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A MIXED TYPE SYSTEM OF THREE EQUATIONS MODELLING REACTING FLOWS

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ABSTRACT. In this paper we contrast two approaches for proving the validity of relaxation limits $\alpha \to \infty$ of systems of balance laws

$$u_t + f(u)_x = \alpha g(u) \quad .$$

In one approach this is proven under some suitable stability condition; in the other approach, one adds artificial viscosity to the system

$$u_t + f(u)_x = \alpha g(u) + \epsilon u_{xx}$$

and lets $\alpha \to \infty$ and $\epsilon \to 0$ together with $M\alpha \leq \epsilon$ for a suitable large constant M. We illustrate the usefulness of this latter approach by proving the convergence of a relaxation limit for a system of mixed type, where a subcharacteristic condition is not available.

1. Viscosity joining forces with relaxation

In this paper we are interested in combining the zero relaxation with the zero dissipation limit of the Cauchy problem for models of reaction flow:

$$\left.\begin{aligned} v_t - u_x &= \epsilon v_{xx} \\ u_t - \sigma(v,s)_x &= \epsilon u_{xx} \\ s_t + c_1 s_x + \frac{s - h(v)}{\tau} &= \epsilon s_{xx} \end{aligned}\right\} \tag{1.1}$$

with initial data

$$\big(v,u,s\big)_{|t=0} = \big(v_0,u_0,s_0\big) \quad . \tag{1.2}$$

The third equation in (1.1) contains a relaxation mechanism with $h(v)$ as the equilibrium value for s, τ the relaxation time, ϵ the viscous parameter and c_1 a constant.

The problem (1.1) can be considered as a singular perturbation problem as τ tends to zero. When $\tau \leq \frac{\epsilon}{M}$, with M a suitable large constant which depends only on the initial data, and ϵ goes to zero, the convergance of the solutions $\big(v^{\epsilon,\delta}, u^{\epsilon,\delta}, s^{\epsilon,\delta}\big)$ for the Cauchy problem (1.1), (1.2) was proven in [Lu1], [Lu2] for two particular cases. In [Lu1] we consider $h(v) = v$, $\sigma(v,s) = \sigma(v) + s$, and in [Lu2] the case $\sigma(v,s) = c_1 v + c_2 s$ and $h(v)$ is a nondecreasing function. The relaxation

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and dissipation limit of $\big(v, u, s\big)$ in (1.1) satisfies $s = h(v)$ and (v, u) is an entropy solution of the equilibrium system

$$\left.\begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, h(v))_x &= 0 \end{aligned}\right\} \tag{1.3}$$

When $\epsilon = 0$ and $\tau \to 0$ the relaxation limit for the Cauchy problem (1.1), (1.2) was obtained in [T] for the special case $\sigma(v, s) = c_1 v + c_2 s$, $0 < h'(v) \leq c$. The equilibrium solution (v, u) is bounded in L^2 as is proven in [Lu2]. An important early contribution of conservation laws with relaxation [Liu] introduced the subcharacteristic condition which is needed for the stability of the relaxation limit for many physical models. This condition can be illustrated with a simple example:

$$\left.\begin{aligned} v_t - u_x &= 0 \\ u_t - cv_x + \frac{u - h(v)}{\tau} &= 0 \end{aligned}\right\} \tag{1.4}$$

The two eigenvalues for the system

$$\left.\begin{aligned} v_t - u_x &= 0 \\ u_t - cv_x &= 0 \end{aligned}\right\} \tag{1.5}$$

are $\lambda_1 = -\sqrt{c}$, $\lambda_2 = \sqrt{c}$ and the eigenvalue of the equilibrium equation of (1.4) is $\lambda^* = -h'(v)$. Then the subcharacteristic condition ($c > 0$)

$$\lambda_1 < \lambda^* < \lambda_2 \tag{1.6}$$

will ensure the stability of solutions (v^τ, u^τ) of (1.4) as τ goes to zero. However when $c < 0$, the condition (1.6) fails to hold. In fact (1.4) is ill posed for $c < 0$ as we proceed to show. By the first equation in (1.4) there must exist a function w such that

$$w_x = v, \quad w_t = u \quad .$$

Thus the second equation in (1.4) can be put in the form

$$w_{tt} - cw_{xx} + \frac{1}{\tau}(w_t - h(w_x)) = 0 \quad . \tag{1.7}$$

This is an elliptic equation. Its Dirichlet problem in the domain $t > 0$ can be solved with the data

$$w(x, 0) = \int_0^x v_0(s) ds \quad ,$$

but then $u_0(x)$ cannot be chosen independently of $v_0(x)$ since in this case we must have

