Master's thesis
in Mathematical Physics

Linear Landau Damping Coupled with Relaxation for the Two-Species Vlasov-Poisson-BGK System

submitted by
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Abstract

The aim of this work is to show the linear Landau damping of the electric field coupled with the relaxation of a two-species system towards an equilibrium solution. The physical setting is a plasma, or more precisely a gas consisting of ions and electrons. Lev D. Landau showed in 1946 that the electric field of such a gas gets damped because of the force term in the Vlasov-Poisson system. He described the gas considering only the electrons. In contrast, the model in this work is the two-species Vlasov-Poisson-BGK system describing electrons as well as ions. Furthermore, it involves a collision kernel describing how the particles collide within a species but also includes interspecies collisions. We consider the case of a weakly collisional regime and the limiting case of long wavelengths. By linearising the mathematical system and examining the singularities of the Fourier transformed electric field we find a long-term expression for the electric field depending on an exponential function with a product of the time and a parameter being the argument. For one thing, we show analytically that this parameter is negative and therefore that the electric field decays exponentially which proves the Landau damping effect. Secondly, we investigate this property numerically by means of a Python code by Eric Sonnendrücker. By rearranging the dispersion relation of the system and by using the plasma dispersion function we find the zeros of the dispersion relation and with that the parameter of the exponential function which we show to be negative.

Zusammenfassung

Ziel dieser Arbeit ist es, die lineare Landau-Dämpfung des elektrischen Feldes in Verbindung mit der Relaxation eines Zwei-Spezies-Systems in Richtung einer Gleichgewichtslösung nachzuweisen. Das physikalische Setting ist ein Plasma, genauer gesagt ein Gas, das aus Ionen und Elektronen besteht. Lev D. Landau zeigte 1946, dass das elektrische Feld eines solchen Gases aufgrund des Kraftterms im Vlasov-Poisson-System gedämpft wird. Er beschrieb das Gas nur unter Berücksichtigung der Elektronen. Im Gegensatz dazu ist das Modell in dieser Arbeit das Vlasov-Poisson-BGK-System für zwei Spezies, das sowohl Elektronen als auch Ionen beschreibt. Darüber hinaus beinhaltet es einen Kollisionskern, der beschreibt, wie die Teilchen innerhalb einer Spezies zusammenstoßen, aber auch Kollisionen zwischen den zwei Spezies einschließt. Wir betrachten den Fall eines schwach kollidierenden Systems und den Grenzfall langer Wellenlängen. Durch Linearisierung des mathematischen Systems und Untersuchung der Singularitäten der Fourier-Transformierten des elektrischen Feldes finden wir einen Ausdruck für das Langzeitverhalten des elektrischen Feldes, der von einer Exponentialfunktion über ein Produkt aus der Zeit mit einem Parameter als Argument abhängt. Zum einen zeigen wir analytisch, dass dieser Parameter negativ ist und somit das elektrische Feld exponentiell abfällt, was die Landau-Dämpfung beweist. Zum anderen untersuchen wir diese Eigenschaft numerisch mit Hilfe eines Python-Codes von Eric Sonnendrücker. Durch Umstellen der Dispersionsrelation des Systems und durch Verwendung der Plasma-Dispersionsfunktion finden wir die Nullstellen der Dispersionrelation des Systems und damit den Parameter der Exponentialfunktion, der sich als negativ erweist.

Contents

1 Introduction

Nowadays, it is essential to research models and properties for plasmas. Plasma physics has become an integral part of today's world. Plasma is no theoretical construct or a substance which is only located far away in the solar system, for example during plasma eruptions on the surface of our sun. It can be created on earth and is used every day, for example in photography, for street lighting, or plasma television [25]. Furthermore, the composition of the stars in our universe from plasma and their generation of energy from a process called nuclear fusion inspires the search for a new source of energy which could be an important part for the energy transition.

These and possible new applications are the motivation to further investigate mathematical and physical characteristics of plasmas.

1.1 Plasmas

The calculations and proofs of this thesis are based on the Vlasov-Poisson-BGK equations. These equations are differential equations which are used to describe a plasma.

A plasma is a gas consisting of positively charged ions and negatively charged electrons and therefore it is an ionized gas. In general, there might also be some neutral particles which means that a plasma can be partially ionized. Next to solid, liquid and gaseous, plasma was mentioned by D. A. Frank-Kamenetskii as the fourth state of matter [14]. This term seems to make sense since plasma can be generated from gas, for example by increasing the temperature to such an extent that the neutral gas atoms split into ions and electrons. The ionization energy barrier must therefore be overcome. At the same time, it might be misleading to consider plasma as another state of matter since one can observe gaseous, liquid or solid phases [25].

The properties of gases and plasmas differ completely. Unlike gas, a plasma has a high electrical conductivity. Furthermore, the interactions are dominated by the long-range Coulomb forces [28] in contrast to the short-range van der Waals forces occurring during collisions in a gas [25]. Due to the charged particles moving in space, an electric field is created, which in turn strongly influences the movement of the particles. In total, a plasma is quasi-neutral which means that there is the same amount of electrons and ions such that the total charge is zero.

To describe a plasma one can use several models: the microscopic, the macroscopic and the kinetic model. The macroscopic model uses physical quantities which can be seen or measured. Therefore, for example, the density or the temperature of a system can be used to describe it. Since there are a lot of informations of a gas which can not be described by the macroscopic quantities this model is too imprecise for the research of this thesis. Microscopically, one has to describe the movement of each single particle. Since a plasma consists of a high number of particles, in the order of 10^{23}, we use the kinetic model. It is a mesoscopic model which is at an intermediate level. One can explain macroscopic quantities as well as the statistical behaviour of the particles by means of the particle distributions.

In preparation for the next chapters, some characteristic values of plasmas are listed in the following. One quantity describing a plasma is the *thermal velocity* [28]

$$v_{\mathrm{th}} = \sqrt{\frac{T}{m}}$$

of a particle with mass m. Note that we use a notation in which the Boltzmann constant k_{B} is contained in the temperature variable T. The thermal velocity refers to a variable linked to the temperature of a species and does not describe the velocity of the particles themselves.

The *plasma oscillation frequency* or *plasma frequency*

$$\omega_{\mathrm{p}} = \sqrt{\frac{nq^2}{\epsilon_0 m}}$$

is a quantity describing the *plasma oscillations* or *Langmuir waves* in a plasma consisting of particles with number density n, charge q and mass m. ϵ_0 is the vacuum permittivity. Plasma oscillations are periodic oscillations of the charge density.

For further informations and characteristics of plasmas see [25] or [28].

1.2 Emergence of the Research Question

The starting point of the history of the Landau damping phenomenon could be placed in the 19$^{\mathrm{th}}$ century, with the development of the so-called Boltzmann equation

$$\partial_t f(x,v,t) + v\nabla_x f(x,v,t) = Q(f(x,v,t))$$

by the Austrian physicist Ludwig Boltzmann. It characterises the evolution of the probability density function $f(x,v,t)$ of electrons in the setting of a gas. The transport part on the left-hand side belongs to the free movement of the particles. On the right-hand side the collision kernel describes how electrons collide with each other. Later we will insert an explicit expression for $Q(f(x,v,t))$. The equation and some of its properties are described in more detail in Chapter 2.2.

Under the condition that the particles of a gas are homogeneously distributed in the space, one can prove a theorem which states that an isolated system tends towards a state of equilibrium. This H-theorem, explained more precisely in Chapter 2.2.2, was the proof that the second law of thermodynamics holds since it showed that the gas evolution is an irreversible process. This relaxation of the system only occurs because of the collision kernel on the right-hand side of the Boltzmann equation.

The next historical step was the proof of linear Landau damping in 1946 by Lev D. Landau [19]. As opposed to the Boltzmann equation the setting was a plasma or more precisely a dilute gas consisting of ions and electrons with charge $q = \pm e$. Landau described this plasma by means of the Vlasov-Poisson system

$$\partial_t f(x,v,t) + v\nabla_x f(x,v,t) - \frac{e}{m}E(x,t)\cdot\partial_v f(x,v,t) = 0$$
$$\partial_x E(x,t) = \frac{\rho(x,t)}{\epsilon_0} = \frac{e}{\epsilon_0}(n_0 - n(x,t))$$

including the electric field E caused by the electrons and ions themselves without considering an external field. Only the electrons with mass m and number density $n(x,t)$ are described by the equations while ions are treated as neutralizing background with constant number density n_0.

Landau's approach to prove the Landau damping was to linearise the equations around a homogeneous equilibrium function and after performing a Fourier and a Laplace transform to study singularities in the complex plane. Since the proof of the two-species case involves the same mathematical steps, we will come back to the approach in Chapter 3.2. One important point is that Landau studied the limiting case of long waves such that he let the wave vector k go to zero. Finally he showed the exponential decay of the electric field E and therefore a damping effect even without considering collisions explicitly. This was also a proof for the irreversibility of time just like the relaxation effect of the Boltzmann equation.

In 1964, twenty years after its mathematical proof, the Landau damping was proven experimentally by John H. Malmberg and Charles B. Wharton [21].

Clément Mouhot and Cédric Villani showed in 2010 that Landau damping can also be proven analytically in the non-linearised case of the Vlasov-Poisson equation [23]. They showed the damping effect by using a functional analysis approach and by means of the Newton algorithm, an iterative approximation scheme.

In addition to many other projects, some members of the working group of Prof. Klingenberg have also been working on further specifications and characteristics of the plasma model. One important step was done by Marlies Pirner who showed the linear Landau damping for the two-species Vlasov-Poisson equations

$$\partial_t f_1(x,v,t) + v\nabla_x f_1(x,v,t) + \frac{e}{m_1}E(x,t)\cdot\partial_v f_1(x,v,t) = 0$$
$$\partial_t f_2(x,v,t) + v\nabla_x f_2(x,v,t) - \frac{e}{m_2}E(x,t)\cdot\partial_v f_2(x,v,t) = 0$$
$$\partial_x E(x,t) = \frac{\rho(x,t)}{\epsilon_0} = \frac{e}{\epsilon_0}(n_1(x,t) - n_2(x,t))$$

and calculated convergence rates for the damping process [26]. In her model, she did not only consider the electrons but also included the ions as moving particles. Index 1 refers to ions while index 2 belongs to electron quantities.

The last point to be mentioned here is the work of Lena Baumann in 2021 who showed linear Landau damping coupled with relaxation for the Vlasov-Poisson-BGK system [2]

$$\partial_t f(x,v,t) + v\nabla_x f(x,v,t) - \frac{e}{m}E(x,t)\cdot\partial_v f(x,v,t) = Q_{\text{BGK}}(x,v,t)$$
$$\partial_x E(x,t) = \frac{\rho(x,t)}{\epsilon_0} = \frac{e}{\epsilon_0}(n_0 - n(x,t)).$$

This system combines the Boltzmann equation including the collision kernel with the force term including the electric field E and therefore connects relaxation and Landau damping. Lena Baumann chose a simplified expression for the collision operator by Bhatnagar, Gross and Krook, the so-called BGK collision kernel [5]

$$Q_{\text{BGK}} = \nu(M(x,v,t) - f(x,v,t)).$$

$M(x,v,t)$ is the Maxwellian distribution

$$M(x,v,t) = \frac{n(x,t)}{(2\pi T_0/m)^{1/2}} \exp\left(-\frac{|v-u(x,t)|^2}{2T_0/m}\right).$$

On the one hand one has the relaxation effect because of the collision kernel on the right-hand side of the Vlasov-Poisson-BGK system and on the other hand the force term causes the damping effect of the electric field. Lena Baumann found that both effects reinforce each other such that we have a coupling of relaxation and damping effect.

The task of this thesis is to combine the work of Lena Baumann and Marlies Pirner and show that considering the Vlasov-Poisson-BGK system applied to two species, namely ions and electrons, there is a Landau damping effect coupled with relaxation such that the system tends to an equilibrium.

1.3 Structure of the Thesis

This Master's thesis is structured as follows.

At first, Chapter 2 gives some fundamental mathematical concepts which are important for the following work. Starting with the theory about kinetic equations in general and becoming familiar with the famous Boltzmann equation and some of its properties we will see some of the basics of Fourier and Laplace transform. Finally, after introducing the residue theorem, one can find some basic functions and identities which are used in the next chapters.

Chapter 3 contains the main part of the Master's thesis. We consider the physical setting of a gas mixture of electrons and ions or more precisely a plasma. To describe this system we use the two-species Vlasov-Poisson-BGK system consisting of the transport part of the particles, a force term because of charged particles creating an electric field and a collision kernel describing how ions and electrons collide. After describing the mathematical model and specifying the one-dimensional system we want to show the Landau damping effect of the electric field coupled with the relaxation of the two-species system towards an equilibrium solution. Therefore we proceed as follows.

The first step will be to linearise the equations with respect to a known equilibrium function. Then we will perform a Fourier and a Laplace transform to eliminate spatial and time derivatives. The next step will be to find a dispersion relation of the system. This step is based on our aim to find an expression for the electric field as we want to prove its damping effect. If we decompose the electric field into its single waves each with a specific frequency we get an equation with the variables frequency and wave vector and therefore a dispersion relation. The task will then be to determine the frequency with the largest imaginary part and examine whether it causes the electric field to grow exponentially. We will determine an expression for the electric field depending on this frequency with the largest imaginary part and therefore have an expression for the long-term behaviour of the electric field.

To finally show the damping effect coupled with relaxation we will do two different approaches. The first one, the analytical calculation, is to show that the long-term expression for the electric field decays exponentially by neglecting higher order terms and examining the argument of the exponential function of the electric field. The second approach will be to use numerical methods to show the damping effect. Therefore, we will modify a Python code which was created by Eric Sonnendrücker [31] and already used by Lena Baumann to show the damping effect for the one-species model [2].

The last part of this chapter will be to interpret the Landau damping effect physically and expand the interpretation to the two-species Vlasov-Poisson-BGK model.

The last Chapter 4 will summarize the results and give a conclusion of the work of this Master's thesis.

2 Fundamentals

2.1 Distribution Functions for Kinetic Equations

To describe and give a prediction of physical systems it is crucial to set up a mathematical model which is precise enough to represent the reality but also simple enough so that one can calculate states and characteristics of the system. When examining gases consisting of N particles, a common way to create a model is the imagination of particles as hard spheres colliding elastically with each other. Each particle or sphere has a position $x \in \mathbb{R}^3$ and a velocity $v \in \mathbb{R}^3$ at a time $t \in \mathbb{R}_0^+$. As the number of particles in a gas is of order 10^{23} the physicists J. C. Maxwell and L. Boltzmann introduced the concept of distribution functions. With these functions it is possible to describe the state of a gas by giving the number of particles which have a certain position and velocity (x, v) at time t.

Definition 2.1 (Distribution function, [26]). A function

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad (x, v, t) \longmapsto f(x, v, t)$$

is called a *distribution function* if and only if $f(x, v, t)dxdv$ is the number of particles with velocities in $(v, v + dv)$ located in the interval $(x, x + dx)$ at time t.

The distribution function of a system can not be measured but there are other quantities which can physically be determined and also be calculated with the distribution function. These quantities are macroscopic values and are defined as follows.

Definition 2.2 (Number density, [26]). Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \longrightarrow \mathbb{R},\ \Lambda \subset \mathbb{R}^3$ with $f \in L^1(dv),\ f \geq 0$ be a distribution function. Then, the function

$$n : \Lambda \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad (x, t) \longmapsto n(x, t) = \int_{\mathbb{R}^3} f(x, v, t)dv$$

is called the *number density*.

Definition 2.3 (Mean velocity, [26]). Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \longrightarrow \mathbb{R},\ \Lambda \subset \mathbb{R}^3$ with $vf \in L^1(dv),\ f \geq 0$ be a distribution function. Then, we define the function

$$n \cdot u : \Lambda \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}^3, \quad (x, t) \longmapsto (n \cdot u)(x, t) = \int_{\mathbb{R}^3} vf(x, v, t)dv.$$

If $n > 0$, the function $u(x, t) = \frac{(n \cdot u)(x,t)}{n(x,t)}$ is called the *mean velocity*.

Definition 2.4 (Internal energy and temperature, [26]). Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \longrightarrow \mathbb{R},\ \Lambda \subset \mathbb{R}^3$ with $(1 + |v|^2)f \in L^1(dv),\ f \geq 0$ be a distribution function. Then, the function

$$e : \Lambda \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad (x, t) \longmapsto e(x, t) = \frac{1}{2}m \int_{\mathbb{R}^3} |v - u(x, t)|^2 f(x, v, t)dv$$

is called the *internal energy* of a gas consisting of particles with mass m. If we are in an ideal gas and $n > 0$, the function $T(x,t) = \frac{2}{3}\frac{e(x,t)}{n(x,t)}$ is called the *temperature*.

Definition 2.5 (Entropy, [26]). Let $f : \Lambda \times \mathbb{R}^3 \times \mathbb{R}_0^+ \longrightarrow \mathbb{R},\ \Lambda \subset \mathbb{R}^3$ with $f \geq 0$ be a distribution function. The quantity

$$H(f(x,v,t)) = \int f(x,v,t) \cdot \ln(f(x,v,t))dv$$

defines the negative of the physical *entropy* in statistical mechanics.

It is sensible to specify special distribution functions that are often used. One example is the Maxwellian distribution.

Definition 2.6 (Maxwell-Boltzmann distribution, [26]). A *Maxwell-Boltzmann distribution* is a distribution of the form

$$M(v) = A \exp\left(-\beta \left|v - u\right|^2\right)$$

where $A, \beta \in \mathbb{R}^+$ and $u \in \mathbb{R}^3$. If A, β and u are functions of x and t, $M(x,v,t)$ is called a *local Maxwell distribution*.

2.2 Kinetic Equations

One of the first persons who investigated the motion of gas particles and applied the atomic constitution of matter to gases was Daniel Bernoulli. With him, the kinetic theory of gases was born in 1738. His idea was to consider gas molecules as hard spheres moving rapidly and colliding with each other in a perfect elastic manner [7]. Another person who made an important contribution to this subject was Ludwig Boltzmann. He developed an evolution equation for the velocity distribution of a gas in equilibrium, the Boltzmann transport equation [6]

$$\partial_t f + v\nabla_x f = Q(f), \tag{2.1}$$

where $Q(f)$ is called the collision kernel and $f := P(x,v,t)$ is the probability of finding one particle of the gas, let us say particle 1, at time t at position $x_1 =: x$ with velocity $v_1 =: v$. Until today there exist only solutions of this equation for special cases and numerical approximations.

Under some assumptions one can give a more explicit expression for the collision kernel. There must be only few collisions such that the particles have a large mean free path. One collision partner, in this case particle 1, is assumed to have twice the diameter and to be at rest while the other particles are considered to be point masses. Furthermore one takes the so-called Boltzmann-Grad limit which means that we suppose an infinite number of particles in the gas while collisions are assumed to be rare. To get the collision kernel one integrates over the velocities of all particles and over the sphere of the unit ball around particle 1

$$\partial_t f + v\nabla_x f = \int_{\mathbb{R}^3}\int_{S^2} (f' f'_* - f\ f_*) \left| n \cdot (v - v_*) \right| dv_*\, dn \tag{2.2}$$

with $f := P(x, v, t)$, $f_* := P(x_k, v_k, t)$, $f' := P(x, v', t)$ and $f'_* := P(x_k, v'_k, t)$ where the variables, which have no dash, refer to the velocities before and the dashed variables to the velocities after the collision of particle 1 and $k \in \{2, ..., N\}$ [6].

2.2.1 Collision Invariants

One interesting issue of kinetic equations is to find collision invariants. These are functions ϕ satisfying

$$\int_{\mathbb{R}^3} \phi(v) Q(f) dv = 0 \tag{2.3}$$

which is fulfilled for every $f \in C_c^\infty$ if and only if [6]

$$\phi(v) + \phi(v_k) - \phi(v') - \phi(v'_k) = 0. \tag{2.4}$$

Lemma 2.1 ([6])**.** Let $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^2 function that satisfies (2.4). Then

$$\phi(v) = a + b \cdot v + c \cdot |v|^2 \tag{2.5}$$

for suitable constants $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$.

From a physical point of view, a collision invariant is a conservation function, as the time derivative of $\int_{\mathbb{R}^3} \phi(v) f(v) dv$ becomes zero when the homogeneous Boltzmann equation is considered and the collision invariant is therefore conserved. The most relevant conservative laws are the conservation of mass with $\phi(v) \equiv 1$, the conservation of momentum with $\phi(v) \equiv v$ and the conservation of energy with $\phi(v) \equiv \frac{1}{2} |v|^2$.

2.2.2 H-Theorem

The H-theorem indicates that a physical system always tends to decrease its entropy H defined in Definition 2.5. Considering the collision integral (2.3) with the special test function $\phi(v) := \log(f(v))$ one can show the Boltzmann inequality [7]

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \log(f)Q(f)dv \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \log(f)(f'f'_* - f\ f_*)\,|n\cdot(v-v_*)|\,dv\,dv_*\,dn \\
&= \frac{1}{4}\iiint \big(\log(f)+\log(f_*)-\log(f')-\log(f'_*)\big)(f'f'_* - f\ f_*)\,|n\cdot(v-v_*)|\,dv\,dv_*\,dn \\
&= \frac{1}{4}\iiint \log\left(\frac{f\ f_*}{f'f'_*}\right)(f'f'_* - f\ f_*)\,|n\cdot(v-v_*)|\,dv\,dv_*\,dn \\
&\leq 0
\end{aligned}
\tag{2.6}
$$

by performing the variable transformations

$$
v \longleftrightarrow v_*, \quad v \longleftrightarrow v' = v - n(n\cdot(v-v*)), \quad v_* \longleftrightarrow v'_* = v_* + n(n\cdot(v-v*))
$$

and using

$$
(x-y)\log\left(\frac{y}{x}\right) \leq 0 \qquad \forall x,y \geq 0. \tag{2.7}
$$

If $f'f'_* = f\ f_*$ is satisfied the collision integral (2.6) vanishes and $\phi(v) = \log(f(v))$ is a collision invariant. According to Lemma 2.1 f must be a Maxwell-Boltzmann velocity distribution

$$
f(v) = \exp(a + b\cdot v + c\cdot |v|^2) = A\cdot\exp(-\beta\cdot|v-u|^2)
$$

with constants $A, \beta, -c > 0$, $a \in \mathbb{R}$ and $b, u \in \mathbb{R}^3$.

With the Boltzmann equation (2.1) one can find the macroscopic balance equation [6]

$$
\begin{aligned}
\partial_t H := \partial_t \int f\cdot\log(f)dv &= \int \partial_t f(\log(f)+1)dv \\
&= \int Q(f)(\log(f)+1)dv - \int (v\nabla_x f)(\log(f)+1)dv \\
&= \int Q(f)\log(f)dv - \int \nabla_x(vf\log(f))dv =: S - \nabla_x I
\end{aligned}
\tag{2.8}
$$

using that $\phi(v) = 1$ is a collision invariant in the second last step. According to the Boltzmann inequality (2.6) it is valid that $S \leq 0$. Assuming a homogeneous equation so that $f \neq f(x)$ the spatial derivative $\nabla_x I$ vanishes and we are left with

$$
\frac{d}{dt}H \leq 0, \tag{2.9}
$$

the so-called H-theorem.

This theorem is a proof of the second law of thermodynamics as it states that an isolated physical system described by the Boltzmann equation (2.1) tends towards a state of equilibrium which means that the process is irreversible in time. This process is called relaxation.

2.3 Maxwell's Equations

Maxwell's equations link the electric field $E(x,t)$ and the magnetic field $B(x,t)$ of a system including external sources and boundary conditions like charge distributions or currents. The four equations in differential forms are

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad \text{(Gauß's law)} \tag{2.10}$$

$$\nabla \cdot B = 0 \qquad \text{(Gauß's law for magnetism)} \tag{2.11}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's law of induction)} \tag{2.12}$$

$$\nabla \times B = \mu_0 (J + \epsilon_0 \frac{\partial E}{\partial t}) \qquad \text{(Ampère's circuital law)} \tag{2.13}$$

The Gauß's law states the orientation of the electric field in the presence of a charge distribution $\rho(x,t)$. The second law states that magnetic field lines are always closed and that there are no magnetic monopoles. Faraday's law shows that the time evolution of the magnetic field depends on the spatial change of the electric field, while the time evolution of the electric field depends not only on the curl of the magnetic field but also on the electric current $J(x,t)$. ϵ_0 and μ_0 are the permittivity and the permeability of free space.

