kz collisions which can occur, and we have seen that in no case does an approximate centered rarefaction wave arise. Also in each case the collision resulted in two discontinuities, and F remained constant precisely in those cases where the z wave "passed through" the k wave. \square

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