$$v_0(x) = w_t(0, x)$$

and w depends on $v_0(x)$. Nevertheless, if we add a viscosity to the right-hand side of (1.4)

$$\left.\begin{aligned} v_t - u_x &= \epsilon v_{xx} \\ u_t - cv_x + \frac{u - h(v)}{\tau} &= \epsilon u_{xx} \quad , \end{aligned}\right\} \tag{1.8}$$

the system (1.8) is well posed. Applying the approach given in [L3] to this simple situation one can show the compactness of the viscous-relaxation approximation $\big(v^{\epsilon,\tau}, u^{\epsilon,\tau}\big)$ when $\tau \leq \dfrac{\epsilon}{M}$, $\epsilon \to 0$, for a suitable large constant M.

This can be seen through an asymptotic expansion of the Chapman-Enskog type. Let

$$u = h(v) + \tau u_1 + O(\tau^2) \quad . \tag{1.9}$$

Then from the second equation in (1.8) we obtain

$$\begin{aligned} u_1 &= \epsilon u_{xx} - u_t + c v_x + O(\tau) \\ &= \epsilon u_{xx} - h'(v) v_t + c v_x + O(\tau) \\ &= \epsilon u_{xx} - \epsilon h'(v) v_{xx} - h'(v) u_x + c v_x + O(\tau) \\ &= \epsilon u_{xx} - \epsilon h'(v) v_{xx} - {h'}^2(v) v_x + c v_x + O(\tau) \quad . \end{aligned} \tag{1.10}$$

Substituting (1.10) into the first equation in (1.8), we get

$$\begin{aligned} v_t - h(v)_x &= \epsilon v_{xx} + \tau u_{1x} + O(\tau^2) \\ &= \Big(\big(\epsilon + \tau(c - {h'}^2(v))\big) v_x \Big)_x + O(\tau^2) + O(\tau\epsilon) \quad . \end{aligned} \tag{1.11}$$

Thus if $\epsilon > \tau\big({h'}^2(v) - c\big)$, the equation (1.11) is well posed. This is also the effect of the subcharacteristic condition in (1.11) with $\epsilon = 0$.

This illustrates that viscosity is not only of mathematical expedience when acting together with relaxation but may also be a necessary stability mechanism.

Returning back to the 3×3 system (1.1), setting $\epsilon = 0$, the three eigenvalues are

$$\lambda_1 = -\sqrt{\sigma_v(v,s)}, \quad \lambda_2 = c_1, \quad \lambda_3 = \sqrt{\sigma_v(v,s)} \quad . \tag{1.12}$$

If $\sigma_v(v,s) \geq 0$, then (1.1) is a hyperbolic system. For the particular cases of (1.1) considered in [Lu1] [Lu2], it is shown that there is a strictly convex entropy which gives the necessary L^2 estimates and also compactness using compensated compactness. However for the case $\sigma_v(v,s) < 0$ we do not have an entropy in the strict sense. In this paper, our main interest is to deal with this elliptic-hyperbolic case under the weaker restriction $\frac{d\sigma}{dv}\big(v, h(v)\big) > 0$. In order to avoid technical details throughout this paper we assume $h(v) = cv, c$ is a constant. In fact all the steps in this paper work just as well for a more general $h(v)$ which satisfies $h'(v) \geq c > 0$. This can be seen by writing the third term in (1.1) equivalently as

$$\frac{s - h(v)}{\tau} = \frac{h'(\alpha)\big(h^{-1}(s) - v\big)}{\tau}$$

where h^{-1} is the inverse function of h and α takes on a value between $h^{-1}(s)$ and v.

2. Viscous relaxation limit

Theorem 2.1. • **part 1:**

If the initial data $\big(v_0, u_0, s_0\big)$ *are smooth functions satisfying the condition*

(c_1) $\big|v_0, u_0, s_0\big|_{L^2 \cap L^\infty(R)} \leq M_1$

$\lim_{|x| \to \pm\infty} \Big(\frac{d^i v_0}{dx^i}, \frac{d^i u_0}{dx^i}, \frac{d^i s_0}{dx^i}\Big) = (0,0,0), \quad i = 0, 1,$

and $h(v) = cv$, $\sigma(v,s)$ *satisfies the condition*

(c_2) $\big|\sigma_s(v,s)\big| \leq M_2$, $\bar\sigma'(v) \geq d > max\{0, c^2 - c + \frac{2c^2c_1^2}{(M_2+1)^2}\}$, *where* $\bar\sigma(v) = \sigma(v, cv)$,