2.4 Fourier Transform

In physics as well as in mathematics, the Fourier Transform is indispensable. It has numerous applications, for example to solve certain differential equations. Since we will use this concept introduced by J. B. J. Fourier in 1822 [13], the most relevant definitions for the Fourier transform are indicated in the following.

Definition 2.7 (Schwartz space, [29])**.** The *Schwartz space* $S(\mathbb{R}^n)$ of rapidly decaying functions is defined as

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : |f|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| < \infty \text{ for any } \alpha, \beta \in \mathbb{N}_0^n \right\}$$

where $n \geq 1$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ with $\partial_i = \frac{\partial}{\partial x_i} \quad \forall i \in \{1, ..., n\}$.

Lemma 2.2 ([29])**.** If $f \in S(\mathbb{R}^n)$, then

$$|f(x)| \leq c_m (1 + |x|)^{-m}$$

for every $m \in \mathbb{N}$. The converse is not true.

Definition 2.8 (Fourier transform, [29]). The *Fourier transform* $\hat{f}(v)$ of the function $f(x) \in S(\mathbb{R}^n)$ is defined by

$$\hat{f}(v) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp(-\mathrm{i}x \cdot v) dx$$

with $v \in \mathbb{R}^n$ and scalar product $x \cdot v$.

Lemma 2.3 ([29]). The integral in Definition 2.8 is well defined, since

$$\left|\hat{f}(v)\right| \leq c_m (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |x|)^{-m} dx < \infty$$

for $m > n$.

Definition 2.9 (Inverse Fourier transform, [29]). The *inverse Fourier transform* of a function $G(v) \in S(\mathbb{R}^n)$ is defined as

$$\breve{g}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} G(v) \exp(\mathrm{i}x \cdot v) dv$$

with $x \in \mathbb{R}^n$ and scalar product $x \cdot v$.

Lemma 2.4 (Fourier inversion formula, [29]). Let f be a function from $S(\mathbb{R}^n)$. Then

$$\breve{\hat{f}} = f = \hat{\breve{f}}.$$

This is called the *Fourier inversion formula*.

2.5 Laplace Transform

The Laplace transform introduced by P.-S. Laplace in the 18^{th} century is often used to solve differential equations of physical systems. The main requirement is that the variable to transform covers a range from zero to infinity which is why one often transforms the time variable. The most important definitions are as follows.

Definition 2.10 (Functions of exponential order, [1]). Let $\mathcal{C}$ denote the class of real valued continuous functions $f(t)$, $t \in \mathbb{R}$, such that there exists $a \in \mathbb{R}$ and $C > 0$ satisfying the inequality

$$|f(t)| \leq C \exp(at), \qquad \forall t \in \mathbb{R}.$$

Functions satisfying this equation are usually named *functions of exponential order*.

Definition 2.11 (Laplace transform, [1], [4]). We define the *Laplace transform* of $f \in \mathcal{C}$ to be the function

$$\tilde{f}(s) = \int_0^\infty f(t) \exp(-st) dt$$

where s is a complex parameter.

Lemma 2.5 ([1])**.** If $f \in \mathcal{C}$ then the function $|f(t)\exp(-st)|$ is integrable on $[0,\infty)$ for any $s \in \mathbb{C}, \mathrm{Re}(s) > a$. From Calculus one knows that if $|g|$ is integrable then so is g. Thus if $\mathrm{Re}(s) > a$, the integral

$$\int_0^\infty f(t)\exp(-st)dt$$

exists and is finite.

Lemma 2.6 (Laplace transform of the derivative, [1])**.** If f is continuously differentiable and $f' \in \mathcal{C}$ then one has

$$\widetilde{f'}(s) = -f(0) + s \cdot \tilde{f}(s)$$

$\forall s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

Theorem 2.1 (Inverse Laplace transform, [4])**.** If $F(s)$ is analytic for $\mathrm{Re}(s) > a$ and $F(s) = \frac{C}{s} + \mathcal{O}\left(\frac{1}{|s|^2}\right)$ as $|s| \longrightarrow \infty$ along $s = b + \mathrm{i}t$ with $b, t \in \mathbb{R}$, $b > a$, then

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} F(s)\exp(st)ds$$

exists and $F(s) = \tilde{f}(s)$. This expression is called the *inverse Laplace transform.*

2.6 Residue Theorem

Definition 2.12 (Index, [24])**.** Let C be a closed contour in $\mathbb{C}$. The *index* or *winding number* of C is the function $\mathrm{Ind}_C : \mathbb{C} \setminus C \longrightarrow \mathbb{C}$ given by

$$\mathrm{Ind}_C z_0 = \frac{1}{2\pi\mathrm{i}} \int_C \frac{dz}{z - z_0}.$$

Definition 2.13 (Annulus, [24])**.** Let $a \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. The *annulus* centred at a with *inner radius* r and *outer radius* R is the set

$$A(a; r, R) = \{z \in \mathbb{C} : r < |z - a| < R\}.$$

Lemma 2.7 (Laurent series expansion, [24])**.** Let $a \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. If $f \in H(A(a; r, R))$ is a holomorphic function on the annulus centred at a, then for all $z \in A(a; r, R)$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{n=-\infty}^{-1} c_n(z-a)^n$$

where $c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$ for all $n \in \mathbb{Z}$ and any positively oriented simple closed contour $C \subseteq A(a; r, R)$ surrounding a.

Definition 2.14 (Residue, [24])**.** If $f \in H(A(a; 0, r))$ for some $a \in \mathbb{C}$ and $r > 0$, then the *residue* of f at a is given by

$$\mathrm{Res}_{z=a} f(z) = c_{-1}$$

where c_{-1} is the coefficient of $(z-a)^{-1}$ in the Laurent series expansion of f given in Lemma 2.7.

Theorem 2.2 (Residue theorem, [24])**.** Let $\Omega \subseteq \mathbb{C}$ be open, and suppose that f is analytic on Ω except for a finite number of distinct isolated singularities $a_1, ..., a_n \in \Omega$. If C is a closed contour in $\Omega \setminus \{a_1, ..., a_n\}$ such that $\mathrm{Ind}_C z = 0 \quad \forall z \in \mathbb{C} \setminus \Omega$, then

$$\frac{1}{2\pi \mathrm{i}} \int_C f(z) dz = \sum_{j=1}^{n} \mathrm{Ind}_C a_j \cdot \mathrm{Res}_{z=a_j} f(z).$$

2.7 Basic Functions and Identities

Definition 2.15 (Cauchy principal value, [16])**.** Let $f(x)$ be a function that becomes infinite at an interior point $x = c$ of the range (a, b). The *Cauchy principal value* of f is the limit

$$\lim_{\epsilon \longrightarrow 0} \int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^{b} f(x) dx$$

where $0 < \epsilon \leq \min(c-a, b-c)$. Such a limit is usually denoted as

$$\mathcal{P} \int_a^b f(x) dx.$$

Definition 2.16 (Jackson's identity, [15])**.** Let $I(z)$ be the analytic function of the complex variable $z =: x + \mathrm{i}y$, $x, y \in \mathbb{R}$ defined by the integral along the real v-axis

$$I(z) = \int_{-\infty}^{\infty} \frac{f(v)}{v - z} dv$$

where $f(v)$ is a real function of v such that the integral exists for finite z. $I(z)$ can be expanded as follows:

$$I(z) = I(x + \mathrm{i}y) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{i}y)^n \left(\mathcal{P} \int_{-\infty}^{\infty} \frac{f^{(n)}(v)}{v - x} dv + \mathrm{i}\pi f^{(n)}(x) \right)$$

where $f^{(n)}$ is the n^{th} derivative of f and $\mathcal{P}$ denotes the Cauchy principal value of Definition 2.15.

Definition 2.17 (Gamma function, [27]). For $0 < x < \infty$ let

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt$$

which is the *Gamma function*. The integral converges for these x. It yields for $x \in \mathbb{R}^+$ and integer $n \in \mathbb{N}_0$

$$\begin{aligned}
&\Gamma(x+1) = x\Gamma(x)\\
&\Gamma(n+1) = n!\\
&\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x+1}{2}\right)\\
&\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{n! 4^n} \sqrt{\pi}.
\end{aligned}$$

Definition 2.18 ((Complex) Error function, [18], [31]). For $z \in \mathbb{C}$ the *error function* is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt$$

while the *complex error function* is defined as

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(t^2) dt.$$

The two functions are related by $\operatorname{erfi}(z) = -\mathrm{i} \cdot \operatorname{erf}(\mathrm{i}z)$ and $\operatorname{erf}(z) = -\mathrm{i} \cdot \operatorname{erfi}(\mathrm{i}z)$. Obviously, $\operatorname{erf}(0) = \operatorname{erfi}(0) = 0$. For real z both functions are real and have the same sign as $z \in \mathbb{R}$. The complex error function is a classical special function available in most of standard numeric and symbolic computation software.

Definition 2.19 (Gaussian integral, [12], [30]). The *Gaussian integral* that is also known as *probability integral* is the integral

$$\int_{-\infty}^{\infty} \exp(-x^2) dx.$$

By means of coordinate transformation to polar coordinates one can find that this integral is equal to $\sqrt{\pi}$. A more general form of the Gaussian integral is the integral

$$\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}$$

where $a \in \mathbb{R}$, $a \neq 0$.

3 The Vlasov-Poisson-BGK System for Two Species

The two-species Vlasov-Poisson-BGK system can be used to describe the motion of a gas consisting of two types of particles. In the following we consider a mixture of electrons and ions which is also the structure of plasmas. After describing the mathematical specification of the one-dimensional system we want to show a damping phenomenon of the electric field by small perturbations of the equilibrium state. L. D. Landau was the first person who showed the so-called Landau damping effect of an electronic plasma for the limiting case of long waves and small collision frequencies in 1965 [19]. Since he considered the Vlasov-Poisson equation where one has a force term caused by the electric field of the charged particles while the collision kernel is neglected, it is of interest how the relaxation and the damping effect influence each other.

There are several ways to show the damping effect of the electric field. For example, one way is to use the method of characteristics and determine convergence rates like it was done by Lena Baumann and Marlies Pirner in [3] for the two-species Vlasov-Poisson system. For the approach of this thesis, it will be drawn on the work of Lena Baumann who studied Landau damping for the one-species Vlasov-Poisson model as well as Landau damping coupled with relaxation for the one-species Vlasov-Poisson-BGK model [2]. Another important basis for the following considerations and calculations is the two-species Vlasov-BGK model, which was presented by Prof. Dr. Marlies Pirner in her dissertation [26].

3.1 The Model

The Vlasov-Poisson-BGK equations to describe the time evolution of a mixture of electrons and ions in one dimension were presented by Marlies Pirner in [26] and are given by

$$\begin{aligned} \partial_t f_1 + v\partial_x f_1 + \frac{eE}{m_1}\partial_v f_1 &= \nu_{11} n_1 (M_1 - f_1) + \nu_{12} n_2 (M_{12} - f_1) \\ \partial_t f_2 + v\partial_x f_2 - \frac{eE}{m_2}\partial_v f_2 &= \nu_{22} n_2 (M_2 - f_2) + \nu_{21} n_1 (M_{21} - f_2) \end{aligned} \tag{3.1}$$

Here the index 1 is related to the ions and index 2 refers to electrons. The functions $f_1(x, v, t)$ and $f_2(x, v, t)$ are distribution functions mapping the phase space variables, namely the position $x \in \mathbb{R}$ and the velocity $v \in \mathbb{R}$, and the time $t \in \mathbb{R}_0^+$ to the probability to find the ion or the electron at time t at position x with the velocity v.

The first summands of the left-hand sides

$$\partial_t f_i + v\partial_x f_i$$

are the transport parts for the two particles $i = 1$ and $i = 2$ referring to an undisturbed movement. We extend these movements by a force term. If a particle with charge q is located within an electric and a magnetic field then there is a Lorentz force

$$F = q(E + v \times B) \tag{3.2}$$

acting on the charged particle. If we assume that the impact of the magnetic field B is negligible compared to the electric field E we get the force term on the left-hand sides of (3.1). It describes the movement of particles with charge $q = \pm e$ and mass m_i influenced by the electric field which is caused by the charged particles themselves. It is also possible to consider ions with charge $q = +Z \cdot e$ where $Z \in \mathbb{N}$ but we will restrict ourselves to the case $Z = 1$. We assume that there is no external electromagnetic field acting on the gas particles.

Furthermore the two types of particles with number densities n_i are assumed to not only collide with themselves but also with the other species. Therefore we need the BGK collision operators

$$Q_{\mathrm{BGK},i} = \nu_{ii} n_i (M_i - f_i) + \nu_{ij} n_j (M_{ij} - f_i)$$

with $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$ which include the collision frequencies ν_{11} and ν_{22} from ion-ion and electron-electron collisions and ν_{12} and ν_{21} belonging to interspecies collisions. The operator was introduced in 1954 by Bhatnagar, Gross and Krook [5] and is a simplification of the integral representation (2.2) including the relaxation towards an equilibrium function. As the H-theorem indicated in Chapter 2.2.2 is valid for the BGK model [26] the system tends to an equilibrium state corresponding to the Maxwellian distributions

$$\begin{aligned} M_i(x,v,t) &= \frac{n_i(x,t)}{(2\pi T_i(x,t)/m_i)^{1/2}} \exp\left(-\frac{|v - u_i(x,t)|^2}{2T_i(x,t)/m_i}\right) \\ M_{ij}(x,v,t) &= \frac{n_{ij}(x,t)}{(2\pi T_{ij}(x,t)/m_i)^{1/2}} \exp\left(-\frac{|v - u_{ij}(x,t)|^2}{2T_{ij}(x,t)/m_i}\right). \end{aligned} \tag{3.3}$$

To ensure the conservation of mass, momentum and energy the macroscopic quantities number density n_i, mean velocity u_i and temperature T_i of M_1 and M_2 are the same as for the distribution functions f_1 and f_2. For the quantities with double indices one can calculate relations to simplify the system, see [17] and [26]. In case of $\frac{m_1}{m_1+m_2} \longrightarrow 1$ one has found

$$\begin{aligned} &\nu_{22} \approx \nu_{21} \approx \sqrt{\frac{m_1}{m_2}} \nu_{11} \approx \frac{m_1}{m_2} \nu_{12} \\ &n_{ij} = n_i \\ &u_{12} = u_1 = u_{21} \\ &T_{12} = T_1 = T_{21}. \end{aligned} \tag{3.4}$$

As an electron has a mass of $m_2 = 9.1 \cdot 10^{-31}\,\mathrm{kg}$ and ions $m_1 \geq 1.7 \cdot 10^{-27}\,\mathrm{kg}$ [22]

beginning with the lightest hydrogen ion we can take these relations.

Assuming the impact of the magnetic field B to be negligible, Maxwell's equations are simplified and we are left with

$$\begin{aligned} \nabla \cdot E &= \frac{\rho}{\epsilon_0} \\ \nabla \times E &= 0 \end{aligned} \tag{3.5}$$

describing a conservative electric field caused by a charge distribution $\rho(x,t) = q \cdot n(x,t)$. In the one-dimensional case and considering the plasma to be composed of ions with charge $q = +e$ and electrons with $q = -e$ one obtains

$$\frac{\partial E}{\partial x} = \nabla \cdot E = \frac{\rho}{\epsilon_0} = \frac{e}{\epsilon_0}(n_1 - n_2) \tag{3.6}$$

The equations (3.1) and (3.6) together are called the Vlasov-Poisson-BGK system for two species.

3.2 Linear Landau Damping Coupled with Relaxation

3.2.1 Linearisation

In the following we expect the gas to be near by its equilibrium state. Such a state is defined as follows.

Definition 3.1 (Equilibrium solution, [26])**.** We call a pair of functions $(f_1^{\text{equ}}, f_2^{\text{equ}})$ an *equilibrium solution* to (3.1) if and only if $(f_1^{\text{equ}}, f_2^{\text{equ}})$ satisfies (3.1) and

$$\frac{\partial}{\partial t} f_1^{\text{equ}} = 0 = \frac{\partial}{\partial t} f_2^{\text{equ}}.$$

We define our equilibrium solution with the distribution functions

$$f_i^{\text{equ}}(x,v,t) = \frac{n_{i0}}{(2\pi T_{i0}/m_i)^{1/2}} \exp\left(-\frac{|v-u_{i0}|^2}{2T_{i0}/m_i}\right) = f_i^{\text{equ}}(v), \qquad i = 1,2 \tag{3.7}$$

with the constants n_{i0}, T_{i0} and u_{i0}. We assume that the distribution functions $f_i(x,v,t)$ of both species near by the equilibrium state can be written as a function depending on the equilibrium solutions $f_i^{\text{equ}}(v)$ with small perturbation functions h_i according to

$$f_i(x,v,t) = f_i^{\text{equ}}(v) \cdot [1 + h_i(x,v,t)]. \tag{3.8}$$

Furthermore, we state that near equilibrium the temperatures of both species are the same, so that $T_{10} = T_{20} =: T_0$, and that the mean velocity can be set to $u_{10} = u_{20} = 0$ by scaling and shifting the coordinate system [20].

With the distribution function depending on the perturbation function the macroscopic quantities, namely number densities and mean velocities, become

$$
\begin{aligned}
n_i(x,t) &= \int_{\mathbb{R}} f_i(x,v,t)dv = \int_{\mathbb{R}} f_i^{\text{equ}}(v)dv + \int_{\mathbb{R}} f_i^{\text{equ}}(v)\cdot h_i(x,v,t)dv \\
&=: n_{i0} + \sigma_i(x,t) \\
n_i(x,t)u_i(x,t) &= \int_{\mathbb{R}} vf_i(x,v,t)dv = \int_{\mathbb{R}} vf_i^{\text{equ}}(v)dv + \int_{\mathbb{R}} vf_i^{\text{equ}}(v)\cdot h_i(x,v,t)dv \\
&=: u_{i0} + \mu_i(x,t) = \mu_i(x,t)
\end{aligned}
\tag{3.9}
$$

We assume the total charge of the gas to be zero in the equilibrium so that the condition of quasi-neutrality

$$
n_{10} = \int_{\mathbb{R}} f_1^{\text{equ}}(v)dv \overset{!}{=} n_{20} = \int_{\mathbb{R}} f_2^{\text{equ}}(v)dv =: n_0
$$

is fulfilled [26] which is a characteristic of plasmas.

If we insert the distribution function depending on the perturbation function h into the Vlasov-Poisson-BGK system (3.1) and (3.6) we are left with

$$
\begin{aligned}
&\partial_t(f_i^{\text{equ}} + f_i^{\text{equ}}\cdot h_i) + v\partial_x(f_i^{\text{equ}} + f_i^{\text{equ}}\cdot h_i) \pm \frac{eE}{m_i}\partial_v(f_i^{\text{equ}} + f_i^{\text{equ}}\cdot h_i) \\
&\qquad = \nu_{ii}n_i(M_i - f_i^{\text{equ}} - f_i^{\text{equ}}\cdot h_i) + \nu_{ij}n_j(M_{ij} - f_i^{\text{equ}} - f_i^{\text{equ}}\cdot h_i)
\end{aligned}
\tag{3.10}
$$

$$
\partial_x E = \frac{e}{\epsilon_0}(n_1 - n_2) = \frac{e}{\epsilon_0}(n_{10} + \sigma_1 - n_{20} - \sigma_2) = \frac{e}{\epsilon_0}(\sigma_1 - \sigma_2). \tag{3.11}
$$

We can neglect higher order terms in h_i and since f_i^{equ} is independent of x and t, we get for the left-hand side of (3.10) the linearised equation

$$
\partial_t(f_i^{\text{equ}}\cdot h_i) + v\partial_x(f_i^{\text{equ}}\cdot h_i) \pm \frac{eE}{m_i}\partial_v f_i^{\text{equ}}.
$$

As the electric field also depends on h_i according to (3.11) where $\sigma_i = \int_{\mathbb{R}} f_i^{\text{equ}}(v)\cdot h_i(x,v,t)dv$, we can drop the term including $E\cdot h_i$.

For the linearisation of the right-hand side we have to examine the Maxwell distributions M_i and M_{ij} from (3.3). With the linearised macroscopic quantities (3.9) and the relations for the quantities with the double indices (3.4) we have

$$
\begin{aligned}
M_i(x,v,t) &= \frac{n_0 + \sigma_i(x,t)}{(2\pi T_0/m_i)^{1/2}} \exp\left(-\frac{\left|v - \frac{\mu_i(x,t)}{n_0+\sigma_i(x,t)}\right|^2}{2T_0/m_i}\right) \\
M_{ij}(x,v,t) &= \frac{n_0 + \sigma_i(x,t)}{(2\pi T_0/m_i)^{1/2}} \exp\left(-\frac{\left|v - \frac{\mu_1(x,t)}{n_0+\sigma_1(x,t)}\right|^2}{2T_0/m_i}\right)
\end{aligned}
\tag{3.12}
$$

where $M_{ij}(x,v,t)$ no longer depends on index j and $M_{12}(x,v,t) = M_1(x,v,t)$.

We now use the Taylor expansion for the Maxwell distributions around $\sigma_i \approx 0 \approx \mu_i$ up to first order

$$M_i(x,v,t) \approx f_i^{\text{equ}}(v) + \frac{1}{n_0} f_i^{\text{equ}}(v) \cdot \sigma_i(x,t) + \frac{m_i v}{T_0 n_0} f_i^{\text{equ}}(v) \cdot \mu_i(x,t)$$
$$M_{ij}(x,v,t) \approx f_i^{\text{equ}}(v) + \frac{1}{n_0} f_i^{\text{equ}}(v) \cdot \sigma_i(x,t) + \frac{m_i v}{T_0 n_0} f_i^{\text{equ}}(v) \cdot \mu_1(x,t)$$

to find the linear expression for the right-hand side of (3.10)

$$\nu_{ii}(n_0 + \sigma_i)\left(\frac{1}{n_0}\sigma_i + \frac{m_i v}{T_0 n_0}\mu_i - h_i\right) f_i^{\text{equ}} + \nu_{ij}(n_0 + \sigma_j)\left(\frac{1}{n_0}\sigma_i + \frac{m_i v}{T_0 n_0}\mu_1 - h_i\right) f_i^{\text{equ}}$$

Neglecting the terms with higher order perturbation functions, dividing the whole equation by $f_i^{\text{equ}}(v)$ and using that $\partial_v f_i^{\text{equ}}(v) = -\frac{m_i v}{T_0} \cdot f_i^{\text{equ}}(v)$ one gets the linearised Vlasov-Poisson-BGK system

$$\partial_t h_i + v\partial_x h_i \mp \frac{ev}{T_0} E = \nu_{ii}\left(\sigma_i + \frac{m_i v}{T_0}\mu_i - n_0 h_i\right) + \nu_{ij}\left(\sigma_i + \frac{m_i v}{T_0}\mu_1 - n_0 h_i\right) \tag{3.13}$$
$$\partial_x E = \frac{e}{\epsilon_0}(\sigma_1 - \sigma_2) \tag{3.14}$$

for ions with $i = 1$ and $j = 2$ and for electrons with $i = 2$ and $j = 1$.

3.2.2 Fourier Transform

To eliminate the spatial derivatives in the differential equations (3.13) and (3.14) we multiply both sides with $\frac{1}{\sqrt{2\pi}} \exp(-\mathrm{i}kx)$ and integrate with respect to x along the whole real axis. Using that $\widehat{f'}(k) = \mathrm{i}k\hat{f}(k)$ by applying integration by parts we get a system containing Fourier transforms as described in Chapter 2.4:

$$\partial_t \hat{h}_i + \mathrm{i}kv\hat{h}_i \mp \frac{ev}{T_0}\hat{E} = \nu_{ii}\left(\hat{\sigma}_i + \frac{m_i v}{T_0}\hat{\mu}_i - n_0\hat{h}_i\right) + \nu_{ij}\left(\hat{\sigma}_i + \frac{m_i v}{T_0}\hat{\mu}_1 - n_0\hat{h}_i\right) \tag{3.15}$$
$$\mathrm{i}k\hat{E} = \frac{e}{\epsilon_0}(\hat{\sigma}_1 - \hat{\sigma}_2) \tag{3.16}$$

with

$$\hat{\sigma}_i(k,t) = \int_{\mathbb{R}} f_i^{\text{equ}}(v) \cdot \hat{h}_i(k,v,t)dv$$
$$\hat{\mu}_i(k,t) = \int_{\mathbb{R}} v f_i^{\text{equ}}(v) \cdot \hat{h}_i(k,v,t)dv.$$

3.2.3 Laplace Transform

As there is still a time derivative in the system (3.15), (3.16) we apply a Laplace transform according to Chapter 2.5. Therefore we assume that $\hat{h}_i$, $\hat{\sigma}_i$, $\hat{\mu}_i$ and $\hat{E}$ satisfy the requirements of Lemma 2.5. Instead of using the parameter s of Definition 2.11 we insert a new parameter ϖ which is related to s by $s = -\mathrm{i}\varpi$. Multiplying both equations with $\exp(\mathrm{i}\varpi t)$, integrating with respect to t in the range of 0 to ∞ and using the Laplace transform of the derivative in Lemma 2.6 one gets

$$\begin{aligned}
&(-\mathrm{i}\varpi + \mathrm{i}kv)\tilde{\hat{h}}_i - \hat{h}_i(t=0) \mp \frac{ev}{T_0}\tilde{\hat{E}} \\
&\qquad = \nu_{ii}\left(\tilde{\hat{\sigma}}_i + \frac{m_i v}{T_0}\tilde{\hat{\mu}}_i - n_0\tilde{\hat{h}}_i\right) + \nu_{ij}\left(\tilde{\hat{\sigma}}_i + \frac{m_i v}{T_0}\tilde{\hat{\mu}}_1 - n_0\tilde{\hat{h}}_i\right)
\end{aligned} \tag{3.17}$$

$$\mathrm{i}k\tilde{\hat{E}} = \frac{e}{\epsilon_0}(\tilde{\hat{\sigma}}_1 - \tilde{\hat{\sigma}}_2) \tag{3.18}$$

where we assume that $\hat{h}_i \longrightarrow 0$ as $t \longrightarrow \infty$ and with

$$\begin{aligned}
\tilde{\hat{\sigma}}_i(k,\varpi) &= \int_{\mathbb{R}} f_i^{\mathrm{equ}}(v)\cdot\tilde{\hat{h}}_i(k,v,\varpi)dv \\
\tilde{\hat{\mu}}_i(k,\varpi) &= \int_{\mathbb{R}} v f_i^{\mathrm{equ}}(v)\cdot\tilde{\hat{h}}_i(k,v,\varpi)dv.
\end{aligned} \tag{3.19}$$

In the following the notation $\breve{g} := \tilde{\hat{g}}$ will be used for the application of the Fourier and the Laplace transform.

3.2.4 Dispersion Relation

The aim of this work is to find an expression for the electric field as we want to prove an exponential decay. Decomposing the electric field into its single waves each with a specific frequency we can examine the one frequency which might prevent the electric field from being damped. We will see that this is the frequency with the largest imaginary part. In this section we determine the dispersion relation of the system (3.17) and (3.18) which gives a relation between the frequency and the wave vector by finding the algebraic expression for the transformed perturbations of the densities $\breve{\sigma}_1$ and $\breve{\sigma}_2$ and therefore for the electric field $\breve{E}$. Then we can study the behaviour of the electric field.

$\breve{E}$, $\breve{\sigma}_i$ and $\breve{\mu}_i$ depend explicitly or implicitly on $\breve{h}_i$. We want to eliminate $\breve{E}$ and $\breve{\mu}_i$ by giving expressions depending on $\breve{\sigma}_i$. Therefore we rewrite equation (3.17)

$$\begin{aligned}
&(-\mathrm{i}\varpi + \mathrm{i}kv + n_0\nu_{ii} + n_0\nu_{ij})\breve{h}_i \\
&\qquad = \nu_{ii}\left(\breve{\sigma}_i + \frac{m_i v}{T_0}\breve{\mu}_i\right) + \nu_{ij}\left(\breve{\sigma}_i + \frac{m_i v}{T_0}\breve{\mu}_1\right) \pm \frac{ev}{T_0}\breve{E} + \hat{h}_i(t=0)
\end{aligned} \tag{3.20}$$

and multiply it with $f_i^{\mathrm{equ}}(v)$ followed by the integration with respect to v along the real axis. With the expressions for the perturbed quantities (3.19) and $\lambda_1 := \nu_{11} + \nu_{12}$ and

$\lambda_2 := \nu_{22} + \nu_{21}$, respectively, this leads to

$$-\mathrm{i}\overline{\omega}\breve{\sigma}_i + \mathrm{i}k\breve{\mu}_i + n_0\lambda_i\breve{\sigma}_i = n_0\nu_{ii}\breve{\sigma}_i + n_0\nu_{ij}\breve{\sigma}_i + \int_{\mathbb{R}} \hat{h}_i(t=0) f_i^{\mathrm{equ}} dv$$

using the definition of the number density and the mean velocity where $u_{i0} = 0$. With $\hat{\sigma}_i(k,0) = \int_{\mathbb{R}} \hat{h}_i(k,v,0) f_i^{\mathrm{equ}}(v)dv$ we get an expression for $\breve{\mu}_i$ depending only on $\breve{\sigma}_i$

$$\breve{\mu}_i(k,\overline{\omega}) = \frac{\mathrm{i}\overline{\omega}\breve{\sigma}_i(k,\overline{\omega}) + \hat{\sigma}_i(k,0)}{\mathrm{i}k} \tag{3.21}$$

We already have the relation between $\breve{E}$ and $\breve{\sigma}_i$ by equation (3.18), therefore we can insert (3.18) and (3.21) into (3.20)

$$\begin{aligned}(-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_i)\breve{h}_i = &\left(\lambda_i + \nu_{ii}\frac{m_i v\overline{\omega}}{kT_0}\right)\breve{\sigma}_i + \nu_{ii}\frac{m_i v}{\mathrm{i}kT_0}\hat{\sigma}_i(t=0) + \nu_{ij}\frac{m_i v\overline{\omega}}{kT_0}\breve{\sigma}_1 \\ &+ \nu_{ij}\frac{m_i v}{\mathrm{i}kT_0}\hat{\sigma}_1(t=0) \pm \frac{e^2 v}{\mathrm{i}\epsilon_0 kT_0}(\breve{\sigma}_1 - \breve{\sigma}_2) + \hat{h}_i(t=0)\end{aligned}$$

and after dividing the equation by $(-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_i)$, multiplying it with $f_i^{\mathrm{equ}}(v)$ and integrating with respect to v we get the system of two equations

$$\begin{aligned}&\breve{\sigma}_1 \cdot \left(1 - \int_{\mathbb{R}} \frac{\lambda_1 + \lambda_1\frac{m_1 v\overline{\omega}}{kT_0} - \mathrm{i}\frac{e^2 v}{\epsilon_0 kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv\right) - \breve{\sigma}_2 \cdot \int_{\mathbb{R}} \frac{\mathrm{i}\frac{e^2 v}{\epsilon_0 kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \\ &\qquad = -\hat{\sigma}_1(k,0) \cdot \int_{\mathbb{R}} \frac{\mathrm{i}\lambda_1\frac{m_1 v}{kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv + \int_{\mathbb{R}} \frac{\hat{h}_1(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \\ &\breve{\sigma}_2 \cdot \left(1 - \int_{\mathbb{R}} \frac{\lambda_2 + \nu_{22}\frac{m_2 v\overline{\omega}}{kT_0} - \mathrm{i}\frac{e^2 v}{\epsilon_0 kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv\right) - \breve{\sigma}_1 \cdot \int_{\mathbb{R}} \frac{\nu_{21}\frac{m_2 v\overline{\omega}}{kT_0} + \mathrm{i}\frac{e^2 v}{\epsilon_0 kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \\ &\qquad = -\mathrm{i}\int_{\mathbb{R}} \frac{[\nu_{22}\hat{\sigma}_2(k,0) + \nu_{21}\hat{\sigma}_1(k,0)]\frac{m_2 v}{kT_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \\ &\qquad\quad + \int_{\mathbb{R}} \frac{\hat{h}_2(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv\end{aligned}$$

which is only depending on $\breve{\sigma}_i$.

A system of equations of the form

$$\begin{aligned}a_1 \cdot x + b_1 \cdot y &= c_1 \\ a_2 \cdot x + b_2 \cdot y &= c_2\end{aligned}$$

has the solution

$$x = \frac{b_1 c_2 - c_1 b_2}{a_1 b_2 - b_1 a_2}, \qquad y = \frac{c_1 a_2 - a_1 c_2}{a_1 b_2 - b_1 a_2}$$

which can be applied to our system of equations. The solutions for $\breve{\sigma}_i$ are

$$\breve{\sigma}_1(k,\overline{\omega}) = \frac{N_1(k,\overline{\omega})}{D_1(k,\overline{\omega})}, \qquad \breve{\sigma}_2(k,\overline{\omega}) = \frac{N_2(k,\overline{\omega})}{D_2(k,\overline{\omega})} \tag{3.22}$$

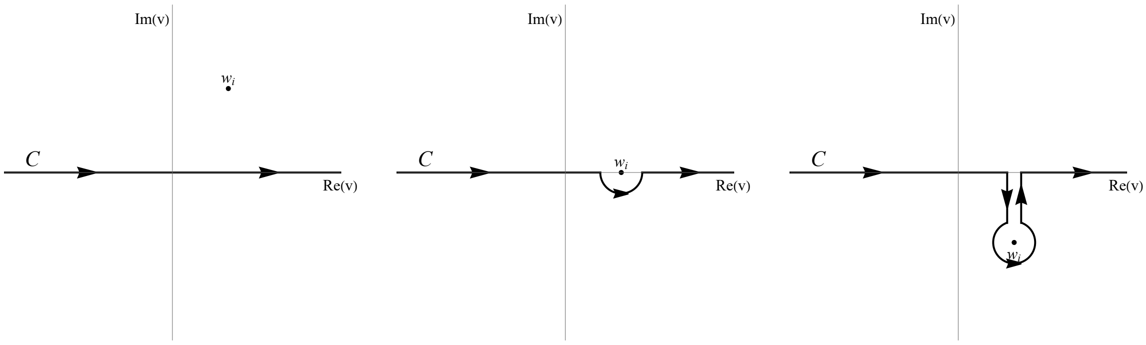
with

$$\begin{aligned}
N_1(k,\overline{\omega}) = & \left(- \int_{\mathbb{R}} \frac{\mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \\
& \left(\int_{\mathbb{R}} \frac{-\mathrm{i}[\nu_{22}\hat{\sigma}_2(k,0) + \nu_{21}\hat{\sigma}_1(k,0)]m_2 \frac{v}{kT_0} + \hat{h}_2(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right) \\
& + \left(\int_{\mathbb{R}} \frac{\mathrm{i}\lambda_1 m_1 \frac{v}{kT_0}\ \hat{\sigma}_1(k,0) - \hat{h}_1(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \\
& \left(1 - \int_{\mathbb{R}} \frac{\lambda_2 + \nu_{22} m_2 \overline{\omega} \frac{v}{kT_0} - \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right) \\
N_2(k,\overline{\omega}) = & \left(1 - \int_{\mathbb{R}} \frac{\lambda_1 + \lambda_1 m_1 \overline{\omega} \frac{v}{kT_0} - \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \\
& \left(\int_{\mathbb{R}} \frac{\mathrm{i}[\nu_{22}\hat{\sigma}_2(k,0) + \nu_{21}\hat{\sigma}_1(k,0)]m_2 \frac{v}{kT_0} - \hat{h}_2(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right) \\
& + \left(\int_{\mathbb{R}} \frac{\mathrm{i}\lambda_1 m_1 \frac{v}{kT_0}\ \hat{\sigma}_1(k,0) - \hat{h}_1(k,v,0)}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \\
& \left(\int_{\mathbb{R}} \frac{\nu_{21} m_2 \overline{\omega} \frac{v}{kT_0} + \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right) \\
D(k,\overline{\omega}) := & D_1(k,\overline{\omega}) = D_2(k,\overline{\omega}) = \left(1 - \int_{\mathbb{R}} \frac{\lambda_1 + \lambda_1 m_1 \overline{\omega} \frac{v}{kT_0} - \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \\
& \left(1 - \int_{\mathbb{R}} \frac{\lambda_2 + \nu_{22} m_2 \overline{\omega} \frac{v}{kT_0} - \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right) \\
& + \left(\int_{\mathbb{R}} \frac{\mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}}(v)dv \right) \cdot \left(- \int_{\mathbb{R}} \frac{\nu_{21} m_2 \overline{\omega} \frac{v}{kT_0} + \mathrm{i}e^2 \frac{v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}}(v)dv \right)
\end{aligned}$$

and the electrical field (3.18) takes the form of

$$\breve{E}(k,\overline{\omega}) = \frac{e}{\mathrm{i}k\epsilon_0}(\breve{\sigma}_1(k,\overline{\omega}) - \breve{\sigma}_2(k,\overline{\omega})) = \frac{e}{\mathrm{i}k\epsilon_0} \cdot \frac{N_1(k,\overline{\omega}) - N_2(k,\overline{\omega})}{D(k,\overline{\omega})}.$$

$D(k,\overline{\omega}) = 0$ is called the dispersion relation because one can rewrite this equation and get a relation between $\overline{\omega}$ and k.



(a) $\mathrm{Im}(w_i) > 0, \forall i \in \{1,2\}$ (b) $\exists i = \{1,2\} : \mathrm{Im}(w_i) = 0$ (c) $\exists i = \{1,2\} : \mathrm{Im}(w_i) < 0$

Fig. 1: Integration paths of the integrals in $\breve{E}$ provided that $k > 0$ so that the residue Theorem 2.2 is applicable. One must distinguish between the cases $\mathrm{Im}(w_i) = \mathrm{Im}(\frac{\overline{\omega}+\mathrm{i}n_0\lambda_i}{k}) > 0, \forall i \in \{1,2\}$ and $\exists i = \{1,2\} : \mathrm{Im}(w_i) \leq 0$. If $k < 0$ the integration paths must pass the pole w_i from above.

3.2.5 Inverse Laplace Transform of the Electric Field

The intention of this work is to show the damping phenomenon of the electric field. Therefore we want to apply the inverse Laplace transform to get a time-dependent expression in order to investigate the long-term behaviour. According to Theorem 2.1 $\breve{E}$ has to fulfil some requirements, namely the analyticity for $\mathrm{Re}(s) = b = \mathrm{Im}(\overline{\omega}) > a$.

It is obvious that for $\mathrm{Im}(w_i) := \mathrm{Im}(\frac{\overline{\omega}+\mathrm{i}n_0\lambda_i}{k}) > 0, \forall i = \{1,2\}$, $\breve{E}$ is analytic because $f_i^{\mathrm{equ}}(v)$ is analytic and $N_1(k,\overline{\omega})$, $N_2(k,\overline{\omega})$ and $D(k,\overline{\omega})$ are well-defined as the denominators $-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_i = \mathrm{i}k(v - \frac{\overline{\omega}+\mathrm{i}n_0\lambda_i}{k})$ do not vanish along the integration path, the real axis. Therefore we can apply the inverse transform with the lower bound $a \geq -n_0\lambda_i$.

Since we are interested in an explicit expression for the electric field after applying the inverse Laplace transform we need to use a tool to calculate the integral, namely the residue theorem, see Chapter 2.6. The condition for the application of Theorem 2.2 is the analyticity of the function except for a finite number of distinct isolated singularities. If there exists at least one $i \in \{1,2\}$ such that $\mathrm{Im}(w_i) \leq 0$ we have to perform an analytical continuation of $N_1(k,\overline{\omega})$, $N_2(k,\overline{\omega})$ and $D(k,\overline{\omega})$ like it was for example done by Prof. Dr. Eric Sonnendrücker in [31]. One redefines the integration path C by deforming the path along the real axis at the poles in such a way that all w_i are located above the integration contour. The new path is sketched in Figure 1. For $k > 0$ the line should pass below the poles of the denominator and for $k < 0$ the integration path should lie above the poles. Therefore the expression for $\breve{E}(k,\overline{\omega})$ exists and can be analytically continued.

Another condition for the residue theorem is that we need to modify the integration path so that it is a closed contour γ as shown in Figure 2. We take a line parallel to the real axis which lies above all singularities $\overline{\omega}_1, ..., \overline{\omega}_j$ of $\breve{E}(k,\overline{\omega})\exp(-\mathrm{i}\overline{\omega}t)$ and close the path by adding a semi-sphere with infinite radius so that all singularities are located inside of the contour. If we assume the analyticity of $\breve{E}$ apart from the singularities we

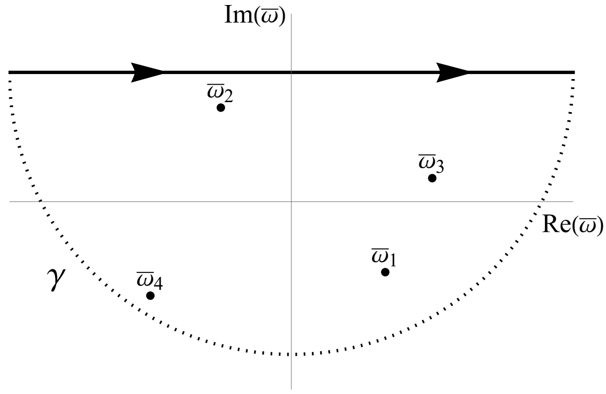


Fig. 2: Integration path of the inverse Laplace transform integral. All singularities $\overline{\omega}_1, ..., \overline{\omega}_j$ of $\breve{E}(k, \overline{\omega}) \exp(-\mathrm{i}\overline{\omega}t)$ must be located inside of the contour. The radius of the semi-sphere goes to infinity.

get the expression of the Fourier transformed electric field [31]

$$\begin{aligned}\hat{E}(k,t) &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty+\mathrm{i}a}^{\infty+\mathrm{i}a} \breve{E}(k,\overline{\omega}) \exp(-\mathrm{i}\overline{\omega}t) d\overline{\omega} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \breve{E}(k,\overline{\omega}) \exp(-\mathrm{i}\overline{\omega}t) d\overline{\omega} \\ &= \sum_{j=1}^{n} \mathrm{Res}_{\overline{\omega}=\overline{\omega}_j} \left(\breve{E}(k,\overline{\omega}) \exp(-\mathrm{i}\overline{\omega}t) \right) = \sum_{j=1}^{n} \mathrm{Res}_{\overline{\omega}=\overline{\omega}_j} \left(\breve{E}(k,\overline{\omega}) \right) \exp(-\mathrm{i}\overline{\omega}_j t)\end{aligned} \tag{3.23}$$

where the winding numbers of all singularities is one. Let $\overline{\omega}_k$ be the singularity with the largest imaginary part and write $\overline{\omega}_k =: \omega + \mathrm{i}\gamma$ where $\omega, \gamma \in \mathbb{R}$. Letting the time t go to infinity one gets the asymptotic behaviour of the electric field

$$\hat{E}(k,t) \longrightarrow \mathrm{Res}_{\overline{\omega}=\overline{\omega}_k} \left(\breve{E}(k,\overline{\omega}) \right) \exp(-\mathrm{i}\omega t) \exp(\gamma t)$$

which means that γ indicates the long-term behaviour. If $\gamma < 0$ the Fourier component of the electric field gets damped while otherwise the value might explode for $t \longrightarrow \infty$. As we want to get this property of the singularity $\overline{\omega}_k$ we investigate in the following the dispersion relation since the singularities $\overline{\omega}_j$ correspond exactly to the zeros of $D(k, \overline{\omega})$.

3.2.6 Analytical Calculation of the Damping Effect

Let us recall the dispersion relation

$$
\begin{aligned}
&D(k,\overline{\omega}) = 0\\
\iff &\left(1 - \int_{\mathbb{R}} \frac{\lambda_1 + \frac{\lambda_1 m_1 \overline{\omega} v}{kT_0} - \mathrm{i}\frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}} dv\right) \cdot \left(1 - \int_{\mathbb{R}} \frac{\lambda_2 + \frac{\nu_{22} m_2 \overline{\omega} v}{kT_0} - \mathrm{i}\frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}} dv\right)\\
&+ \left(\int_{\mathbb{R}} \frac{\mathrm{i}\frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} f_1^{\mathrm{equ}} dv\right) \cdot \left(- \int_{\mathbb{R}} \frac{\frac{\nu_{21} m_2 \overline{\omega} v}{kT_0} + \mathrm{i}\frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} f_2^{\mathrm{equ}} dv\right) = 0
\end{aligned} \tag{3.24}
$$

with the equilibrium distributions

$$
f_i^{\mathrm{equ}}(v) = \frac{n_0}{(2\pi T_0/m_i)^{1/2}} \exp\left(-\frac{v^2}{2T_0/m_i}\right). \tag{3.25}
$$

The quantities $\sqrt{\frac{T_0}{m_i}}$ can be expressed as the thermal velocities $v_{\mathrm{th},1}$ of ions and $v_{\mathrm{th},2}$ of electrons. Next we perform the substitution

$$
u_i := \frac{v}{\sqrt{2} v_{\mathrm{th},i}}
$$

in the integrals in (3.24) so that the arguments of the exponential functions get simplified and get

$$
\begin{aligned}
&\left(1 - \frac{n_0}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\lambda_1 + \frac{\sqrt{2}\lambda_1 \overline{\omega} u_1}{k v_{\mathrm{th},1}} - \mathrm{i}\frac{\sqrt{2} e^2 u_1}{\epsilon_0 m_1 k v_{\mathrm{th},1}}}{-\mathrm{i}\overline{\omega} + \mathrm{i}\sqrt{2} k v_{\mathrm{th},1} u_1 + n_0\lambda_1} \exp(-u_1^2) du_1\right) \cdot\\
&\left(1 - \frac{n_0}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\lambda_2 + \frac{\sqrt{2}\nu_{22} \overline{\omega} u_2}{k v_{\mathrm{th},2}} - \mathrm{i}\frac{\sqrt{2} e^2 u_2}{\epsilon_0 m_2 k v_{\mathrm{th},2}}}{-\mathrm{i}\overline{\omega} + \mathrm{i}\sqrt{2} k v_{\mathrm{th},2} u_2 + n_0\lambda_2} \exp(-u_2^2) du_2\right)\\
+&\left(\frac{n_0}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\mathrm{i}\frac{\sqrt{2} e^2 u_1}{\epsilon_0 m_1 k v_{\mathrm{th},1}}}{-\mathrm{i}\overline{\omega} + \mathrm{i}\sqrt{2} k v_{\mathrm{th},1} u_1 + n_0\lambda_1} \exp(-u_1^2) du_1\right) \cdot\\
&\left(-\frac{n_0}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{\frac{\sqrt{2}\nu_{21} \overline{\omega} u_2}{k v_{\mathrm{th},2}} + \mathrm{i}\frac{\sqrt{2} e^2 u_2}{\epsilon_0 m_2 k v_{\mathrm{th},2}}}{-\mathrm{i}\overline{\omega} + \mathrm{i}\sqrt{2} k v_{\mathrm{th},2} u_2 + n_0\lambda_2} \exp(-u_2^2) du_2\right) = 0.
\end{aligned}
$$