then *for fixed* ϵ, τ *satisfying* $\tau(M_2+1)^2 \le \epsilon$, *the solutions* $(u,v,s) \in C^2$ *of the Cauchy problem* (1.1), (1.2) *exist in* $(-\infty,\infty)\times[0,T]$ *for any given* $T>0$ *and satisfy*

$$|v(x,t)|, \quad |u(x,t), \quad |s(x,t)| \quad \le M(\epsilon,\tau,T), \tag{2.1}$$

$$|u^2(.,t)|_{L^1(R)}, \quad |v^2(.,t)|_{L^1(R)}, \quad |s^2(.,t)|_{L^1(R)} \quad \le M, \tag{2.2}$$

$$|(s-cv)^2|_{L^1(R\times R^+)} \le \tau M, \tag{2.3}$$

$$|\epsilon u_x^2|_{L^1(R\times R^+)}, |\epsilon v_x^2|_{L^1(R\times R^+)}, |\epsilon s_x^2|_{L^1(R\times R^+)} \le M.$$

- **part 2:**
  If $\bar\sigma(v) = \sigma(v,cv)$ *satisfies the condition*
  (c_3) $\bar\sigma(v_0)=0$ *and* $\bar\sigma''(v)\ne 0$ *for* $v\ne v_0$, $\bar\sigma''$, $\bar\sigma''' \in L^2\cap L^\infty$,
  then *there exists a subsequence* $\big(v^{\epsilon,\tau},u^{\epsilon,\tau},s^{\epsilon,\tau}\big)$ *of the solutions to the Cauchy problem* (1.1), (1.2) *and there exists* L^2 *bounded functions* $\big(\bar v,\bar u,\bar s\big)$ *such that*

$$\big(v^{\epsilon,\tau},u^{\epsilon,\tau},s^{\epsilon,\tau}\big) \to \big(\bar v,\bar u,\bar s\big) \quad a.e.(x,t) \tag{2.4}$$

as $(\epsilon,\tau)\to(0,0)$ *subject to the condition* $\tau(M_2+1)^2\le\epsilon$, *and* $(\bar v,\bar u)$ *is an entropy solution of the equilibrium system* (1.3) *with the initial data* $\big(v_0(x),u_0(x)\big)$.

Proof. To prove part 1, we use the following local existence lemma and the L^∞ estimates given in (2.1).

Lemma 2.2 (Local existence)**.** *If the initial data satisfies the condition* $\big(c_1\big)$ *in Theorem 2.1, then for any fixed* ϵ *and* $\tau>0$ *the Cauchy problem* (1.1), (1.2) *admits a unique smooth local solution* (u,v,s) *which satisfies*

$$|\frac{\partial^i v}{\partial x^i}| + |\frac{\partial^i u}{\partial x^i}| + |\frac{\partial^i s}{\partial x^i}| \le M(t_1,\epsilon,\tau) < +\infty, \quad i=0,1,2, \tag{2.5}$$

where $M(t_1,\epsilon,\tau)$ *is a positive constant that depends only on* t_1,ϵ,τ *and* t_1 *depends on* $|v_0|_{L^\infty}, |u_0|_{L^\infty}, |s_0|_{L^\infty}$.

Moreover

$$\lim_{|x|\to\pm\infty}\Big(\frac{d^i v}{dx^i},\frac{d^i u}{dx^i},\frac{d^i s}{dx^i}\Big) = (0,0,0), \quad i=0,1, \tag{2.6}$$

uniformly in $t\in[0,t_1]$.

The proof of Lemma 2.2 is standard.

To derive the crucial estimates given in (2.1), we need the necessary condition $\tau(M_2+1)^2\le\epsilon$ and condition (c_2) in the Theorem 2.1.

Multiply $\bar\sigma(v)+cv-cs$ to the first equation in (1.1), multiply u to the second and $s-cv$ to the third equation and add the results. This gives

$$\begin{aligned}\Big(\int_0^v \bar\sigma(v)+cvdv+\frac{u^2}{2}-csv+\frac{s^2}{2}\Big)_t + \Big(cus-u(\bar\sigma(v)+cv)+\frac{c_1 s^2}{2}\Big)_x \\ - cc_1 v s_x - u\Big(\sigma(v,s)+s-\big(\sigma(v,cv)+cv\big)\Big)_x + \frac{(s-cv)^2}{\tau} \\ = \epsilon\Big(\int_0^v \bar\sigma(v)+cvdv+\frac{u^2}{2}-csv+\frac{s^2}{2}\Big)_{xx} \\ - \epsilon\big(\bar\sigma'(v)+c\big)v_x^2 - \epsilon u_x^2 - \epsilon s_x^2 + 2c\epsilon s_x v_x \quad .\end{aligned} \tag{2.7}$$