To simplify the denominators we use the notation

$$
X_i := \frac{\mathrm{i}\overline{\omega} - n_0\lambda_i}{\sqrt{2}\mathrm{i} k v_{\mathrm{th},i}}
$$

which gives

$$
\begin{aligned}
&\left(1+\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},1}}\int_{\mathbb{R}}\left(\lambda_1+\frac{\sqrt{2}\lambda_1\overline{\omega}u_1}{kv_{\mathrm{th},1}}-\mathrm{i}\frac{\sqrt{2}e^2u_1}{\epsilon_0 m_1 kv_{\mathrm{th},1}}\right)\frac{\exp(-u_1^2)}{u_1-X_1}du_1\right)\cdot\\
&\left(1+\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},2}}\int_{\mathbb{R}}\left(\lambda_2+\frac{\sqrt{2}\nu_{22}\overline{\omega}u_2}{kv_{\mathrm{th},2}}-\mathrm{i}\frac{\sqrt{2}e^2u_2}{\epsilon_0 m_2 kv_{\mathrm{th},2}}\right)\frac{\exp(-u_2^2)}{u_2-X_2}du_2\right)\\
+&\left(-\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},1}}\int_{\mathbb{R}}\left(\mathrm{i}\frac{\sqrt{2}e^2u_1}{\epsilon_0 m_1 kv_{\mathrm{th},1}}\right)\frac{\exp(-u_1^2)}{u_1-X_1}du_1\right)\cdot\\
&\left(\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},2}}\int_{\mathbb{R}}\left(\frac{\sqrt{2}\nu_{21}\overline{\omega}u_2}{kv_{\mathrm{th},2}}+\mathrm{i}\frac{\sqrt{2}e^2u_2}{\epsilon_0 m_2 kv_{\mathrm{th},2}}\right)\frac{\exp(-u_2^2)}{u_2-X_2}du_2\right)=0
\end{aligned}
$$

As we are interested in the long-behaviour we can insert the definition of $\overline{\omega}_k=\omega+\mathrm{i}\gamma$ which was the singularity with the largest imaginary part γ. With this we can split X_i in a real and an imaginary part

$$X_i=\frac{\omega}{\sqrt{2}kv_{\mathrm{th},i}}+\mathrm{i}\frac{n_0\lambda_i+\gamma}{\sqrt{2}kv_{\mathrm{th},i}}=:\eta_i+\mathrm{i}\xi_i \tag{3.26}$$

where $\eta_i,\xi_i\in\mathbb{R}$. To get rid of the constant for the following calculations we rewrite

$$
\begin{aligned}
A_1&=\frac{\sqrt{2}e^2}{\epsilon_0 m_1 kv_{\mathrm{th},1}}, & B_1&=\frac{\sqrt{2}\lambda_1\overline{\omega}}{kv_{\mathrm{th},1}}\\
A_2&=\frac{\sqrt{2}e^2}{\epsilon_0 m_2 kv_{\mathrm{th},2}}, & B_2&=\frac{\sqrt{2}\nu_{22}\overline{\omega}}{kv_{\mathrm{th},2}}, & C_2&=\frac{\sqrt{2}\nu_{21}\overline{\omega}}{kv_{\mathrm{th},2}}
\end{aligned}
$$

and substitute

$$
\begin{aligned}
g_1(u_1)&=\exp(-u_1^2), & G_1(u_1)&=u_1\cdot\exp(-u_1^2)\\
g_2(u_2)&=\exp(-u_2^2), & G_2(u_2)&=u_2\cdot\exp(-u_2^2).
\end{aligned}
$$

With this we can rewrite the dispersion relation as

$$
\begin{aligned}
&\left(1+\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},1}}\left[\lambda_1\int_{\mathbb{R}}\frac{g_1(u_1)}{u_1-X_1}du_1+(B_1-\mathrm{i}A_1)\int_{\mathbb{R}}\frac{G_1(u_1)}{u_1-X_1}du_1\right]\right)\cdot\\
&\left(1+\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},2}}\left[\lambda_2\int_{\mathbb{R}}\frac{g_2(u_2)}{u_2-X_2}du_2+(B_2-\mathrm{i}A_2)\int_{\mathbb{R}}\frac{G_2(u_2)}{u_2-X_2}du_2\right]\right)\\
+&\left(-\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},1}}\mathrm{i}A_1\int_{\mathbb{R}}\frac{G_1(u_1)}{u_1-X_1}du_1\right)\cdot\left(\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\mathrm{th},2}}(C_2+\mathrm{i}A_2)\int_{\mathbb{R}}\frac{G_2(u_2)}{u_2-X_2}du_2\right)=0.
\end{aligned}
\tag{3.27}
$$

For every single integral in this expression we can apply Jackson's identity 2.16 since they have the form

$$I(X) = \int_{\mathbb{R}} \frac{g(u)}{u - X} du$$

with complex X. For each such an expression it holds

$$I(X) = I(\eta + \mathrm{i}\xi) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\xi)^n}{n!} \left(\mathcal{P} \int_{\mathbb{R}} \frac{g^{(n)}(u)}{u - \eta} du + \mathrm{i}\pi g^{(n)}(\eta) \right)$$

where $g^{(n)}$ is the n^{th} derivative and $\mathcal{P}$ is the Cauchy principal value.

Just like Landau we consider the limiting case of long waves, such that $k \longrightarrow 0$, and also assume that $|\gamma|$ is small. This is why we restrict ourselves to the first order of Jackson's identity and get for the dispersion relation

$$\begin{aligned}
&\Bigg(1 + \frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\text{th},1}} \Bigg[\lambda_1 \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{g_1(u_1)}{u_1 - \eta_1} du_1 + \mathrm{i}\pi g_1(\eta_1) + \mathrm{i}\xi_1 \mathcal{P} \int_{\mathbb{R}} \frac{g_1'(u_1)}{u_1 - \eta_1} du_1 - \pi\xi_1 g_1'(\eta_1) \Bigg) \\
&\quad + (B_1 - \mathrm{i}A_1) \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{G_1(u_1)}{u_1 - \eta_1} du_1 + \mathrm{i}\pi G_1(\eta_1) + \mathrm{i}\xi_1 \mathcal{P} \int_{\mathbb{R}} \frac{G_1'(u_1)}{u_1 - \eta_1} du_1 - \pi\xi_1 G_1'(\eta_1) \Bigg) \Bigg] \Bigg) \cdot \\
&\Bigg(1 + \frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\text{th},2}} \Bigg[\lambda_2 \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{g_2(u_2)}{u_2 - \eta_2} du_2 + \mathrm{i}\pi g_2(\eta_2) + \mathrm{i}\xi_2 \mathcal{P} \int_{\mathbb{R}} \frac{g_2'(u_2)}{u_2 - \eta_2} du_2 - \pi\xi_2 g_2'(\eta_2) \Bigg) \\
&\quad + (B_2 - \mathrm{i}A_2) \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{G_2(u_2)}{u_2 - \eta_2} du_2 + \mathrm{i}\pi G_2(\eta_2) + \mathrm{i}\xi_2 \mathcal{P} \int_{\mathbb{R}} \frac{G_2'(u_2)}{u_2 - \eta_2} du_2 - \pi\xi_2 G_2'(\eta_2) \Bigg) \Bigg] \Bigg) \\
&+ \Bigg(- \frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\text{th},1}} \mathrm{i}A_1 \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{G_1(u_1)}{u_1 - \eta_1} du_1 + \mathrm{i}\pi G_1(\eta_1) + \mathrm{i}\xi_1 \mathcal{P} \int_{\mathbb{R}} \frac{G_1'(u_1)}{u_1 - \eta_1} du_1 \\
&\quad - \pi\xi_1 G_1'(\eta_1) \Bigg) \Bigg) \cdot \Bigg(\frac{\mathrm{i}n_0}{\sqrt{2\pi}kv_{\text{th},2}} (C_2 + \mathrm{i}A_2) \Bigg(\mathcal{P} \int_{\mathbb{R}} \frac{G_2(u_2)}{u_2 - \eta_2} du_2 + \mathrm{i}\pi G_2(\eta_2) \\
&\quad + \mathrm{i}\xi_2 \mathcal{P} \int_{\mathbb{R}} \frac{G_2'(u_2)}{u_2 - \eta_2} du_2 - \pi\xi_2 G_2'(\eta_2) \Bigg) \Bigg) = 0.
\end{aligned} \tag{3.28}$$

We can express the derivatives by

$$\begin{aligned}
g_i'(u_i) &= -2u_i \exp(-u_i^2) = -2G_i(u_i) \\
G_i'(u_i) &= (1 - 2u_i^2) \exp(-u_i^2).
\end{aligned}$$

To calculate the values of the integrals we use the definition of the Gamma function 2.17.

Consider the Cauchy principal value integrals separately. By means of the geometric series one gets

$$\mathcal{P}\int_{\mathbb{R}} \frac{g_i(u_i)}{u_i-\eta_i}du_i = -\frac{1}{\eta_i}\mathcal{P}\int_{\mathbb{R}} \frac{\exp(-u_i^2)}{1-\frac{u_i}{\eta_i}}du_i = -\frac{1}{\eta_i}\int_{-\infty}^{\infty} \exp(-u_i^2)\sum_{n=0}^{\infty}\left(\frac{u_i}{\eta_i}\right)^n du_i$$

where the principal value integral has a finite value for η_1 and η_2 sufficiently large since

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\exp(-u^2)}{u-\eta}du \overset{v:=u-\eta}{=} \int_{-\infty}^{\infty} \frac{\exp(-(v+\eta)^2)}{v}dv \\
&= \exp(-\eta^2)\left(\int_{-\infty}^{0} \frac{1}{v}\exp(-v^2)\exp(-2\eta v)dv + \int_{0}^{\infty} \frac{1}{v}\exp(-v^2)\exp(-2\eta v)dv\right) \\
&= \exp(-\eta^2)\int_{0}^{\infty} \frac{\exp(-v^2)}{v}\left(\exp(-2\eta v)-\exp(2\eta v)\right)dv \\
&= -2\exp(-\eta^2)\int_{0}^{\infty} \frac{\sinh(2\eta v)}{v}\exp(-v^2)dv \\
&= -2\exp(-\eta^2)\int_{0}^{\infty} \left(2\eta + \frac{(2\eta)^3 v^2}{3!} + ...\right)\exp(-v^2)dv
\end{aligned}$$

which is an analytic integral and has no poles. In the last step we used the series expansion of the hyperbolic sine. As $\sinh(2\eta v)$ behaves like the linear function $2\eta v$ around $v=0$ and with the exponential function $\exp(-\eta^2)$ we can say that the value of the integral is finite for large η.

For our Cauchy principal value we can further use the integral representation of the Gamma function and the integer formulas in Definition 2.17 and get

$$\begin{aligned}
&\mathcal{P}\int_{\mathbb{R}} \frac{g_i(u_i)}{u_i-\eta_i}du_i = -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\int_{-\infty}^{\infty} u_i^n \exp(-u_i^2)du_i \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left((-1)^n\int_{0}^{\infty} u_i^n\exp(-u_i^2)du_i + \int_{0}^{\infty} u_i^n\exp(-u_i^2)du_i\right) \\
&\overset{w_i:=u_i^2}{=} -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^n\int_{0}^{\infty} w_i^{\frac{n-1}{2}}\exp(-w_i)dw_i + \frac{1}{2}\int_{0}^{\infty} w_i^{\frac{n-1}{2}}\exp(-w_i)dw_i\right) \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^n\Gamma\left(\frac{n+1}{2}\right) + \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\right) \\
&= -\frac{\sqrt{\pi}}{\eta_i}\left(1+\frac{1}{2\eta_i^2}+\frac{3}{4\eta_i^4}+\mathcal{O}\left(\frac{1}{\eta_i^6}\right)\right)
\end{aligned}$$

Analogously one calculates

$$
\begin{aligned}
\mathcal{P}\int_{\mathbb{R}} \frac{G_i(u_i)}{u_i-\eta_i}du_i &= -\frac{1}{\eta_i}\mathcal{P}\int_{\mathbb{R}} \frac{u_i\exp(-u_i^2)}{1-\frac{u_i}{\eta_i}}du_i = -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\int_{-\infty}^{\infty} u_i^{n+1}\exp(-u_i^2)du_i \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^{n+1}\int_0^{\infty} w_i^{\frac{n}{2}}\exp(-w_i)dw_i + \frac{1}{2}\int_0^{\infty} w_i^{\frac{n}{2}}\exp(-w_i)dw_i\right) \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^{n+1}\Gamma\left(\frac{n+2}{2}\right)+\frac{1}{2}\Gamma\left(\frac{n+2}{2}\right)\right) \\
&= -\frac{\sqrt{\pi}}{\eta_i}\left(\frac{1}{2\eta_i}+\frac{3}{4\eta_i^3}+\frac{15}{8\eta_i^5}+\mathcal{O}\left(\frac{1}{\eta_i^7}\right)\right) \\
\mathcal{P}\int_{\mathbb{R}} \frac{g_i'(u_i)}{u_i-\eta_i}du_i &= -2\mathcal{P}\int_{\mathbb{R}} \frac{G_i(u_i)}{u_i-\eta_i}du_i = \frac{\sqrt{\pi}}{\eta_i}\left(\frac{1}{\eta_i}+\frac{3}{2\eta_i^3}+\frac{15}{4\eta_i^5}+\mathcal{O}\left(\frac{1}{\eta_i^7}\right)\right) \\
\mathcal{P}\int_{\mathbb{R}} \frac{G_i'(u_i)}{u_i-\eta_i}du_i &= \mathcal{P}\int_{\mathbb{R}} \frac{g_i(u_i)}{u_i-\eta_i}du_i - 2\mathcal{P}\int_{\mathbb{R}} \frac{u_i^2\exp(-u_i^2)}{u_i-\eta_i}du_i \\
&= \frac{\sqrt{\pi}}{\eta_i}\left(\frac{1}{\eta_i^2}+\frac{3}{\eta_i^4}+\mathcal{O}\left(\frac{1}{\eta_i^6}\right)\right)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{P}\int_{\mathbb{R}} \frac{u_i^2\exp(-u_i^2)}{u_i-\eta_i}du_i &= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\int_{-\infty}^{\infty} u_i^{n+2}\exp(-u_i^2)du_i \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^{n+2}\int_0^{\infty} w_i^{\frac{n+1}{2}}\exp(-w_i)dw_i + \frac{1}{2}\int_0^{\infty} w_i^{\frac{n+1}{2}}\exp(-w_i)dw_i\right) \\
&= -\sum_{n=0}^{\infty}\frac{1}{\eta_i^{n+1}}\left(\frac{1}{2}(-1)^{n+2}\Gamma\left(\frac{n+3}{2}\right)+\frac{1}{2}\Gamma\left(\frac{n+3}{2}\right)\right) \\
&= -\frac{\sqrt{\pi}}{\eta_i}\left(\frac{1}{2}+\frac{3}{4\eta_i^2}+\frac{15}{8\eta_i^4}+\mathcal{O}\left(\frac{1}{\eta_i^6}\right)\right).
\end{aligned}
$$

We can now insert the principal values in the dispersion relation (3.28). As we consider large values for η_1 and η_2 we only take terms up to order η_i^{-4} into account and get after simplifying

$$
\begin{aligned}
&\frac{\exp(-\eta_1^2-\eta_2^2)}{64k^2\eta_1^4\eta_2^4 v_{\mathrm{th},1}v_{\mathrm{th},2}}\Bigg\{\Bigg(\exp(\eta_1^2)\Bigg[8k\eta_1^4 v_{\mathrm{th},1}-\sqrt{2}n_0\Bigg(A_1\left(2\eta_1^2-4\mathrm{i}\eta_1\xi_1+3\right)\\
&\quad+B_1\left(2\mathrm{i}\eta_1^2+4\eta_1\xi_1+3\mathrm{i}\right)+2\lambda_1\left(2\mathrm{i}\eta_1^3+2\eta_1^2\xi_1+\mathrm{i}\eta_1+3\xi_1\right)\Bigg)\Bigg]\\
&\quad+4\sqrt{2\pi}\eta_1^4 n_0\Bigg[(A_1+\mathrm{i}B_1)\left(2\eta_1^2\xi_1+\mathrm{i}\eta_1-\xi_1\right)+\lambda_1(-1+2\mathrm{i}\eta_1\xi_1)\Bigg]\Bigg)\cdot\\
&\Bigg(\exp(\eta_2^2)\Bigg[8k\eta_2^4 v_{\mathrm{th},2}-\sqrt{2}n_0\Bigg(A_2\left(2\eta_2^2-4\mathrm{i}\eta_2\xi_2+3\right)+B_2\left(2\mathrm{i}\eta_2^2+4\eta_2\xi_2+3\mathrm{i}\right)\\
&\quad+2\lambda_2\left(2\mathrm{i}\eta_2^3+2\eta_2^2\xi_2+\mathrm{i}\eta_2+3\xi_2\right)\Bigg)\Bigg]+4\sqrt{2\pi}\eta_2^4 n_0\Bigg[(A_2+\mathrm{i}B_2)\left(2\eta_2^2\xi_2+\mathrm{i}\eta_2-\xi_2\right)\\
&\quad+\lambda_2(-1+2\mathrm{i}\eta_2\xi_2)\Bigg]\Bigg)\\
&-2A_1 n_0^2(A_2-\mathrm{i}C_2)\Bigg(\exp(\eta_1^2)\left(-2\eta_1^2+4\mathrm{i}\eta_1\xi_1-3\right)+4\sqrt{\pi}\eta_1^4\left(2\eta_1^2\xi_1+\mathrm{i}\eta_1-\xi_1\right)\Bigg)\cdot\\
&\Bigg(\exp(\eta_2^2)\left(-2\eta_2^2+4\mathrm{i}\eta_2\xi_2-3\right)+4\sqrt{\pi}\eta_2^4\left(2\eta_2^2\xi_2+\mathrm{i}\eta_2-\xi_2\right)\Bigg)\Bigg\}=0
\end{aligned}
\tag{3.29}
$$

The expanded expression of (3.29) can be found in the Appendix A.1.

We want to examine the real part of the expanded expression of the dispersion relation in Appendix A.1. After inserting the definition for ξ_1 and ξ_2 from (3.26) we can neglect some of the summands. For one thing we consider only small collision frequencies such that terms with higher order than linear in ν_{ii} and ν_{ij} are neglectable. This includes also λ_i as $\lambda_i = \nu_{ii} + \nu_{ij}$ was introduced to simplify terms containing both of the collision frequencies. Since γ is also a small quantity we can further neglect terms with γ^2 or higher order and $\nu_{ij} \cdot \gamma$ [2]. In addition as $\eta_i^2 = \frac{\omega^2}{2k^2 v_{\mathrm{th},i}^2}$ becomes large for large wavelengths we can delete terms with $\exp(-\eta_1^2)$ and $\exp(-\eta_2^2)$. We get for the real part of the dispersion relation

$$1-\frac{3n_0e^2}{4\epsilon_0 m_1 k^2 v_{\mathrm{th},1}^2\eta_1^4}-\frac{n_0e^2}{2\epsilon_0 m_1 k^2 v_{\mathrm{th},1}^2\eta_1^2}-\frac{3n_0e^2}{4\epsilon_0 m_2 k^2 v_{\mathrm{th},2}^2\eta_2^4}-\frac{n_0e^2}{2\epsilon_0 m_2 k^2 v_{\mathrm{th},2}^2\eta_2^2}=0. \tag{3.30}$$

With the definition of η_i and the so-called plasma oscillation frequency $\omega_{\mathrm{p},i} = \sqrt{\frac{n_0e^2}{\epsilon_0 m_i}}$ we obtain

$$1-\frac{3k^2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2}{\omega^4}-\frac{\omega_{\mathrm{p},1}^2}{\omega^2}-\frac{3k^2\omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2}{\omega^4}-\frac{\omega_{\mathrm{p},2}^2}{\omega^2}=0 \tag{3.31}$$

Solving this equation for ω^2 and performing a Taylor expansion around $k = 0$ one has the relation

$$\omega^2 = \left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right) + \frac{3\left(\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 + \omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2\right)}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} k^2 + O\left(k^3\right) \tag{3.32}$$

which is related to the Bohm-Gross dispersion relation $\omega^2 \approx \omega_{\mathrm{p}}^2 + 3k^2 v_{\mathrm{th}}^2$ for longitudinal electron plasma waves. This equation was already introduced by Landau in [19] and could also be determined in Lena Baumann's one-species model [2].

We can use this relation now for the imaginary part of the expanded dispersion relation in Appendix A.1. Like before we insert the definition of ξ_i, η_i and $\omega_{\mathrm{p},i}$ and neglect terms with order two or higher in the collision frequencies and γ. This gives

$$\begin{aligned}
&-\frac{\lambda_2 n_0}{\omega} - \frac{\nu_{22} n_0}{\omega} - \frac{2\lambda_1 n_0}{\omega} + \frac{2\gamma\omega_{\mathrm{p},1}^2}{\omega^3} + \frac{2\gamma\omega_{\mathrm{p},2}^2}{\omega^3} + \frac{\lambda_2\omega_{\mathrm{p},1}^2 n_0}{\omega^3} - \frac{4k^2\lambda_1 v_{\mathrm{th},1}^2 n_0}{\omega^3} - \frac{k^2\lambda_2 v_{\mathrm{th},2}^2 n_0}{\omega^3} \\
&- \frac{3k^2\nu_{22} v_{\mathrm{th},2}^2 n_0}{\omega^3} + \frac{2\lambda_1\omega_{\mathrm{p},1}^2 n_0}{\omega^3} + \frac{2\lambda_1\omega_{\mathrm{p},2}^2 n_0}{\omega^3} + \frac{2\lambda_2\omega_{\mathrm{p},2}^2 n_0}{\omega^3} + \frac{\nu_{22}\omega_{\mathrm{p},1}^2 n_0}{\omega^3} + \frac{\omega_{\mathrm{p},1}^2\nu_{21} n_0}{\omega^3} \\
&+ \frac{4k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 n_0}{\omega^5} + \frac{6k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2 n_0}{\omega^5} + \frac{3k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 n_0}{\omega^5} + \frac{k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^5} \\
&+ \frac{3k^2\nu_{22}\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 n_0}{\omega^5} + \frac{3k^2\nu_{22}\omega_{\mathrm{p},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^5} + \frac{3k^2\omega_{\mathrm{p},1}^2\nu_{21} v_{\mathrm{th},1}^2 n_0}{\omega^5} + \frac{3k^2\omega_{\mathrm{p},1}^2\nu_{21} v_{\mathrm{th},2}^2 n_0}{\omega^5} \\
&+ \frac{12k^4\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^7} + \frac{3k^4\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^7} \\
&+ \frac{9k^4\nu_{22}\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^7} + \frac{9k^4\omega_{\mathrm{p},1}^2\nu_{21} v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^7} \\
&+ \frac{\sqrt{\frac{\pi}{2}}\omega_{\mathrm{p},1}^2\omega\exp(-\frac{\omega^2}{2k^2 v_{\mathrm{th},1}^2})}{k^3 v_{\mathrm{th},1}^3} + \frac{\sqrt{\frac{\pi}{2}}\omega_{\mathrm{p},2}^2\omega\exp(-\frac{\omega^2}{2k^2 v_{\mathrm{th},2}^2})}{k^3 v_{\mathrm{th},2}^3} - \frac{\pi\lambda_2\omega_{\mathrm{p},1}^2\omega n_0\exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{2k^4 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}} \\
&- \frac{\pi\lambda_1\omega_{\mathrm{p},2}^2\omega n_0\exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{2k^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^3} - \frac{\pi\lambda_1\omega_{\mathrm{p},2}^2\omega^3 n_0\exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{2k^6 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}^3} \\
&- \frac{\pi\nu_{22}\omega_{\mathrm{p},1}^2\omega^3 n_0\exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{2k^6 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}^3} - \frac{\pi\omega_{\mathrm{p},1}^2\nu_{21}\omega^3 n_0\exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{2k^6 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}^3} = 0.
\end{aligned}$$

Solving this equation for γ and using $\nu_{21} = \lambda_2 - \nu_{22}$ one obtains

$$\begin{aligned}\gamma = &+ \frac{\lambda_1 \omega^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} + \frac{\lambda_2 \omega^2 n_0}{2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} + \frac{\nu_{22} \omega^2 n_0}{2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{\lambda_1 \omega_{\mathrm{p},1}^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} - \frac{\lambda_1 \omega_{\mathrm{p},2}^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} \\
&- \frac{\lambda_2 \omega_{\mathrm{p},1}^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} - \frac{\lambda_2 \omega_{\mathrm{p},2}^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} + \frac{2k^2 \lambda_1 v_{\mathrm{th},1}^2 n_0}{\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2} + \frac{k^2 \lambda_2 v_{\mathrm{th},2}^2 n_0}{2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} + \frac{3k^2 \nu_{22} v_{\mathrm{th},2}^2 n_0}{2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} \\
&- \frac{2k^2 \lambda_1 \omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 n_0}{\omega^2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{3k^2 \lambda_1 \omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2 n_0}{\omega^2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{3k^2 \lambda_2 \omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 n_0}{\omega^2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{2k^2 \lambda_2 \omega_{\mathrm{p},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^2\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} \\
&- \frac{6k^4 \lambda_1 \omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^4\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{6k^4 \lambda_2 \omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\omega^4\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} \\
&- \frac{\sqrt{\frac{\pi}{2}} \omega_{\mathrm{p},1}^2 \omega^4 \exp(-\frac{\omega^2}{2k^2 v_{\mathrm{th},1}^2})}{2k^3 v_{\mathrm{th},1}^3\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} - \frac{\sqrt{\frac{\pi}{2}} \omega_{\mathrm{p},2}^2 \omega^4 \exp(-\frac{\omega^2}{2k^2 v_{\mathrm{th},2}^2})}{2k^3 v_{\mathrm{th},2}^3\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} \\
&+ \frac{\pi \lambda_1 \omega_{\mathrm{p},2}^2 \omega^4 n_0 \exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{4k^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^3\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} + \frac{\pi \lambda_2 \omega_{\mathrm{p},1}^2 \omega^4 n_0 \exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{4k^4 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} \\
&+ \frac{\pi \lambda_1 \omega_{\mathrm{p},2}^2 \omega^6 n_0 \exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{4k^6 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}^3\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)} + \frac{\pi \lambda_2 \omega_{\mathrm{p},1}^2 \omega^6 n_0 \exp(-\frac{\omega^2\left(v_{\mathrm{th},1}^2 + v_{\mathrm{th},2}^2\right)}{2k^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2})}{4k^6 v_{\mathrm{th},1}^3 v_{\mathrm{th},2}^3\left(\omega_{\mathrm{p},1}^2 + \omega_{\mathrm{p},2}^2\right)}.\end{aligned} \tag{3.33}$$

This is an expression for the imaginary part of the singularity $\overline{\omega}_k$ with the largest imaginary part which means that γ shows the long-term behaviour of the electric field. The solution depends on the constraints of our system. Landau showed that in the case of the one-species Vlasov-Poisson system, where the movement of the particles is influenced by the electric field without considering collisions of the particles, the field is damped with $\gamma \approx -\sqrt{\frac{\pi}{8}} \frac{\omega_{\mathrm{p}}^4}{(k v_{\mathrm{th}})^3} \exp\left(-\frac{\omega^2}{2k^2 v_{\mathrm{th}}^2}\right) =: -\gamma_{\mathrm{L}}$ which is strictly negative [19]. Lena Baumann showed in her Master's thesis that if one expands the system by the BGK collision operator it also gets damped with $\gamma \approx -\gamma_{\mathrm{L}} - \frac{\nu k^2 v_{\mathrm{th}}^2}{\omega_{\mathrm{p}}^2} =: -\gamma_{\mathrm{L}} - \gamma_{\mathrm{BGK}}$ [2]. The damping effect for the two-species Vlasov equations without BGK-kernel was determined in [3] with the methods of characteristics and the Landau-Penrose stability criterion. In the two-species Vlasov-Poisson-BGK case (3.33) one can see that the solution for γ includes the Landau damping coefficient γ_{L} as well as the BGK collision relaxation γ_{BGK}, referred to the two species ions and electrons but also mixed terms.

It remains to prove the negative sign of γ for the two-species Vlasov-Poisson-BGK system. For this purpose we further proceed like with the real part and neglect the exponential terms. Furthermore we can use the expansion of ω^2 up to first order from (3.32) and the relation of the collision frequencies (3.4) which means that $\nu_{22} \approx \nu_{21}$ and

$\nu_{22} \approx \frac{1}{2}\lambda_2$. We get

$$
\begin{aligned}
\gamma =& -\frac{\lambda_2\omega_{\mathrm{p},1}^2 n_0}{4\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} - \frac{\lambda_2\omega_{\mathrm{p},2}^2 n_0}{4\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} + \frac{2k^2\lambda_1 v_{\mathrm{th},1}^2 n_0}{\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2} + \frac{5k^2\lambda_2 v_{\mathrm{th},2}^2 n_0}{4\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} \\
&- \frac{2k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2} - \frac{3k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2} - \frac{6k^4\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^3} \\
&- \frac{2k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},2}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2} - \frac{3k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2} - \frac{6k^4\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2 n_0}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^3} \\
=& \frac{n_0}{\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2}\Bigg(\left(1-\frac{\lambda_2\omega_{\mathrm{p},1}^2}{8k^2\lambda_1 v_{\mathrm{th},1}^2}\right)2k^2\lambda_1 v_{\mathrm{th},1}^2 + \left(1-\frac{\lambda_2\omega_{\mathrm{p},2}^2}{5k^2\lambda_2 v_{\mathrm{th},2}^2}\right)\frac{5k^2\lambda_2 v_{\mathrm{th},2}^2}{4} \\
&- \frac{2k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} - \frac{3k^2\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},2}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} - \frac{6k^4\lambda_1\omega_{\mathrm{p},2}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2} \\
&- \frac{2k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},2}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} - \frac{3k^2\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)} - \frac{6k^4\lambda_2\omega_{\mathrm{p},1}^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}{\left(\omega_{\mathrm{p},1}^2+\omega_{\mathrm{p},2}^2\right)^2}\Bigg).
\end{aligned}
$$

The first two summands with the factors $1-\frac{\lambda_2\omega_{\mathrm{p},i}^2}{8k^2\lambda_i v_{\mathrm{th},i}^2}$ are negative because the brackets only depend on physical constants describing the particles and the gas apart from k. Since we consider the case of long wavelengths such that $k \longrightarrow 0$ the fraction becomes large and the brackets become negative. In total we have a negative expression for γ which means that the damping effect of the electric field is proven. In addition we can say that the Vlasov-Poisson-BGK system experiences a stronger damping than the Vlasov-Poisson or the BGK system itself, since the damping coefficient consists of contributions of both parts, the electric field part and the BGK collision part, as well as further mixed contributions.

To investigate the magnitude of the damping effect of the two-species BGK model compared to Lena Baumann's one-species BGK model [2] we pursue in the following a numerical approach which means that we determine the zeros of the dispersion relation by means of a Python algorithm by Eric Sonnendrücker [31].

3.2.7 Numerical Calculation of the Damping Effect

Eric Sonnendrücker wrote a Python code that can be used to calculate the zeros of an analytical function [31]. He applied it to calculate the zeros of the dispersion relation of the linearised Vlasov-Poisson equation and therefore showed the linear Landau damping effect numerically. One can find the Python script in Appendix A.2 where the code is modified to be applicable to the two-species Vlasov-Poisson-BGK case.

To be able to use the code, the plasma dispersion function is introduced in the following, since we will have to express the dispersion functions by means of it. It was initially introduced by Fried and Conte to define an analytic continuation of integrals including linear combinations of Maxwellian functions which are often used in plasma physics [9].

Definition 3.2 (Plasma dispersion function, [31])**.** The *plasma dispersion function* is defined as

$$\mathcal{Z}_{\pm}(\zeta) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{-z^2}}{z-\zeta} dz$$

where C is any open contour parallel to the real axis at infinity that passes below the pole $z = \zeta$ for $\mathcal{Z}_-$ and above the pole for $\mathcal{Z}_+$.

There are several ways to rewrite the plasma dispersion function:

Lemma 3.1 ([31])**.** The plasma dispersion functions $\mathcal{Z}_-$ (respectively $\mathcal{Z}_+$) are independent of the contour C of the form $t \longmapsto t + \mathrm{i}u$ for $|t|$ sufficiently large and passing below (respectively above) the pole $z = \zeta$. Moreover we have the following expression for $\mathcal{Z}_{\pm}$:

$$\begin{aligned}\mathcal{Z}_{\pm}(\zeta) &= \frac{1}{\sqrt{\pi}} \left[\mathcal{P} \int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u} du + \mathrm{i}\pi e^{-\zeta^2} \right] \\ &= \sqrt{\pi} e^{-\zeta^2} \left[\mathrm{i} - \mathrm{erfi}(\zeta) \right]\end{aligned}$$

where $\mathrm{erfi}(\zeta)$ is the complex error function specified in Definition 2.18 and $\mathcal{P} \int_{\mathbb{R}} \frac{e^{-(u+\zeta)^2}}{u} du$ denotes the Cauchy principal value from Definition 2.15.

Recall the dispersion relation

$$\begin{aligned}&D(k, \overline{\omega}) = 0 \\ \iff &\left(1 - \int_{\mathbb{R}} \frac{\lambda_1 + \frac{\lambda_1 m_1 \overline{\omega} v}{k T_0} - \mathrm{i} \frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0 \lambda_1} f_1^{\mathrm{equ}} dv \right) \cdot \left(1 - \int_{\mathbb{R}} \frac{\lambda_2 + \frac{\nu_{22} m_2 \overline{\omega} v}{k T_0} - \mathrm{i} \frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0 \lambda_2} f_2^{\mathrm{equ}} dv \right) \\ &+ \left(\int_{\mathbb{R}} \frac{\mathrm{i} \frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0 \lambda_1} f_1^{\mathrm{equ}} dv \right) \cdot \left(- \int_{\mathbb{R}} \frac{\frac{\nu_{21} m_2 \overline{\omega} v}{k T_0} + \mathrm{i} \frac{e^2 v}{\epsilon_0 k T_0}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0 \lambda_2} f_2^{\mathrm{equ}} dv \right) = 0\end{aligned}$$

from the analytical calculation and insert the equilibrium functions

$$f_i^{\mathrm{equ}}(v) = \frac{n_0}{(2\pi T_0/m_i)^{1/2}} \exp\left(-\frac{v^2}{2T_0/m_i} \right), \qquad i = 1, 2$$

and the thermal velocities

$$v_{\mathrm{th},i} = \sqrt{\frac{T_0}{m_i}}, \qquad i = 1, 2.$$

This results in the equation

$$D(k,\overline{\omega}) = \left(1 - \frac{n_0}{\sqrt{2\pi}v_{\text{th},1}} \int_{\mathbb{R}} \frac{\lambda_1 + \frac{\lambda_1\overline{\omega}v}{kv_{\text{th},1}^2} - \mathrm{i}\frac{e^2 v}{\epsilon_0 m_1 k v_{\text{th},1}^2}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} \exp(-\frac{v^2}{2v_{\text{th},1}^2}) dv\right) \cdot$$
$$\left(1 - \frac{n_0}{\sqrt{2\pi}v_{\text{th},2}} \int_{\mathbb{R}} \frac{\lambda_2 + \frac{\nu_{22}\overline{\omega}v}{kv_{\text{th},2}^2} - \mathrm{i}\frac{e^2 v}{\epsilon_0 m_2 k v_{\text{th},2}^2}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} \exp(-\frac{v^2}{2v_{\text{th},2}^2}) dv\right)$$
$$+ \left(\frac{n_0}{\sqrt{2\pi}v_{\text{th},1}} \int_{\mathbb{R}} \frac{\mathrm{i}\frac{e^2 v}{\epsilon_0 m_1 k v_{\text{th},1}^2}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_1} \exp(-\frac{v^2}{2v_{\text{th},1}^2}) dv\right) \cdot$$
$$\left(-\frac{n_0}{\sqrt{2\pi}v_{\text{th},2}} \int_{\mathbb{R}} \frac{\frac{\nu_{21}\overline{\omega}v}{kv_{\text{th},2}^2} + \mathrm{i}\frac{e^2 v}{\epsilon_0 m_2 k v_{\text{th},2}^2}}{-\mathrm{i}\overline{\omega} + \mathrm{i}kv + n_0\lambda_2} \exp(-\frac{v^2}{2v_{\text{th},2}^2}) dv\right) = 0.$$

In the next step we divide the integrals into the single summands and extract the factor ik from the denominator which gives

$$D(k,\overline{\omega}) = \left(1 + \frac{\mathrm{i}n_0\lambda_1}{\sqrt{2\pi}kv_{\text{th},1}} \int_{\mathbb{R}} \frac{\exp(-\frac{v^2}{2v_{\text{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_1}{k}} dv + \frac{\mathrm{i}n_0\lambda_1\overline{\omega}}{\sqrt{2\pi}k^2 v_{\text{th},1}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_1}{k}} dv\right.$$
$$\left. + \frac{n_0 e^2}{\sqrt{2\pi}\epsilon_0 m_1 k^2 v_{\text{th},1}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_1}{k}} dv\right) \cdot$$
$$\left(1 + \frac{\mathrm{i}n_0\lambda_2}{\sqrt{2\pi}kv_{\text{th},2}} \int_{\mathbb{R}} \frac{\exp(-\frac{v^2}{2v_{\text{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_2}{k}} dv + \frac{\mathrm{i}n_0\nu_{22}\overline{\omega}}{\sqrt{2\pi}k^2 v_{\text{th},2}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_2}{k}} dv\right.$$
$$\left. + \frac{n_0 e^2}{\sqrt{2\pi}\epsilon_0 m_2 k^2 v_{\text{th},2}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_2}{k}} dv\right)$$
$$+ \left(\frac{n_0 e^2}{\sqrt{2\pi}\epsilon_0 m_1 k^2 v_{\text{th},1}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_1}{k}} dv\right) \cdot$$
$$\left(\frac{\mathrm{i}n_0\nu_{21}\overline{\omega}}{\sqrt{2\pi}k^2 v_{\text{th},2}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_2}{k}} dv - \frac{n_0 e^2}{\sqrt{2\pi}\epsilon_0 m_2 k^2 v_{\text{th},2}^3} \int_{\mathbb{R}} \frac{v\exp(-\frac{v^2}{2v_{\text{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{in_0\lambda_2}{k}} dv\right)$$
$$= 0$$

To express this equation by means of the plasma dispersion function and the Gaussian integral from Definition 2.19 we add a zero by $v = (v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_i}{k}) + (\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_i}{k})$ and get

$$\begin{aligned}
D(k,\overline{\omega}) = 0 = &\Bigg(1 + \frac{\mathrm{i}n_0\lambda_1}{\sqrt{2\pi}kv_{\mathrm{th},1}}\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_1}{k}}dv \\
&+ \frac{\mathrm{i}n_0\lambda_1\overline{\omega}}{\sqrt{2\pi}k^2v_{\mathrm{th},1}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_1}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_1}{k}}dv\right) \\
&+ \frac{n_0e^2}{\sqrt{2\pi}\epsilon_0m_1k^2v_{\mathrm{th},1}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_1}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_1}{k}}dv\right)\Bigg)\cdot \\
&\Bigg(1 + \frac{\mathrm{i}n_0\lambda_2}{\sqrt{2\pi}kv_{\mathrm{th},2}}\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_2}{k}}dv \\
&+ \frac{\mathrm{i}n_0\nu_{22}\overline{\omega}}{\sqrt{2\pi}k^2v_{\mathrm{th},2}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_2}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_2}{k}}dv\right) \\
&+ \frac{n_0e^2}{\sqrt{2\pi}\epsilon_0m_2k^2v_{\mathrm{th},2}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_2}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_2}{k}}dv\right)\Bigg) \\
&+\Bigg(\frac{n_0e^2}{\sqrt{2\pi}\epsilon_0m_1k^2v_{\mathrm{th},1}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_1}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},1}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_1}{k}}dv\right)\Bigg)\cdot \\
&\Bigg(\frac{\mathrm{i}n_0\nu_{21}\overline{\omega}}{\sqrt{2\pi}k^2v_{\mathrm{th},2}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_2}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_2}{k}}dv\right) \\
&- \frac{n_0e^2}{\sqrt{2\pi}\epsilon_0m_2k^2v_{\mathrm{th},2}^3}\left(\int_{\mathbb{R}}\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})dv + \left(\frac{\overline{\omega}}{k} + \frac{\mathrm{i}n_0\lambda_2}{k}\right)\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},2}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_2}{k}}dv\right)\Bigg)
\end{aligned}$$

By Definition 2.19 with $a = \frac{1}{2v_{\mathrm{th},i}^2}$ it yields

$$\int_{\mathbb{R}}\exp\left(-\frac{v^2}{2v_{\mathrm{th},i}^2}\right)dv = \sqrt{2\pi}v_{\mathrm{th},i}$$

and with $u_i = \frac{v}{\sqrt{2}v_{\mathrm{th},i}}$ one has

$$\begin{aligned}
\int_{\mathbb{R}}\frac{\exp(-\frac{v^2}{2v_{\mathrm{th},i}^2})}{v - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_i}{k}}dv &= \int_{\mathbb{R}}\frac{\sqrt{2}v_{\mathrm{th},i}\cdot\exp(-u_i^2)}{\sqrt{2}v_{\mathrm{th},i}u_i - \frac{\overline{\omega}}{k} - \frac{\mathrm{i}n_0\lambda_i}{k}}du_i = \int_{\mathbb{R}}\frac{\exp(-u_i^2)}{u_i - \frac{\overline{\omega}}{\sqrt{2}kv_{\mathrm{th},i}} - \frac{\mathrm{i}n_0\lambda_i}{\sqrt{2}kv_{\mathrm{th},i}}}du_i \\
&= \sqrt{\pi}\cdot\mathcal{Z}\left(\frac{\overline{\omega} + \mathrm{i}n_0\lambda_i}{\sqrt{2}kv_{\mathrm{th},i}}\right).
\end{aligned}$$

Introducing the plasma frequencies $\omega_{\mathrm{p},i}^2 = \frac{n_0 e^2}{\epsilon_0 m_i}$ and rearranging the expression gives the final equation

$$
\begin{aligned}
&D(k,\overline{\omega}) = \\
&\quad\Bigg(1 + \frac{\omega_{\mathrm{p},1}^2 + \mathrm{i} n_0 \lambda_1 \overline{\omega}}{k^2 v_{\mathrm{th},1}^2} \\
&\qquad\qquad + \left(\frac{\overline{\omega}(\omega_{\mathrm{p},1}^2 - n_0^2\lambda_1^2)}{\sqrt{2}k^3 v_{\mathrm{th},1}^3} + \frac{\mathrm{i} n_0 \lambda_1}{\sqrt{2}k v_{\mathrm{th},1}}\left(1 + \frac{\overline{\omega}^2 + \omega_{\mathrm{p},1}^2}{k^2 v_{\mathrm{th},1}^2}\right)\right)\mathcal{Z}\left(\frac{\overline{\omega} + \mathrm{i} n_0 \lambda_1}{\sqrt{2}k v_{\mathrm{th},1}}\right)\Bigg) \cdot \\
&\quad\Bigg(1 + \frac{\omega_{\mathrm{p},2}^2 + \mathrm{i} n_0 \nu_{22} \overline{\omega}}{k^2 v_{\mathrm{th},2}^2} \\
&\qquad + \left(\frac{\overline{\omega}(\omega_{\mathrm{p},2}^2 - n_0^2 \nu_{22}\lambda_2)}{\sqrt{2}k^3 v_{\mathrm{th},2}^3} + \frac{\mathrm{i} n_0}{\sqrt{2}k v_{\mathrm{th},2}}\left(\lambda_2 + \frac{\nu_{22}\overline{\omega}^2 + \lambda_2 \omega_{\mathrm{p},2}^2}{k^2 v_{\mathrm{th},2}^2}\right)\right)\mathcal{Z}\left(\frac{\overline{\omega} + \mathrm{i} n_0 \lambda_2}{\sqrt{2}k v_{\mathrm{th},2}}\right)\Bigg) \\
&- \frac{\omega_{\mathrm{p},1}^2\left(\omega_{\mathrm{p},2}^2 - \mathrm{i} n_0 \nu_{21}\overline{\omega}\right)}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \cdot \\
&\quad\left(1 + \frac{\overline{\omega} + \mathrm{i} n_0 \lambda_1}{\sqrt{2}k v_{\mathrm{th},1}}\mathcal{Z}\left(\frac{\overline{\omega} + \mathrm{i} n_0 \lambda_1}{\sqrt{2}k v_{\mathrm{th},1}}\right)\right) \cdot \left(1 + \frac{\overline{\omega} + \mathrm{i} n_0 \lambda_2}{\sqrt{2}k v_{\mathrm{th},2}}\mathcal{Z}\left(\frac{\overline{\omega} + \mathrm{i} n_0 \lambda_2}{\sqrt{2}k v_{\mathrm{th},2}}\right)\right) = 0.
\end{aligned}
\tag{3.34}
$$

In this form we can use the dispersion relation for the calculation of the zeros with Eric Sonnendrücker's Python algorithm.

The numerical implementation to find the zeros of an analytic function $f(\overline{\omega})$ works as follows. At first one defines a rectangular box in the complex plane of the value $\overline{\omega}$. The box is defined by the parameters $x_{\min}$ and $x_{\max}$ for the horizontal boundaries and by $y_{\min}$ and $y_{\max}$ for the vertical limits.

Then one can find the number of zeros of $f(\overline{\omega})$ in the rectangular box by means of Cauchy's argument principle

Lemma 3.2 (Cauchy's argument principle, [8])**.** Let $f(x)$ be a meromorphic function in a simply connected domain D contained in a Jordan contour ∂D. Suppose that $f(x)$ has neither zero nor a pole on ∂D, but has N_z zeros $z_i \in D$, $i = 1, ..., N_z$, and N_p poles $p_i \in D$, $i = 1, ..., N_p$, all counting the multiplicities. Then for any function $g(x)$ analytic in $\overline{D}$, where $\overline{D}$ is the closure of D

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{f'(x)}{f(x)}g(x)dx = \sum_{i=1}^{N_z} g(z_i) - \sum_{i=1}^{N_p} g(p_i).$$

The *argument principle* corresponds to the case $g(x) = 1$, for which the equation reduces to

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f'(x)}{f(x)} dx = N_z - N_p.$$

In our case, D is the rectangular box defined in the first step. Since we consider our input function $f(\overline{\omega})$ to be analytic, we do not have any poles. Therefore $N_p = 0$ and the argument principle gives the number of zeros $N_z =: N$ of $f(\overline{\omega})$. If N is larger than 5 the box will be subdivided into four smaller boxes and one starts again with calculating the number of zeros in each sub box. Otherwise one proceeds with the code.

The next step is to introduce a polynomial

$$P(\overline{\omega}) = \prod_{j=1}^{N} (\overline{\omega} - \overline{\omega}_j) = \overline{\omega}^N + c_1 \overline{\omega}^{N-1} + ... + c_N$$

with the same zeros $\overline{\omega}_1, ..., \overline{\omega}_N$ as $f(\overline{\omega})$ which are still unknown. This method was introduced by L. M. Delves and J. N. Lyness. For further informations the reader is referred to their paper [10]. One can find the coefficients $c_1, ..., c_N$ by the so-called Newton's identities [31]

$$\begin{aligned}
0 &= s_1 + c_1 \\
0 &= s_2 + s_1 c_1 + 2c_2 \\
&\vdots \\
0 &= s_N + s_{N-1} c_1 + ... + s_1 c_{N-1} + N c_N
\end{aligned}$$

with the k^{th} power sum

$$s_k := \overline{\omega}_1^k + \overline{\omega}_2^k + ... + \overline{\omega}_N^k = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} x^k \frac{f'(x)}{f(x)} dx$$

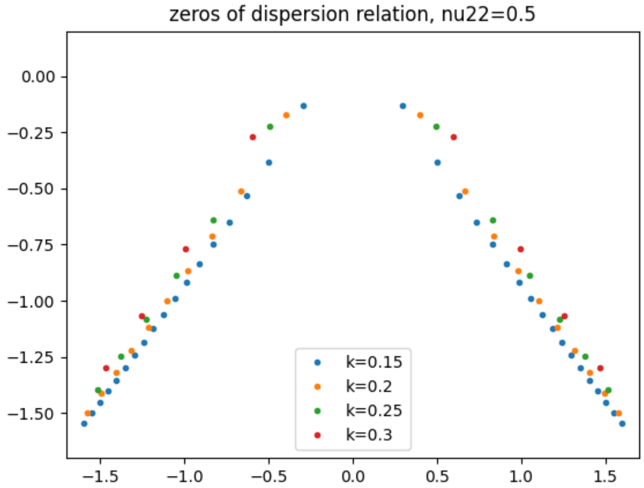
Now the polynomial $P(\overline{\omega})$ is completely defined and the zeros of it can be determined by means of Hankel matrices.