For the third and fourth terms on the left-hand side of (2.7), we have the estimate

$$
\begin{aligned}
&- cc_1 v s_x - u\big(\sigma(v,s) + s - (\sigma(v,cv) + cv)\big)_x \\
&= \Big(\frac{c_1 c^2}{2} v^2 - cc_1 vs\Big)_x - \Big(u\big(\sigma(v,s) + s - (\sigma(v,cv) + cv)\big)\Big)_x \\
&\quad + u_x\big(\sigma_s(v,\alpha) + 1\big)(s - cv) + cc_1 v_x (s - cv)
\end{aligned} \tag{2.8}
$$

where α takes a value between s and cv. The last two terms in (2.8) have the upper bound

$$\frac{3(s-cv)^2}{4\tau} + \frac{\tau(M_2+1)^2 u_x^2}{2} + \tau c^2 c_1^2 v_x^2 \tag{2.9}$$

by the first condition in (c_2).

Combining (2.7), (2.8) and (2.9), we get the following inequality:

$$
\begin{aligned}
&\Big(\int_0^v \bar\sigma(v) + cvdv + \frac{u^2}{2} - csv + \frac{s^2}{2}\Big)_t + \Big(cus - u(\bar\sigma(v) + cv) + \frac{c_1}{2}s^2\Big)_x \\
&\Big(cc_1 vs - \frac{c_1 c^2}{2} v^2\Big)_x - \Big(u\big(\sigma(v,s) + s - (\sigma(v,cv) + c)\big)\Big)_x + \frac{(s-cv)^2}{4\tau} \\
&\le \epsilon\Big(\int_0^v \bar\sigma(v) + cvdv + \frac{u^2}{2} - csv + \frac{s^2}{2}\Big)_{xx} - \epsilon\big(\bar\sigma'(v) + c\big)v_x^2 - \epsilon s_x^2 + 2c\epsilon s_x v_x \\
&\quad + c^2 c_1^2 \tau v_x^2 - \big(\epsilon - \frac{\tau(M_2+1)^2}{2}\big)u_x^2 \quad .
\end{aligned} \tag{2.10}
$$

Noticing the second condition in (c_2) we know that $\Big(\int_0^v \bar\sigma(v) + cvdv + \frac{u^2}{2} - csv + \frac{s^2}{2}\Big)$ is a strictly convex function. If the condition $\tau(M_2+1)^2 \le \epsilon$ is satisfied, we immediately get the estimates (2.2), (2.3) by integrating (2.10) on $R \times [0,T]$ with the behaviour in (2.6). Differentiating the first equation in (1.1) with respect to x, we get

$$\big(v_x\big)_t - u_{xx} = \epsilon\big(v_x\big)_{xx} \quad . \tag{2.11}$$

Multiplying v_x to (2.11) yields

$$\big(\frac{v_x^2}{2}\big)_t - \big(v_x u_x\big)_x + u_x v_{xx} = \epsilon\big(\frac{v_x^2}{2}\big)_{xx} - \epsilon v_{xx}^2 \quad . \tag{2.12}$$

Integrating (2.12) on $R \times [0,T]$ and noticing the bound $|u_x^2|_{L^1(R\times R^+)} \le M(\epsilon)$, we obtain the bound $|v_x^2(.,t)|_{L^1(R)} \le M(\epsilon)$, where $M(\epsilon)$ is a constant depending on ϵ. Therefore

$$v^2 = |\int_{-\infty}^x (v^2)_x dx| \le \int_{-\infty}^\infty v^2 dx + \int_{-\infty}^\infty v_x^2 dx \le M(\epsilon) \quad .$$

Similarily from the second and third equations in (1.1) we get $|s_x^2(.,t)|_{L^1(R)} \le M(\epsilon)$ and $|u_x^2(.,t)|_{L^1(R)} \le M(\epsilon)$. So we get the estimates in (2.1) and the proof of part 1 in Theorem 2.1.

From the estimates in (2.2) and (2.3) it is easy to prove the compactness of $\eta(v^{\epsilon,\tau})_t + q(v^{\epsilon,\tau})_x$ in $H^{-1}_{loc}(R \times R^+)$, where $(v^{\epsilon,\tau}, u^{\epsilon,\tau})$ are the solutions of the Cauchy problem (1.1), (1.2) and (η, q) is any entropy-entropy flux pair constructed in [S]. Using the technique in [S] the convergence of $(v^{\epsilon,\tau}, u^{\epsilon,\tau})$ can be obtained. For details see [Lu1], [Lu2]. From the first estimate in (2.3) we obtain the convergence $s^{\epsilon,\tau} \to \bar{s}$. So Theorem 2.1 is proven.

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