Here one constructs an $2N$-dimensional vector with the values s_k of the power sums. After constructing the Hankel matrices one can find the zeros as the generalized eigenvalues of the matrices.

At the last step we verify that the found zeros are also zeros of the function $f(\overline{\omega})$. If this is not the case one can redefine the rectangular box and start the algorithm again.

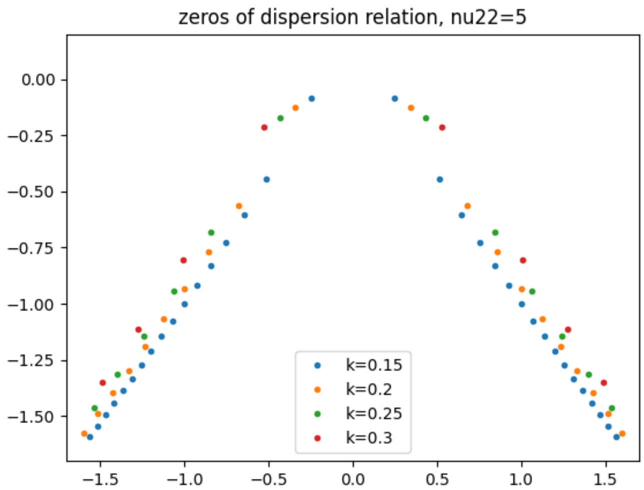
For a more detailed explanation of the code see [2] and [31]. The full Python code for finding the zeros of the dispersion relation for the two-species Vlasov-Poisson-BGK model can be found in Appendix A.2.

The result is shown in Figure 3. The parameters can be seen in the code in Appendix A.2. It is important to take the relations of the physical constants of ions and electrons into account. For example, if we assume the plasma to be composed of hydrogen ions and electrons, the particles have a mass ratio of $m = \frac{m_1}{m_2} = 1800$. This value must



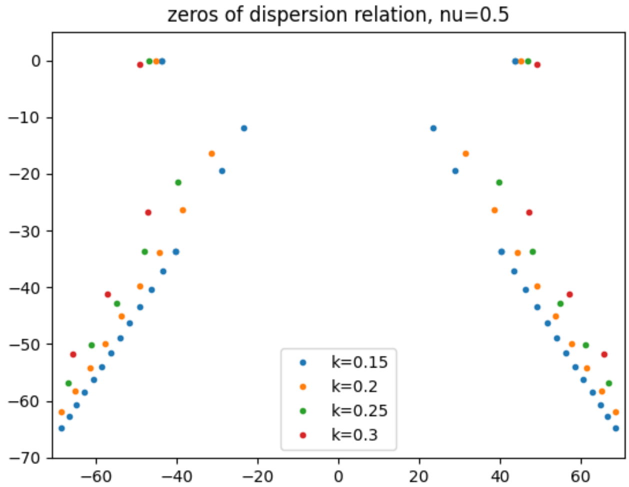
wave vector k	zero $\overline{\omega}_k$ with largest imaginary part
0.15	$\pm 0.29598 - 0.12852\mathrm{i}$
0.20	$\pm 0.39665 - 0.17445\mathrm{i}$
0.25	$\pm 0.49622 - 0.22135\mathrm{i}$
0.30	$\pm 0.59440 - 0.26946\mathrm{i}$

Fig. 3: Representation of the zeros of the dispersion relation (3.34) in the complex plane of $\overline{\omega}$. The collision frequency is set to $\nu_{22} = 0.5$, the other parameters can be found in the code in Appendix A.2. The wave vector k takes values between 0.15 and 0.3 in steps of 0.05. One can see that all zeros are below the x-axis and therefore $\gamma = \mathrm{Im}(\overline{\omega}_k) < 0$ where $\overline{\omega}_k$ is the zero with the largest imaginary part. On the right-hand side one can find the exact values for these zeros.



wave vector k	zero $\overline{\omega}_k$ with largest imaginary part
0.15	$\pm 0.24947 - 0.08603\mathrm{i}$
0.20	$\pm 0.34106 - 0.12712\mathrm{i}$
0.25	$\pm 0.43342 - 0.17026\mathrm{i}$
0.30	$\pm 0.52567 - 0.21508\mathrm{i}$

Fig. 4: Visualization of the zeros of the dispersion relation (3.34) in the complex plane of $\overline{\omega}$. The collision frequency is set to $\nu_{22} = 5$, the other parameters can be found in the code in Appendix A.2. The wave vector k takes values between 0.15 and 0.3 in steps of 0.05. On the right-hand side one can find the exact values for these zeros. All imaginary parts are negative.



wave vector k	zero $\overline{\omega}_k$ with largest imaginary part
0.15	$\pm 43.90393 - 0.01314\mathrm{i}$
0.20	$\pm 45.13996 - 0.03040\mathrm{i}$
0.25	$\pm 46.89901 - 0.14555\mathrm{i}$
0.30	$\pm 49.16410 - 0.60953\mathrm{i}$

Fig. 5: Zeros of the dispersion relation of Lena Baumann's one-species model in the complex plane of $\overline{\omega}$. The collision frequency of the electrons is set to $\nu = 0.5$. The other parameters are $v_{\mathrm{th}} = \sqrt{m}$ and $\omega_{\mathrm{p}} = \sqrt{m}$ where m is the mass ratio between ion and electron and set to $m = 1800$. All zeros are below the x-axis which becomes more clear considering the table on the right-hand side with the values of the highest points.

be included in the parameters for the collision frequencies according to (3.4), for the thermal velocities $v_{\mathrm{th},i} = \sqrt{\frac{T_0}{m_i}}$ and for the plasma oscillation frequencies $\omega_{\mathrm{p},i} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_i}}$.

One can see in Figure 3 that all zeros of the dispersion relation are located below the x-axis which means that all imaginary parts of the zeros are negative, even the one with the largest imaginary part γ. Therefore the mapping shows that $\gamma < 0$ and the damping effect of the electric field with the asymptotic behaviour

$$\hat{E}(k,t) \longrightarrow \operatorname{Res}_{\overline{\omega}=\overline{\omega}_k}\left(\breve{E}(k,\overline{\omega})\right)\exp(-\mathrm{i}\omega t)\exp(\gamma t)$$

is proven numerically.

To compare the magnitude of the damping effect of different models or parameters one can give the exact positions of the zeros with the largest imaginary part in the complex plane. The values can be seen in the table on the right-hand side of Fig. 3.

Increasing the collision frequency ν_{22} one can see in Figure 4 that γ gets closer to the x-axis and the damping effect gets weakened. Of course, it is not reasonable to further increase the collision frequency since we are in the setting of a dilute gas where only few collisions occur. Varying the value for ν_{22} between 0 and 1 does not change the result remarkably, nor does an increase in the mass ratio.

Next we want to examine the differences to Lena Baumann's numerical results with the one-species model where she considered the Vlasov-Poisson-BGK system for the electrons of the plasma. The code can be seen in her Master's thesis [2]. To be able to compare the results the parameters v_{th} and ω_{p} of the one-species model are changed from 1 to the mass ratio $\sqrt{m}$ such that the physical constants belonging to electrons have the same value.

The result for the collision frequency $\nu = 0.5$ can be seen in Figure 5. Regarding the values in the table of Fig. 5 for the zeros with the largest imaginary part one can see that all points are located below the x-axis. At the one-species model the zeros are further apart from each other such that the range of the coordinate axes differ from the upper two graphs.

Taking a look at the exact values of $\overline{\omega}_k$ and comparing the table in Fig. 5 with the one in Fig. 3, one can see that for $k = 0.15$, $k = 0.2$ and $k = 0.25$ the absolute value of the imaginary parts and therefore γ is larger with the two-species model than considering only the electrons. This means that the damping effect is strengthened if we take both species into account. For $k = 3$ it is the other way round where we can see that the theoretical observations in Chapter 3.2.6 only hold for the limiting case of long wavelengths or $k \longrightarrow 0$.

3.3 Physical Interpretation of Landau Damping

There are several possibilities to interpret the Landau damping effect. Landau himself did not give a physical interpretation [23].

One way to visualize the damping effect physically was presented by Marlies Pirner [26]. We assume that a gas is in a state of equilibrium. If we slightly change the position of a particle of the gas, there is a force that pushes the particles back to its original position. We can think of this force as a spring. This spring causes the particle to oscillate around its equilibrium state, and over time this oscillation is damped.

Another common interpretation of the damping effect in the one-species model uses an energetic approach, see [11], [23] and [32]. We illustrate the setting by means of waves in the sea, which correspond to the plasma oscillations and therefore visualize the oscillations of the electric field. In Figure 6 the blue waves move with phase velocity v_{w}. The boats represent the electrons. If the boats have a velocity of approximately the same magnitude as the phase velocity v_{w} of the waves, the boats are 'caught' in the waves and the maximum energy exchange takes place.

We separate the boats into two groups. One group with a slightly slower velocity v than the wave velocity and one group with boats which are slightly faster than v_{w}. The first group gets accelerated because they gain energy from the waves. One can think of this process as if the waves push the boat. Conversely, the boats which are slightly faster push the waves and therefore lose energy to the waves and get decelerated.

In a gas near equilibrium the velocities of particles are distributed according to the Maxwellian distribution function, see Figure 7. As a result, slower particles occur more frequently than faster ones. Assuming that the waves move with phase velocity v_{w}, then more boats are accelerated by the waves than the waves receive energy from the boats. In total, the waves lose energy and are therefore damped.

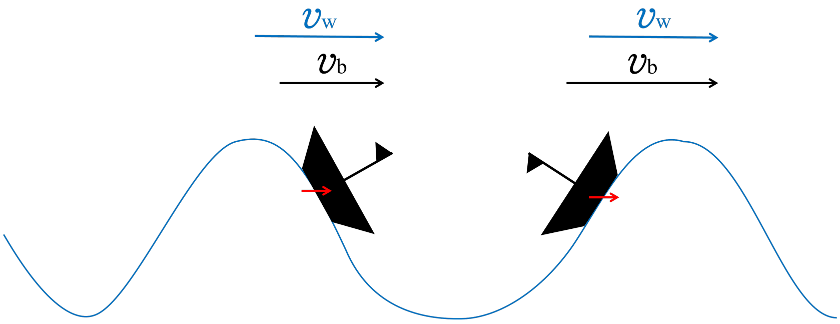


Fig. 6: Visualization of the Landau damping effect. The waves, corresponding to the plasma oscillations, move with the phase velocity v_w. The electrons are represented by the boats and have various velocities v. Boats which are slightly slower than the waves get accelerated while slightly faster boats get decelerated. Because of the Maxwellian distribution of the velocities in a gas, in total, there are more slower particles than faster particles, see Fig. 7. Therefore the waves lose more energy than they gain and hence they are damped.

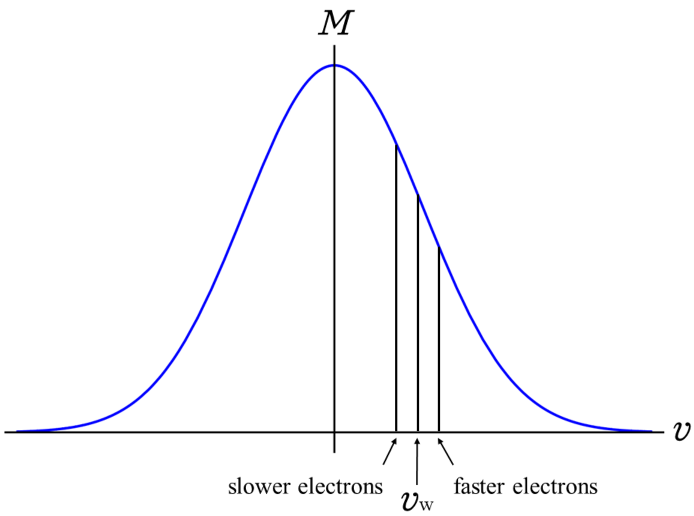


Fig. 7: Velocities in a gas in equilibrium can be considered to be distributed according to the Maxwellian distribution. Consequently there are more slower particles than faster ones. Assuming the phase velocity of the waves in our visualization of the damping effect to be v_w, there are more boats or electrons which get accelerated than decelerated. In total, the waves lose energy and are damped.

This interpretation is based on the one-species considerations. The question that now arises is, why the Landau damping effect is amplified when two species are considered and how this can be visualized. So far, the picture included the electrons as boats and the waves as background. Mathematically, the waves were not considered as acting participant. With the two-species model we also include the background in our calculations. This could therefore be interpreted to mean that the waves also interact with the boats. In the one-species model, the boats give energy to and get energy from the waves. Now, the waves can also be considered as acting partner and push the boats themselves. The energy exchange is larger in this model than in the one-species model and therefore the damping effect is intensified.

4 Conclusion and Outlook

In this Master's thesis we have analysed the two-species Vlasov-Poisson-BGK equations

$$\begin{aligned}
&\partial_t f_1 + v\partial_x f_1 + \frac{eE}{m_1}\partial_v f_1 = \nu_{11} n_1 (M_1 - f_1) + \nu_{12} n_2 (M_{12} - f_1)\\
&\partial_t f_2 + v\partial_x f_2 - \frac{eE}{m_2}\partial_v f_2 = \nu_{22} n_2 (M_2 - f_2) + \nu_{21} n_1 (M_{21} - f_2)\\
&\frac{\partial E}{\partial x} = \frac{e}{\epsilon_0}(n_1 - n_2)
\end{aligned}$$

with respect to linear Landau damping coupled with relaxation. The equations describe a plasma, or more precisely a gas consisting of the two species ions and electrons. In our setting we considered the case of a weakly collisional regime and the limiting case of long wavelengths.

To show linear Landau damping one has to show that the electric field, which oscillates, is damped. For this purpose, we have done several mathematical steps which were based on the work of Lev D. Landau. He studied the damping effect of the electric field for the Vlasov-Poisson equations considering the one-species case where only electrons are described by the equations. In addition to his work, we have considered the electrons as well as the ions and we added a collision kernel or BGK kernel. For this system we wanted to find out if the system still tends to an equilibrium. The steps we have performed were as follows. At first we have linearised the equations with respect to an equilibrium function. By performing Fourier and Laplace transforms we have got a dispersion relation of the system. With this we have been able to find an expression for the Fourier transformed electric field and got the long-term behaviour

$$\hat{E}(k,t) \longrightarrow \mathrm{Res}_{\overline{\omega}=\overline{\omega}_k}\left(\breve{E}(k,\overline{\omega})\right)\exp(-\mathrm{i}\omega t)\exp(\gamma t)$$

which depends exponentially on the time t. The last step has been to investigate if the electric field increases or decreases exponentially.

In order to show this we have chosen two different ways. At first we have calculated the sign of the damping coefficient γ analytically. We have expanded the dispersion relation and neglected higher order terms since we have considered the case of weakly collisional regime and wave vector $k \longrightarrow 0$. The second approach has been to rewrite the dispersion relation by means of the plasma dispersion function. Then we have been able to use the python code by Eric Sonnendrücker which calculates the zeros of a given dispersion function [31].

For both approaches we have got the result that $\gamma < 0$ and therefore a proof for the damping effect of the electric field in the limiting case of long wavelengths. Compared to the one-species Vlasov-Poisson-BGK model examined by Lena Baumann [2] we have received the result that the damping effect is intensified since there are terms concerning ions and electrons and mixed terms contributing to the damping coefficient. In addition to the damping effect due to the BGK equations we have got further terms from the relaxation effect of the collision kernel. This collision term itself tends to an equilibrium which can be explained with the H-theorem. Both of the dampening effects reinforce themselves.

In the future, a new aspect could be to consider the BGK system for two species, though without linearising the model. In 2011 Clément Mouhot and Cédric Villani published their work where they proved non-linear Landau damping for the Vlasov-Poisson equation for one species [23]. One could extend their model by the BGK collision kernel or by considering two species or even a combination of the two extensions.

A Appendix

A.1 Expanded Expression of the Dispersion Relation

The real part of the dispersion relation (3.29) after applying the first order of Jackson's identity as done in Chapter 3.2.6 is:

$$
\begin{aligned}
0 = {} & \frac{\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\pi\eta_2\nu_{21}\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{3\lambda_1\omega\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_2 \eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\gamma\nu_{22}\sqrt{\pi}\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{\nu_{22}\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\nu_{21}\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_2^2\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{3\nu_{21}\omega\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\xi_1 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
& + \frac{\lambda_2\xi_1 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\lambda_2\xi_1 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\lambda_1\xi_1 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
& + \frac{3\lambda_1\xi_1 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^4\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{3\lambda_1\xi_1 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{9\lambda_1\xi_1 n_0^2 e^2}{8\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^4\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
& + \frac{\exp(-\eta_1^2-\eta_2^2)\pi\eta_1\nu_{21}\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\nu_{22}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\lambda_1\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{\nu_{22}\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\nu_{21}\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{\lambda_1\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_2 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\nu_{22} w\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1^2\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_2 \eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\nu_{22} w\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\nu_{21}\omega\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\lambda_1\omega\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_2 \eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
& + \frac{\gamma\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_2\xi_2 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{3\lambda_2\xi_2 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
& + \frac{3\lambda_2\xi_2 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{9\lambda_2\xi_2 n_0^2 e^2}{8\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\xi_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda_1\xi_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_2\eta_1\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\lambda_1\xi_2 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_2\eta_1^3\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1\eta_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\lambda_2\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{\epsilon_0 k^3 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_1^2)\lambda_2\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_2^2)\lambda_1\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_2\eta_1^4 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\lambda_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_2^2)\lambda_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{4\epsilon_0 k^3 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_2\eta_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{\epsilon_0 k^3 m_1\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2)\lambda_2\eta_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_2^2)\lambda_1\eta_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{\epsilon_0 k^3 m_2\eta_1 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\eta_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_2\eta_1^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\lambda_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_2\eta_2^2 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\lambda_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{4\epsilon_0 k^3 m_2\eta_2^4 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\nu_{22}\eta_2\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\eta_2\nu_{21}\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\eta_2\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\nu_{22}\eta_2\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_2^2)\eta_2\nu_{21}\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\lambda_1\eta_2\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\nu_{22}\eta_1\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\eta_1\nu_{21}\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\eta_1\omega n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\nu_{22}\eta_1\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_1^2)\eta_1\nu_{21}\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\lambda_1\eta_1\omega n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\nu_{22}\eta_2\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\gamma\eta_2\nu_{21}\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\lambda_1\eta_2\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\nu_{22}\eta_1^2\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\gamma\eta_1^2\nu_{21}\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\eta_1^2\xi_1 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\gamma\nu_{22}\eta_1^2\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_1^2)\gamma\eta_1^2\nu_{21}\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\gamma\lambda_1\eta_1^2\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\sqrt{\pi}\xi_1 n_0 e^2}{\epsilon_0 k^2 m_1 v_{\mathrm{th},1}^2}\\
&+\frac{\exp(-\eta_2^2)\nu_{21}\omega\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\omega\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\nu_{22}w\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\nu_{21}\omega\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\omega\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\sqrt{2}\epsilon_0 k^3 m_2\eta_1^2 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\gamma\nu_{22}\eta_2^2\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\eta_2^2\nu_{21}\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\lambda_1\eta_2^2\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_2^2)\gamma\nu_{22}\eta_2^2\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\gamma\eta_2^2\nu_{21}\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{n_0 e^2}{2\epsilon_0 k^2 m_1\eta_1^2 v_{\mathrm{th},1}^2}\\
&+\frac{\exp(-\eta_1^2)\gamma\nu_{22}\eta_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\eta_1\nu_{21}\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\eta_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{2\exp(-\eta_1^2)\eta_1^2\xi_1 n_0\sqrt{\pi}e^2}{\epsilon_0 k^2 m_1 v_{\mathrm{th},1}^2}+\frac{2\exp(-\eta_2^2)\eta_2^2\xi_2 n_0\sqrt{\pi}e^2}{\epsilon_0 k^2 m_2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\nu_{22}w\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
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\begin{aligned}
&+\frac{3\exp(-\eta_2^2)\gamma\lambda_1\eta_2^2\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{3n_0e^2}{4\epsilon_0 k^2 m_1\eta_1^4 v_{\mathrm{th},1}^2}-\frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\eta_1^2\xi_1 n_0^2 e^2}{\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&-\frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\eta_1\eta_2\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{\exp(-\eta_2^2)\lambda_2\sqrt{2\pi}\eta_2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{n_0e^2}{2\epsilon_0 k^2 m_2\eta_2^2 v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2)\lambda_2\sqrt{2\pi}\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{3\exp(-\eta_1^2)\lambda_2\sqrt{\frac{\pi}{2}}\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{\exp(-\eta_2^2)\sqrt{\pi}\xi_2 n_0 e^2}{\epsilon_0 k^2 m_2 v_{\mathrm{th},2}^2}\\
&-\frac{3n_0e^2}{4\epsilon_0 k^2 m_2\eta_2^4 v_{\mathrm{th},2}^2}-\frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^3 m_2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}-\frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_2^2\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&-\frac{\sqrt{2}\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}-\frac{3\exp(-\eta_2^2)\lambda_1\sqrt{\frac{\pi}{2}}\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2)\lambda_1\sqrt{2\pi}\eta_1\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1^2\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2-\eta_2^2)\pi\eta_1^2\eta_2\nu_{21}\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1\eta_2^2\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2-\eta_2^2)\pi\eta_1\eta_2^2\nu_{21}\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1^2\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1^2\eta_2^2\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\pi\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1^2\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\eta_2^2\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1^2\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_2^2)\gamma\sqrt{\pi}\nu_{21}\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_2^2)\gamma\nu_{22}\sqrt{\pi}\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_2^2)\gamma\lambda_1\sqrt{\pi}\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_2\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\gamma\lambda_1 n_0^2 e^2}{4\epsilon_0 k^4 m_2\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{3\exp(-\eta_2^2)\gamma\sqrt{\pi}\nu_{21}\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{3\exp(-\eta_2^2)\gamma\lambda_1\sqrt{\pi}\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_2\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma\sqrt{\pi}\nu_{21}\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma\nu_{22}\sqrt{\pi}\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_2^2)\sqrt{\pi}\eta_2^2\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma\lambda_1\sqrt{\pi}\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\gamma\nu_{22} n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2)\nu_{22}\sqrt{\pi}\eta_1^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\gamma\nu_{21} n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{3\gamma\nu_{22} n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&-\frac{3\gamma\nu_{21} n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{3\gamma\lambda_1 n_0^2 e^2}{8\epsilon_0 k^4 m_2\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2)\sqrt{\pi}\eta_1^2\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
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\begin{aligned}
&- \frac{2\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\gamma\sqrt{\pi}\nu_{21}\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\eta_2\omega^2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\gamma\lambda_1\sqrt{\pi}\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_2 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\gamma\nu_{22} n_0^2 e^2}{8\epsilon_0 k^4 m_1 \eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1\omega n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\gamma\nu_{21} n_0^2 e^2}{8\epsilon_0 k^4 m_1 \eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\gamma\lambda_1 n_0^2 e^2}{8\epsilon_0 k^4 m_2 \eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&- \frac{9\gamma\nu_{21} n_0^2 e^2}{16\epsilon_0 k^4 m_1 \eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{9\gamma\lambda_1 n_0^2 e^2}{16\epsilon_0 k^4 m_2 \eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_2\omega n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+ \frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1^2\eta_2\omega\xi_1 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\eta_2^2\omega\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+ \frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1^2\omega^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_2^2\omega^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&- \frac{9\gamma\nu_{22} n_0^2 e^2}{16\epsilon_0 k^4 m_1 \eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\nu_{22} w^2\xi_1\xi_2 n_0^2}{k^4\eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\omega\xi_1\xi_2 n_0^2}{\sqrt{2}k^3\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{3\lambda_1\lambda_2\omega\xi_1\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+ \frac{\lambda_1\nu_{22} w\xi_1\xi_2 n_0^2}{\sqrt{2}k^3\eta_1^2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{3\lambda_1\nu_{22} w\xi_1\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^4\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{2k^2\eta_1^2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{3\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{4k^2\eta_1^4\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+ \frac{3\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{4k^2\eta_1^2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{9\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{8k^2\eta_1^4\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{4\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1^2\eta_2^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi n_0^2}{2k^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{g^2\lambda_1\nu_{22} n_0^2}{4k^4\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+ \frac{3\gamma^2\lambda_1\nu_{22} n_0^2}{8k^4\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\gamma^2\lambda_1\nu_{22} n_0^2}{8k^4\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{9\gamma^2\lambda_1\nu_{22} n_0^2}{16k^4\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\lambda_1 n_0}{2k^2\eta_1^2 v_{\mathrm{th},1}^2} + \frac{3\gamma\lambda_1 n_0}{4k^2\eta_1^4 v_{\mathrm{th},1}^2}\\
&+ \frac{3\gamma\nu_{22} n_0}{4k^2\eta_2^4 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\eta_1^2\xi_1 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\eta_2\xi_1 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+ \frac{\gamma\nu_{22} n_0}{2k^2\eta_2^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_2\omega\xi_1\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\omega\xi_1\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\eta_1\eta_2\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_2^2\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+ \frac{\exp(-\eta_2^2)\lambda_1\lambda_2\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1^2\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+ \frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\lambda_1\lambda_2 n_0^2}{4k^2\eta_1^3\eta_2 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+ \frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1\omega\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\lambda_1\lambda_2\eta_2\omega\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_2^2)\lambda_1\lambda_2\xi_1 n_0^2\sqrt{\pi}}{2k^2\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}}
\end{aligned}
$$

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\begin{aligned}
&+\frac{3\exp(-\eta_2^2)\lambda_1\lambda_2\eta_2\omega\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\omega\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{\lambda_1\lambda_2 n_0^2}{2k^2\eta_1\eta_2 v_{\mathrm{th},1}v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2^2\omega\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\nu_{22}w\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega^2\xi_1 n_0^2\sqrt{\pi}}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\omega\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1^2\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1^2\omega^2\xi_1 n_0^2\sqrt{\pi}}{k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1^2\omega^2\xi_1 n_0^2\sqrt{\pi}}{2k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_2^2)\lambda_1\lambda_2\xi_1 n_0^2\sqrt{\pi}}{4k^2\eta_1^4 v_{\mathrm{th},1}v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\xi_1 n_0^2\sqrt{\pi}}{k^2\eta_2 v_{\mathrm{th},1}v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\xi_1 n_0^2\sqrt{\pi}}{2k^2\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}x n_0^2\sqrt{\pi}}{2k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}x n_0^2\sqrt{\pi}}{4k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2^2\omega^2\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2^2\omega^2\xi_2 n_0^2\sqrt{\pi}}{2k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{4\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\omega\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1\omega^2\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\lambda_2\eta_2\xi_2 n_0^2\sqrt{\pi}}{k^2\eta_1 v_{\mathrm{th},1}v_{\mathrm{th},2}}+\frac{\exp(-\eta_2^2)\lambda_1\lambda_2\eta_2\xi_2 n_0^2\sqrt{\pi}}{2k^2\eta_1^3 v_{\mathrm{th},1}v_{\mathrm{th},2}}\\
&+\frac{4\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1^2\omega\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\xi_2 n_0^2\sqrt{\pi}}{2k^2\eta_2^2 v_{\mathrm{th},1}v_{\mathrm{th},2}}+\frac{3\exp(-\eta_1^2)\lambda_1\lambda_2\xi_2 n_0^2\sqrt{\pi}}{4k^2\eta_2^4 v_{\mathrm{th},1}v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}X n_0^2\sqrt{\pi}}{2k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}X n_0^2\sqrt{\pi}}{4k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\xi_1 n_0\sqrt{\pi}}{k^2 v_{\mathrm{th},1}^2}\\
&+\frac{\exp(-\eta_2^2)\gamma\nu_{22}X n_0\sqrt{\pi}}{k^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1^2\omega\xi_1 n_0^2\sqrt{2\pi}}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2^2\omega\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_1 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\eta_2\xi_1\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1^2\xi_1\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{\exp(-\eta_1^2)\lambda_1\sqrt{\frac{\pi}{2}}n_0}{k v_{\mathrm{th},1}}\\
&+\frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\xi_1\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_1^2 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1\xi_1\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}-\frac{\lambda_1\xi_1 n_0}{\sqrt{2}k\eta_1^2 v_{\mathrm{th},1}}\\
&-\frac{3\lambda_1\xi_1 n_0}{2\sqrt{2}k\eta_1^4 v_{\mathrm{th},1}}-\frac{\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1\omega n_0}{k^2 v_{\mathrm{th},1}^2}-\frac{2\exp(-\eta_1^2)\gamma\lambda_1\sqrt{\pi}\eta_1^2\xi_1 n_0}{k^2 v_{\mathrm{th},1}^2}-\frac{\lambda_1\omega\xi_1 n_0}{k^2\eta_1^3 v_{\mathrm{th},1}^2}\\
&-\frac{\exp(-\eta_2^2)\lambda_2\sqrt{\frac{\pi}{2}}n_0}{k v_{\mathrm{th},2}}-\frac{\lambda_2\xi_2 n_0}{\sqrt{2}k\eta_2^2 v_{\mathrm{th},2}}-\frac{3\lambda_2\xi_2 n_0}{2\sqrt{2}k\eta_2^4 v_{\mathrm{th},2}}-\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1\eta_2\xi_1\xi_2 n_0^2}{k^2 v_{\mathrm{th},1}v_{\mathrm{th},2}}\\
&-\frac{\lambda_1\lambda_2 n_0^2}{4k^2\eta_1\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}}-\frac{\lambda_1\lambda_2 n_0^2}{8k^2\eta_1^3\eta_2^3 v_{\mathrm{th},1}v_{\mathrm{th},2}}+\frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^4 v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&-\frac{2\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1^2\eta_2\omega\xi_1\xi_2 n_0^2}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\xi_1 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}
\end{aligned}
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\begin{aligned}
&- \frac{\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}n_0^2}{2k^3\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}n_0^2}{4k^3\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\eta_1 n_0^2}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\omega\xi_1 n_0^2}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\lambda_1\lambda_2\omega n_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\gamma\lambda_1\lambda_2\xi_1 n_0^2}{\sqrt{2}k^3\eta_1^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{3\lambda_1\lambda_2\omega n_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\xi_1\xi_2 n_0^2}{k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\gamma\lambda_1\lambda_2\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{3\gamma\lambda_1\lambda_2\xi_2 n_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\eta_1 n_0^2}{2k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\xi_1\xi_2 n_0^2}{2k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\omega\xi_1 n_0^2}{2k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\eta_2\omega n_0}{k^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_2^2)\gamma\nu_{22}\sqrt{\pi}\eta_2^2\xi_2 n_0}{k^2 v_{\mathrm{th},2}^2} \\
&- \frac{\gamma\lambda_1\lambda_2\xi_1 n_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\lambda_1\lambda_2\omega n_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\gamma\lambda_1\lambda_2\xi_2 n_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{9\gamma\lambda_1\lambda_2\xi_2 n_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\nu_{22}\omega\xi_2 n_0}{k^2\eta_2^3 v_{\mathrm{th},2}^2} - \frac{2\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\eta_2^2\omega\xi_1\xi_2 n_0^2}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\xi_2 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\eta_2 n_0^2}{k^3\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\omega\xi_2 n_0^2}{k^3\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\xi_1\xi_2 n_0^2}{\sqrt{2}k^3\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\eta_2 n_0^2}{2k^3\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\omega\xi_2 n_0^2}{2k^3\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\xi_1\xi_2 n_0^2}{2k^3\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}n_0^2}{2k^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\lambda_1\nu_{22}wn_0^2}{2\sqrt{2}k^3\eta_1\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\gamma\lambda_1\nu_{22}xn_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\lambda_1\nu_{22}wn_0^2}{4\sqrt{2}k^3\eta_1^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\gamma\lambda_1\nu_{22}xn_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\gamma\lambda_1\nu_{22}Xn_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\lambda_1\nu_{22}wn_0^2}{4\sqrt{2}k^3\eta_1\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1\eta_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}n_0^2}{4k^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\gamma\lambda_1\nu_{22}xn_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_2\omega\xi_1 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{3\lambda_1\nu_{22}wn_0^2}{8\sqrt{2}k^3\eta_1^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{9\gamma\lambda_1\nu_{22}xn_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\omega\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_2^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{4\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1^2\eta_2^2\omega^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\omega^2\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\eta_2\omega n_0^2}{k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_2^2\xi_2 n_0^2}{k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\pi}\omega^2\xi_2 n_0^2}{2k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
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\begin{aligned}
&- \frac{2\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_1\xi_2 n_0^2}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\eta_2\omega n_0^2}{2k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\gamma\lambda_1\nu_{22}X n_0^2}{\sqrt{2}k^3\eta_1\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_2^2\xi_2 n_0^2}{2k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\pi}\omega^2\xi_2 n_0^2}{4k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\lambda_1\lambda_2\omega n_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_2\xi_1 n_0^2}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\eta_1\omega n_0^2}{k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_1^2\xi_1 n_0^2}{k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\sqrt{\pi}\omega^2\xi_1 n_0^2}{2k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\lambda_1\nu_{22}w^2 n_0^2}{4k^4\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\gamma\lambda_1\nu_{22}w\xi_1 n_0^2}{k^4\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\lambda_1\nu_{22}w^2 n_0^2}{8k^4\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_1\xi_2 n_0^2}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_1\xi_2 n_0^2}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\gamma\lambda_1\nu_{22}w\xi_2 n_0^2}{k^4\eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{g^2\lambda_1\nu_{22}x\xi_2 n_0^2}{k^4\eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\gamma\lambda_1\nu_{22}w\xi_2 n_0^2}{2k^4\eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\eta_1\omega n_0^2}{2k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_1^2\xi_1 n_0^2}{2k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\lambda_1\nu_{22}\sqrt{\pi}\omega^2\xi_1 n_0^2}{4k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\lambda_1\nu_{22}w^2 n_0^2}{8k^4\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1\eta_2\nu_{21}n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1\eta_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{3\gamma\lambda_1\nu_{22}w\xi_1 n_0^2}{2k^4\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{9\lambda_1\nu_{22}w^2 n_0^2}{16k^4\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\gamma\nu_{22}\sqrt{\pi}\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + 1
\end{aligned}
$$

The imaginary part of the expanded expression of (3.29) is:

$$
\begin{aligned}
0 = &\frac{\nu_{22}\omega n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\nu_{21}\omega n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\lambda_1\omega n_0^2 e^2}{4\epsilon_0 k^4 m_2\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\nu_{22}\omega n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\nu_{21}\omega n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\lambda_1\omega n_0^2 e^2}{8\epsilon_0 k^4 m_2\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_2\nu_{21}\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\nu_{22}\omega n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\nu_{21}\omega n_0^2 e^2}{8\epsilon_0 k^4 m_1\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\lambda_1\omega n_0^2 e^2}{8\epsilon_0 k^4 m_2\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{9\nu_{22}\omega n_0^2 e^2}{16\epsilon_0 k^4 m_1\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{4\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1^2\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{9\lambda_1\omega n_0^2 e^2}{16\epsilon_0 k^4 m_2\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{4\exp(-\eta_1^2-\eta_2^2)\pi\eta_1^2\eta_2^2\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\gamma\nu_{22}\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\nu_{21}\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{4\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1^2\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\gamma\lambda_1\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_2\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\gamma\nu_{22}\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\pi\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3\gamma\nu_{21}\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\gamma\lambda_1\xi_1 n_0^2 e^2}{4\epsilon_0 k^4 m_2 \eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\nu_{22}\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1\nu_{21}\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\gamma\nu_{21}\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\gamma\lambda_1\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_2 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\gamma\nu_{22}\xi_2 n_0^2 e^2}{4\epsilon_0 k^4 m_1 \eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\eta_1^2\eta_2\xi_1\xi_2 n_0^2\sqrt{2}e^2}{\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&+ \frac{\lambda_2 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^2\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{3\lambda_2 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^4\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\lambda_2 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&+ \frac{3\lambda_2 n_0^2 e^2}{8\sqrt{2}\epsilon_0 k^3 m_1 \eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\lambda_1 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_2 \eta_1\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\lambda_1 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&+ \frac{3\lambda_1 n_0^2 e^2}{4\sqrt{2}\epsilon_0 k^3 m_2 \eta_1\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{3\lambda_1 n_0^2 e^2}{8\sqrt{2}\epsilon_0 k^3 m_2 \eta_1^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\xi_1 n_0 e^2}{\epsilon_0 k^2 m_1 \eta_1^3 v_{\mathrm{th},1}^2} + \frac{\xi_2 n_0 e^2}{\epsilon_0 k^2 m_2 \eta_2^3 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2)\lambda_2\xi_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{\epsilon_0 k^3 m_1 \eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2)\lambda_2\xi_1 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_2^2)\lambda_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{\epsilon_0 k^3 m_2 \eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_2^2)\lambda_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}e^2}{2\epsilon_0 k^3 m_2 \eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\nu_{22}\omega\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\nu_{21}\omega\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2)\lambda_1\omega\xi_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_1^2)\nu_{22}\omega\xi_1 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_1^2)\nu_{21}\omega\xi_1 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\exp(-\eta_1^2)\lambda_1\omega\xi_1 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\nu_{22}\omega\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\nu_{21}\omega\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_2^2)\lambda_1\omega\xi_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_2^2)\nu_{22}\omega\xi_2 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_2^2)\nu_{21}\omega\xi_2 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\exp(-\eta_2^2)\lambda_1\omega\xi_2 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\gamma\nu_{22}\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\gamma\nu_{21}\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_2^2)\gamma\lambda_1\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\nu_{22}\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\nu_{21}\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2)\gamma\lambda_1\xi_1\xi_2 n_0^2\sqrt{\pi}e^2}{\epsilon_0 k^4 m_2 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\gamma\nu_{22}\eta_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\gamma\eta_2\nu_{21} n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_2^2)\gamma\lambda_1\eta_2 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2 \eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_2^2)\gamma\nu_{22}\eta_2 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_2^2)\gamma\eta_2\nu_{21} n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\exp(-\eta_2^2)\gamma\lambda_1\eta_2 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\nu_{22}\eta_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\eta_1\nu_{21} n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2)\gamma\lambda_1\eta_1 n_0^2\sqrt{\pi}e^2}{2\epsilon_0 k^4 m_2 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_1^2)\gamma\nu_{22}\eta_1 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\exp(-\eta_1^2)\gamma\eta_1\nu_{21} n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3\exp(-\eta_1^2)\gamma\lambda_1\eta_1 n_0^2\sqrt{\pi}e^2}{4\epsilon_0 k^4 m_2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\eta_1 n_0\sqrt{\pi}e^2}{\epsilon_0 k^2 m_1 v_{\mathrm{th},1}^2} - \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\eta_2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&+ \frac{\exp(-\eta_2^2)\eta_2 n_0\sqrt{\pi}e^2}{\epsilon_0 k^2 m_2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_2\pi\eta_1 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_2^2)\lambda_2\sqrt{\pi}\eta_2\xi_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_2^2)\lambda_2\sqrt{\frac{\pi}{2}}\xi_1 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\exp(-\eta_2^2)\lambda_2\sqrt{\frac{\pi}{2}}\eta_2\xi_2 n_0^2 e^2}{2\epsilon_0 k^3 m_1\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\lambda_2\sqrt{2\pi}\eta_1^2\xi_1 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_1^2)\lambda_2\sqrt{\pi}\eta_1\xi_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\lambda_2\xi_1\xi_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_1\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\lambda_2\sqrt{\frac{\pi}{2}}\eta_1^2\xi_1 n_0^2 e^2}{\epsilon_0 k^3 m_1\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{3\exp(-\eta_1^2)\lambda_2\sqrt{\frac{\pi}{2}}\eta_1\xi_2 n_0^2 e^2}{2\epsilon_0 k^3 m_1\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\lambda_2\xi_1\xi_2 n_0^2 e^2}{2\sqrt{2}\epsilon_0 k^3 m_1\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_2 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\sqrt{2\pi}\eta_2^2\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2\xi_1 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_2\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\sqrt{\frac{\pi}{2}}\eta_2^2\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\lambda_1\sqrt{\frac{\pi}{2}}\eta_2\xi_1 n_0^2 e^2}{2\epsilon_0 k^3 m_2\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1\xi_1 n_0^2 e^2}{\sqrt{2}\epsilon_0 k^3 m_2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\lambda_1\sqrt{\frac{\pi}{2}}\xi_2 n_0^2 e^2}{\epsilon_0 k^3 m_2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\lambda_1\sqrt{\frac{\pi}{2}}\eta_1\xi_1 n_0^2 e^2}{2\epsilon_0 k^3 m_2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{3\lambda_1\xi_1\xi_2 n_0^2 e^2}{2\epsilon_0\sqrt{2}k^3 m_2\eta_1^4\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1\eta_2\omega n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\eta_2^2\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\eta_2\omega n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1^2\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1^2\eta_2\nu_{21}\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\nu_{22}\pi\eta_1\eta_2^2\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\pi\eta_1\eta_2^2\nu_{21}\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_1^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\nu_{22}\pi\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\pi\eta_1^2\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\pi\eta_2^2\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\eta_2\omega n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1^2\eta_2\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\pi\eta_1\eta_2^2\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_2^2\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_2^2)\sqrt{\pi}\eta_2^2\nu_{21}\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2^2\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
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\begin{aligned}
&- \frac{\exp(-\eta_2^2)\sqrt{\pi}\eta_2\nu_{21}\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_2^2)\gamma\nu_{22}\sqrt{\pi}\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\nu_{22}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_2^2)\gamma\sqrt{\pi}\eta_2^2\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\nu_{21}\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_1^3 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_2^2)\gamma\lambda_1\sqrt{\pi}\eta_2^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\eta_2^2\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\lambda_1\omega\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_1^3 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_2^2)\sqrt{\pi}\eta_2^2\nu_{21}\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\lambda_1\sqrt{\pi}\eta_2^2\omega\xi_2 n_0^2 e^2}{2\epsilon_0 k^4 m_2 \eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\nu_{22}\omega^2\xi_1 n_0^2}{2k^4\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2)\nu_{22}\sqrt{\pi}\eta_1^2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\sqrt{\pi}\eta_1^2\nu_{21}\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1^2\omega\xi_1 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{\exp(-\eta_1^2)\nu_{22}\sqrt{\pi}\eta_1\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\sqrt{\pi}\eta_1\nu_{21}\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\lambda_1\nu_{22}\omega^2\xi_1 n_0^2}{4k^4\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&- \frac{2\exp(-\eta_1^2)\gamma\nu_{22}\sqrt{\pi}\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2)\gamma\sqrt{\pi}\eta_1^2\nu_{21}\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_1 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\omega\xi_1 n_0^2}{\sqrt{2}k^3\eta_1^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1\omega\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{2\exp(-\eta_1^2)\gamma\lambda_1\sqrt{\pi}\eta_1^2\xi_1\xi_2 n_0^2 e^2}{\epsilon_0 k^4 m_2 \eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\omega\xi_1 n_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&- \frac{3\exp(-\eta_1^2)\nu_{22}\sqrt{\pi}\eta_1^2\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_1^2)\sqrt{\pi}\eta_1^2\nu_{21}\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_1 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\nu_{22}\omega\xi_1 n_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&- \frac{3\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\eta_1^2\omega\xi_1 n_0^2 e^2}{2\epsilon_0 k^4 m_2 \eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2-\eta_2^2)\pi\eta_1\eta_2\nu_{21}\omega n_0^2 e^2}{\epsilon_0 k^4 m_1 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\lambda_1\nu_{22}\omega\xi_1 n_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_2\omega^2\xi_1 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\omega\xi_1 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{3\lambda_1\nu_{22}\omega\xi_1 n_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&+ \frac{2\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1^2\eta_2\xi_1 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{9\lambda_1\nu_{22}\omega\xi_1 n_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\xi_1 n_0^2}{2k^2\eta_1^2\eta_2 v_{\mathrm{th},1} v_{\mathrm{th},2}} \\
&+ \frac{3\lambda_1\lambda_2\xi_1 n_0^2}{4k^2\eta_1^4\eta_2 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{3\exp(-\eta_2^2)\lambda_1\lambda_2\omega n_0^2\sqrt{\frac{\pi}{2}}}{4k^3\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\omega^2\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{3\lambda_1\lambda_2\xi_1 n_0^2}{8k^2\eta_1^4\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}} + \frac{\lambda_1\nu_{22}\omega^2\xi_2 n_0^2}{2k^4\eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{3\lambda_1\nu_{22}\omega^2\xi_2 n_0^2}{4k^4\eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\lambda_1\lambda_2\omega\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} \\
&+ \frac{3\lambda_1\lambda_2\omega\xi_2 n_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{3\lambda_1\lambda_2\omega\xi_2 n_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{9\lambda_1\lambda_2\omega\xi_2 n_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{2\gamma\lambda_1\nu_{22}\omega\xi_1\xi_2 n_0^2}{k^4\eta_1^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} \\
&+ \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\omega\xi_2 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\lambda_1\nu_{22}\omega\xi_2 n_0^2}{\sqrt{2}k^3\eta_1\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\lambda_1\nu_{22}\omega\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} \\
&+ \frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1^2\omega\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{4\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_2^2\omega\xi_1\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}
\end{aligned}
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\begin{aligned}
&+\frac{\gamma\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{\sqrt{2}k^3\eta_1^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\gamma\lambda_1\lambda_2\xi_1\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\gamma\lambda_1\nu_{22}\xi_1\xi_2 n_0^2}{\sqrt{2}k^3\eta_1^2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{3\gamma\lambda_1\nu_{22}\xi_1\xi_2 n_0^2}{2\sqrt{2}k^3\eta_1^4\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+\frac{\lambda_1\lambda_2\xi_2 n_0^2}{2k^2\eta_1\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}}+\frac{\lambda_1\lambda_2\xi_2 n_0^2}{4k^2\eta_1^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}}+\frac{3\lambda_1\lambda_2\xi_2 n_0^2}{4k^2\eta_1\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}}+\frac{3\lambda_1\lambda_2\xi_2 n_0^2}{8k^2\eta_1^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{2\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1\eta_2^2\xi_2 n_0^2}{k^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\eta_1 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_2 n_0^2}{\sqrt{2}k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\eta_2\xi_1\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\xi_1\xi_2 n_0^2\sqrt{2}}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\lambda_2\omega n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\lambda_1\lambda_2\xi_1 n_0^2}{4k^2\eta_1^2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\omega n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\eta_1\omega n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_1^2)\lambda_1\nu_{22}\omega n_0^2\sqrt{\frac{\pi}{2}}}{4k^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1^2\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1\xi_1 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2\omega\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1\omega^2 n_0^2\sqrt{\pi}}{2k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_1^2)\lambda_1\lambda_2\omega\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\omega\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega^2 n_0^2\sqrt{\pi}}{2k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\omega\xi_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\eta_2\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\omega n_0^2\sqrt{\pi}}{2\sqrt{2}k^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}\\
&+\frac{3\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{2k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}+\frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\eta_2\omega^2 n_0^2\sqrt{\pi}}{4k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\lambda_2 n_0^2\sqrt{\pi}}{2k^2\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{3\exp(-\eta_1^2)\lambda_1\nu_{22}\eta_1\omega^2 n_0^2\sqrt{\pi}}{4k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{2\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2\omega\xi_1 n_0^2\sqrt{\pi}}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\lambda_2 n_0^2\sqrt{\pi}}{4k^2\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\xi_2 n_0^2\sqrt{\frac{\pi}{2}}}{k^3\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{2\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1^2\omega\xi_1 n_0^2\sqrt{\pi}}{k^4\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2 n_0^2\sqrt{\pi}}{2k^2\eta_2 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{3\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1^2\omega\xi_1 n_0^2\sqrt{\pi}}{k^4\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2\xi_1 n_0^2\sqrt{\pi}}{\sqrt{2}k^3\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_1^2)\lambda_1\lambda_2 n_0^2\sqrt{\pi}}{4k^2\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}}\\
&+\frac{2\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\omega\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\omega\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\lambda_1 n_0}{2\sqrt{2}k\eta_1^3 v_{\mathrm{th},1}}\\
&+\frac{2\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1\omega\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}+\frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\omega^2\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma\lambda_1\sqrt{\pi}\eta_1 n_0}{k^2 v_{\mathrm{th},1}^2}
\end{aligned}
$$

$$+ \frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\omega^2\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{2\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\eta_2^2\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} - \frac{\lambda_1 n_0}{\sqrt{2}k\eta_1 v_{\mathrm{th},1}}$$

$$+ \frac{2\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\eta_1^2\xi_1\xi_2 n_0^2\sqrt{\pi}}{k^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_2^2)\gamma\lambda_1\lambda_2\eta_2\xi_2 n_0^2\sqrt{\pi}}{\sqrt{2}k^3\eta_1^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\lambda_1\omega n_0}{2k^2\eta_1^2 v_{\mathrm{th},1}^2}$$

$$+ \frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\eta_1\xi_1 n_0^2\sqrt{\pi}}{\sqrt{2}k^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1\xi_2 n_0^2\sqrt{\pi}}{\sqrt{2}k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{2\exp(-\eta_1^2)\lambda_1\eta_1^2\omega\xi_1 n_0\sqrt{\pi}}{k^2 v_{\mathrm{th},1}^2}$$

$$+ \frac{2\exp(-\eta_2^2)\nu_{22}\eta_2^2\omega\xi_2 n_0\sqrt{\pi}}{k^2 v_{\mathrm{th},2}^2} + \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\eta_1^2\xi_1 n_0^2\sqrt{2\pi}}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} + \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\eta_2^2\xi_2 n_0^2\sqrt{2\pi}}{k^3\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}$$

$$+ \frac{\exp(-\eta_1^2)\lambda_1\eta_1\xi_1 n_0\sqrt{2\pi}}{k v_{\mathrm{th},1}} + \frac{\exp(-\eta_2^2)\lambda_2\eta_2\xi_2 n_0\sqrt{2\pi}}{k v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\lambda_1\sqrt{\pi}\omega\xi_1 n_0}{k^2 v_{\mathrm{th},1}^2} - \frac{\gamma\lambda_1\xi_1 n_0}{k^2\eta_1^3 v_{\mathrm{th},1}^2}$$

$$- \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1\eta_2\omega\xi_2 n_0^2}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1\xi_1 n_0^2}{k^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{\lambda_2 n_0}{2\sqrt{2}k\eta_2^3 v_{\mathrm{th},2}}$$

$$- \frac{\exp(-\eta_2^2)\lambda_1\lambda_2\sqrt{2\pi}\eta_2\omega\xi_1\xi_2 n_0^2}{k^3\eta_1^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_2\xi_2 n_0^2}{k^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{3\lambda_1\omega n_0}{4k^2\eta_1^4 v_{\mathrm{th},1}^2}$$

$$- \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\lambda_2\pi\eta_1^2\omega\xi_1 n_0^2}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\exp(-\eta_2^2)\lambda_1\lambda_2\sqrt{\pi}\eta_2\xi_1\xi_2 n_0^2}{2k^2\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{\lambda_2 n_0}{\sqrt{2}k\eta_2 v_{\mathrm{th},2}}$$

$$- \frac{\exp(-\eta_2^2)\lambda_1\lambda_2\sqrt{\pi}\eta_2\xi_1\xi_2 n_0^2}{k^2\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{\pi}\eta_1\xi_1\xi_2 n_0^2}{k^2\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\xi_1 n_0^2}{k^3\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}$$

$$- \frac{3\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{\pi}\eta_1\xi_1\xi_2 n_0^2}{2k^2\eta_2^4 v_{\mathrm{th},1} v_{\mathrm{th},2}} - \frac{2\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\lambda_2\pi\eta_1^2\eta_2\xi_1\xi_2 n_0^2}{k^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\gamma\nu_{22}\xi_2 n_0}{k^2\eta_2^3 v_{\mathrm{th},2}^2}$$

$$- \frac{\gamma\lambda_1\lambda_2 n_0^2}{2\sqrt{2}k^3\eta_1^2\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\gamma\lambda_1\lambda_2 n_0^2}{4\sqrt{2}k^3\eta_1^4\eta_2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{2\pi}\eta_1^2\omega\xi_1\xi_2 n_0^2}{k^3\eta_2^2 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}}$$

$$- \frac{\exp(-\eta_1^2)\gamma\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\xi_1 n_0^2}{2k^3\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\gamma\lambda_1\lambda_2 n_0^2}{4\sqrt{2}k^3\eta_1^2\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{3\gamma\lambda_1\lambda_2 n_0^2}{8\sqrt{2}k^3\eta_1^4\eta_2^3 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\nu_{22}\omega n_0}{2k^2\eta_2^2 v_{\mathrm{th},2}^2}$$

$$- \frac{3\exp(-\eta_1^2)\lambda_1\lambda_2\sqrt{\frac{\pi}{2}}\eta_1^2\omega\xi_1\xi_2 n_0^2}{k^3\eta_2^4 v_{\mathrm{th},1}^2 v_{\mathrm{th},2}} - \frac{\exp(-\eta_2^2)\gamma\nu_{22}\sqrt{\pi}\eta_2 n_0}{k^2 v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\nu_{22}\sqrt{\pi}\omega\xi_2 n_0}{k^2 v_{\mathrm{th},2}^2}$$

$$- \frac{3\nu_{22}\omega n_0}{4k^2\eta_2^4 v_{\mathrm{th},2}^2} - \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\eta_2\omega\xi_1 n_0^2}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_2^2\omega\xi_2 n_0^2}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}$$

$$- \frac{2\sqrt{2}\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1\eta_2^2\xi_1\xi_2 n_0^2}{k^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{2\pi}\eta_2^2\omega\xi_1\xi_2 n_0^2}{k^3\eta_1^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}$$

$$- \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\xi_2 n_0^2}{k^3\eta_1 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\xi_2 n_0^2}{2k^3\eta_1^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{3\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\frac{\pi}{2}}\eta_2^2\omega\xi_1\xi_2 n_0^2}{k^3\eta_1^4 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}$$

$$- \frac{\gamma\lambda_1\nu_{22} n_0^2}{2\sqrt{2}k^3\eta_1\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\gamma\lambda_1\nu_{22} n_0^2}{4\sqrt{2}k^3\eta_1^3\eta_2^2 v_{\mathrm{th},1} v_{\mathrm{th},2}^2} - \frac{\exp(-\eta_1^2)\lambda_1\nu_{22}\sqrt{2\pi}\eta_1\omega\xi_1\xi_2 n_0^2}{k^3\eta_2^3 v_{\mathrm{th},1} v_{\mathrm{th},2}^2}$$

$$
\begin{aligned}
&-\frac{3\gamma\lambda_1\nu_{22}n_0^2}{4\sqrt{2}k^3\eta_1\eta_2^4v_{\mathrm{th},1}v_{\mathrm{th},2}^2}-\frac{3\gamma\lambda_1\nu_{22}n_0^2}{8\sqrt{2}k^3\eta_1^3\eta_2^4v_{\mathrm{th},1}v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1^2\eta_2\omega^2\xi_1n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\nu_{22}\pi\eta_1\eta_2^2\omega^2\xi_2n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{8\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\eta_1^2\eta_2^2\omega\xi_1\xi_2n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_2\xi_1n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2-\eta_2^2)\gamma^2\lambda_1\nu_{22}\pi\eta_1\xi_2n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\gamma\lambda_1\nu_{22}\omega n_0^2}{2k^4\eta_1^2\eta_2^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{2\exp(-\eta_1^2-\eta_2^2)\gamma\lambda_1\nu_{22}\pi\omega\xi_1\xi_2n_0^2}{k^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_2^2)\lambda_1\nu_{22}\sqrt{\pi}\eta_2^2\omega^2\xi_1\xi_2n_0^2}{k^4\eta_1^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\gamma^2\lambda_1\nu_{22}\xi_1n_0^2}{2k^4\eta_1^3\eta_2^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_2n_0^2}{k^4\eta_1^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_2n_0^2}{2k^4\eta_1^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\xi_1\xi_2n_0^2}{k^4\eta_1^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{3\exp(-\eta_2^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_2n_0^2}{4k^4\eta_1^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_1n_0^2}{k^4\eta_2^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_1n_0^2}{2k^4\eta_2^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{3\exp(-\eta_2^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_2n_0^2}{2k^4\eta_1^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{3\gamma\lambda_1\nu_{22}\omega n_0^2}{4k^4\eta_1^4\eta_2^2v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{2\exp(-\eta_1^2)\lambda_1\nu_{22}\sqrt{\pi}\eta_1^2\omega^2\xi_1\xi_2n_0^2}{k^4\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\xi_1\xi_2n_0^2}{k^4\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{3\exp(-\eta_1^2)\gamma\lambda_1\nu_{22}\sqrt{\pi}\omega\xi_1n_0^2}{2k^4\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\gamma^2\lambda_1\nu_{22}\xi_2n_0^2}{2k^4\eta_1^2\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{3\gamma^2\lambda_1\nu_{22}\xi_2n_0^2}{4k^4\eta_1^4\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{3\exp(-\eta_1^2)\gamma^2\lambda_1\nu_{22}\sqrt{\pi}\eta_1n_0^2}{4k^4\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{3\gamma\lambda_1\nu_{22}\omega n_0^2}{4k^4\eta_1^2\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&-\frac{9\gamma\lambda_1\nu_{22}\omega n_0^2}{8k^4\eta_1^4\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}+\frac{9\nu_{21}\omega n_0^2e^2}{16\epsilon_0k^4m_1\eta_1^4\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}-\frac{\lambda_1\xi_1\xi_2n_0^2e^2}{\sqrt{2}\epsilon_0k^3m_2\eta_1^2\eta_2^3v_{\mathrm{th},1}v_{\mathrm{th},2}^2}\\
&+\frac{3\gamma\nu_{21}\xi_2n_0^2e^2}{4\epsilon_0k^4m_1\eta_1^4\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}+\frac{3\gamma\lambda_1\xi_2n_0^2e^2}{4\epsilon_0k^4m_2\eta_1^4\eta_2^3v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}\\
&+\frac{2\exp(-\eta_1^2-\eta_2^2)\lambda_1\pi\eta_1\eta_2^2\xi_1\xi_2n_0^2\sqrt{2}e^2}{\epsilon_0k^3m_2v_{\mathrm{th},1}v_{\mathrm{th},2}^2}-\frac{3\gamma^2\lambda_1\nu_{22}\xi_1n_0^2}{4k^4\eta_1^3\eta_2^4v_{\mathrm{th},1}^2v_{\mathrm{th},2}^2}
\end{aligned}
$$

A.2 Python Code for the Numerical Calculation of the Damping Effect

In Chapter 3.2.7 the zeros of the dispersion relation (3.34) of the two-species Vlasov-Poisson-BGK system were determined and represented in Fig. 3 and Fig. 4 in a complex plane. For this purpose, Eric Sonnendrücker's Python code for determining the zeros in case of the Vlasov-Poisson model for one species, namely electrons, was modified [31].

In the first step one defines the plasma dispersion function as well as the class *DefFunctions*. In this class one has the definitions for the functions counting the number of zeros and finding the zeros in a rectangular box by means of the generalized eigenvalues of Hankel matrices. If the determined zeros are not the zeros of the input function or if the box contains more than five zeros, one redefines the rectangular box by dividing it into four smaller boxes.

In the second step one takes the Python file *LandauBGK2Species* and inserts the dispersion relation of one's system, in our case the dispersion relation for the Vlasov-Poisson-BGK system for two species (3.34). After specifying the physical constants one defines the range of k, for which one wants to determine the zeros. As an additional result one gets the amount of the zero with the largest imaginary part.

The pseudo code for the first step looks as follows:

Define plasma dispersion function $\mathcal{Z}$:
    input: x
    output: $\mathcal{Z}(x) = \sqrt{\pi}e^{-x^2}\left[\mathrm{i} - \mathrm{erfi}(x)\right]$

Define **class** *DefFunctions*:
    Initialize

    Define function count_zeros:
        input: rectangular box (xmin, xmax, ymin, ymax)
        output: $s = \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{f'(x)}{f(x)}dx$

    Define function get_zeros:
        input: rectangular box (xmin, xmax, ymin, ymax), error margin (tol, tolK), maximal number of zeros (maxiter)
        s = count_zeros(xmin, xmax, ymin, ymax)
        if $s > 5$ **do**
            redefine box (see function refine)
        else do
            compute zeros by means of Hankel matrices
        check error margin
        output: zeros

    Define function refine:
        input: rectangular box (xmin, xmax, ymin, ymax)
        subdivide box into four smaller boxes of equal size
        output: zeros in smaller boxes

The second step has the following pseudo code:

Define the constants:
    mass ratio between ion and electron (m),
    thermal velocity of ion (vth1) and electron (vth2),
    plasma frequency of ion (omegap1) and electron (omegap2)
    collision frequencies (nu22, nu21, nu12, nu11, lambda2, lambda1)
    number densities (n0)

Define dispersion relation:
    input: omegabar, k

output: dispersion relation according to (3.34)

Define rectangular box:
xmin, xmax, ymin, ymax

for k **in** $[0.15, 0.30]$ in steps of 0.05 **do**:
get_zeros (see class *DefFunctions*)
get zero with the largest imaginary part (omega)
plot zeros with real and imaginary parts as axes

The formulated Python codes for the class *DefFunctions* and the file for the actual calculation *LandauBGK2Species* are the following:

```
import numpy as np
import sympy as sym
import mpmath as mp  # multiple precision arithmetic
from scipy.linalg import hankel, eigvals
from scipy.special import erfi
from scipy import integrate

mp.dps = 30

mods = ['numpy', {'erfi': erfi}]

def Z(x):
    """ Plasma dispersion function """
    return sym.sqrt(sym.pi) * sym.exp(-(x ** 2)) * (1j - sym.erfi(x))

class DefFunctions:
    def __init__(self, D, kmode, max_zeros=20):
        self.kmode = kmode
        self.zeros = []
        omegabar = sym.symbols('omega')
        self.D = sym.lambdify(omegabar, D(omegabar, kmode), 'mpmath')
        self.max_zeros = max_zeros
        self.Dprime_over_D = sym.lambdify(omegabar, sym.diff(D(omegabar
                    , kmode), omegabar) / D(omegabar, kmode), 'mpmath')
        self.D_over_Dprime = sym.lambdify(omegabar, D(omegabar, kmode)
                    / sym.diff(D(omegabar, kmode), omegabar), 'mpmath')

    def count_zeros(self, xmin, xmax, ymin, ymax, tol=1.e-6, deg=3):
        """ Count the number of zeros in the box defined by xmin,xmax,
            ymin,ymax; Returns the number of zeros """
        k = self.kmode
        s1 = mp.quad(lambda t: self.Dprime_over_D(xmax + 1j * t).real,
                    [ymin, ymax], maxdegree=deg)
        s2 = mp.quad(lambda t: self.Dprime_over_D(xmin + 1j * t).real,
                    [ymin, ymax], maxdegree=deg)
        s3 = mp.quad(lambda t: self.Dprime_over_D(t + 1j * ymin).imag,
                    [xmin, xmax], maxdegree=deg)
```

```
    s4 = mp.quad(lambda t: self.Dprime_over_D(t + 1j * ymax).imag,
                 [xmin, xmax], maxdegree=deg)
    return (s1 - s2 + s3 - s4) / (2 * np.pi)

def get_zeros(self, xmin, xmax, ymin, ymax, deg=3, tol=1e-12, tolK=
        0.01, maxiter=10, verbose=False):
    """ Count zeros in the rectangular box """
    if verbose:
        print('Exploring box:' + str(xmin) + ',' + str(xmax) + ','
              + str(ymin) + ',' + str(ymax))
    nzeros = self.count_zeros(xmin, xmax, ymin, ymax)
    K = int(round(nzeros.real))
    if abs(K - nzeros.real) > tolK or K > 5:
        if verbose:
            print('refining: error=', abs(K-nzeros.real), ' K=', K)
        self.refine(xmin, xmax, ymin, ymax, deg, tol, tolK, maxiter
                    , verbose)
    else:
        if verbose:
            print('found ' + str(K) + ' zeros,Error=' + str(abs(K -
                  nzeros.real)))
        # Compute s_m if K>0
        if K > 0:
            s = np.zeros(2 * K, 'complex')
            for m in range(0, 2 * K):
                s1 = mp.quad(lambda t: (xmax + 1j * t) ** m * self.
                        Dprime_over_D(xmax + 1j * t), [ymin,
                        ymax], maxdegree=deg) / (2 * mp.pi)
                s2 = mp.quad(lambda t: (xmin + 1j * t) ** m * self.
                        Dprime_over_D(xmin + 1j * t), [ymin,
                        ymax], maxdegree=deg) / (2 * mp.pi)
                s3 = mp.quad(lambda t: (t + 1j * ymin) ** m * self.
                        Dprime_over_D(t + 1j * ymin), [xmin,
                        xmax], maxdegree=deg) / (2 * 1j * mp.pi)
                s4 = mp.quad(lambda t: (t + 1j * ymax) ** m * self.
                        Dprime_over_D(t + 1j * ymax), [xmin,
                        xmax], maxdegree=deg) / (2 * 1j * mp.pi)
                s[m] = s1 - s2 + s3 - s4
            # Compute zeros as generalised eigenvalues of Hankel
                matrices
            H = hankel(s[0:K], s[K - 1:2 * K - 1])
            H2 = hankel(s[1:K + 1], s[K:2 * K])
            w = eigvals(H2, H)
            # Check error on zero and perform Newton refinement if
                necessary
            error_flag = False
            for i in range(len(w)):
                ww = w[i]
                error = abs(self.D(ww))
                it = 0
                while error > tol and it < maxiter:
                    ww = ww - self.D_over_Dprime(ww)
```

```
                        error = abs(self.D(ww))
                        it = it + 1
                    w[i] = ww
                    if verbose:
                        print(str(ww) + ': error on zero= ' + str(error
                             ) + ' #iter=' + str(it))
                    if (error > tol):
                        error_flag = True
                        break
                if error_flag:
                    self.refine(xmin, xmax, ymin, ymax, deg, tol, tolK,
                          maxiter, verbose)
                else:
                    self.zeros = self.zeros + w.tolist()
        return self.zeros

    def refine(self, xmin, xmax, ymin, ymax, deg, tol, tolK, maxiter,
            verbose):
        """ Get zeros in refined box """
        h = [float(xmax - xmin), float(ymax - ymin)]
        self.get_zeros(xmin, xmin + 0.5 * h[0], ymin + 0.5 * h[1], ymax
              , deg, tol, tolK, maxiter, verbose)
        self.get_zeros(xmin + 0.5 * h[0], xmax, ymin + 0.5 * h[1], ymax
              , deg, tol, tolK, maxiter, verbose)
        self.get_zeros(xmin + 0.5 * h[0], xmax, ymin, ymin + 0.5 * h[1]
              , deg, tol, tolK, maxiter, verbose)
        self.get_zeros(xmin, xmin + 0.5 * h[0], ymin, ymin + 0.5 * h[1]
              , deg, tol, tolK, maxiter, verbose)
```

```
import matplotlib.pyplot as plt
from DefFunctions import *
from pylab import *

# Physical constants
m=1800 # Mass ratio between ion and electron m1/m2
vth1=1  # Thermal velocity ion
vth2=vth1*sym.sqrt(m)  # Thermal velocity electron
omegap1=1 # Plasma frequency ion
omegap2=omegap1*sym.sqrt(m) # Plasma frequency electron
nu22=0.5  # Collision frequency electron with electron
nu21=nu22 # Collision frequency electron with ion
nu11=nu22/sym.sqrt(m) # Collision frequency ion with ion
nu12=nu22/m # Collision frequency ion with electron
lambda1=nu11+nu12 # Collision frequencies ion
lambda2=nu22+nu21 # Collision frequencies electron
n0=1  # number densities

# Dispersion relation
def D(omegabar,k):
   return (1 + ((omegap1**2+1j*n0*lambda1*omegabar)/(k*vth1)**2) + ((
           omegabar*(omegap1**2-n0**2*lambda1**2))/(sym.sqrt(2)*k**3
           *vth1**3)+(1j*n0*lambda1)/(sym.sqrt(2)*k*vth1)*(1+(
```

```
               omegabar**2+omegap1**2)/((k*vth1)**2)))*Z((omegabar+1j*n0
               *lambda1)/(sym.sqrt(2)*k*vth1)))*(1 + ((omegap2**2+1j*n0*
               nu22*omegabar)/(k*vth2)**2) + ((omegabar*(omegap2**2-n0**
               2*nu22*lambda2))/(sym.sqrt(2)*k**3*vth2**3)+(1j*n0)/(sym.
               sqrt(2)*k*vth2)*(lambda2+(nu22*omegabar**2+lambda2*
               omegap2**2)/((k*vth2)**2)))*Z((omegabar+1j*n0*lambda2)/(
               sym.sqrt(2)*k*vth2)))-(omegap1**2*(omegap2**2-1j*n0*nu21*
               omegabar))/((k**2*vth1*vth2)**2)*(1+((omegabar+1j*n0*
               lambda1)/(sym.sqrt(2)*k*vth1))*Z((omegabar+1j*n0*lambda1)
               /(sym.sqrt(2)*k*vth1)))*(1+((omegabar+1j*n0*lambda2)/(sym
               .sqrt(2)*k*vth2))*Z((omegabar+1j*n0*lambda2)/(sym.sqrt(2)
               *k*vth2)))

xmin=-1.6
xmax=-1*xmin
ymin=-1.6
ymax=0.1

for kmode in arange(0.15,0.35,.05):
    print('-------------------------------------')
    print('mode:', kmode.round(decimals=2) )
    zaf=DefFunctions(D,kmode)
    print('number of zeros:', zaf.count_zeros(xmin, xmax, ymin, ymax))
    zeros=zaf.get_zeros(xmin, xmax, ymin, ymax)

    # Determine the zero with the largest imaginary part
    zero_max=zeros[argmax(imag(zeros))]

    print('-------------------------')
    print('k=',kmode.round(decimals=2))
    print ('zero with largest imaginary part (omega):', zero_max)
    plot(real(zeros),imag(zeros),'.',label='k='+str(kmode.round(
              decimals=2)))

axis([xmin-.1,xmax+.1,ymin-.1,ymax+.1])
title('zeros of dispersion relation, nu22={}'.format(nu22))
legend(loc='lower center')
show()
```

The results of the application of the code can be seen in Fig. 3 and Fig. 4 of Chapter 3.2.7.

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Julius-Maximilians-
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WÜRZBURG

Versicherung zur selbständigen Leistungserbringung

Titel der Masterarbeit:

Linear Landau Damping Coupled with Relaxation for the Two-Species Vlasov-Poisson-BGK System

Thema bereitgestellt von (Titel, Vorname, Nachname, Lehrstuhl):
Prof. Dr. Christian Klingenberg, Lehrstuhl für Mathematik VI

Eingereicht durch (Vorname, Nachname, Matrikel):
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