Well-balanced high-order finite difference methods for systems of balance laws.

C. Parés, C. Parés-Pulido.

Universidad de Málaga, ETH Zürich.

Seminar series ”Structure preserving methods for hyperbolic equations”

Málaga, Würzburg University, 03/12/2020.

Outline

Outline

Shallow water equations: formulation

- Let us consider the PDE system

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{g}{2} h^2 \right) = gh\partial_x H, \end{cases} \tag{1}$$

  that governs the evolutions of a shallow layer of fluid, where:
  - the variable x makes reference to the axis of the channel;
  - t is time;
  - $q(x,t)$ is the discharge;
  - $h(x,t)$ is the thickness of the fluid layer;
  - g is the acceleration due to gravity; $H(x)$ is the depth measured from a fixed level of reference;
  - $q(x,t) = h(x,t)u(x,t)$, with u the depth averaged horizontal velocity.

Shallow water equations: stationary solutions

- The eigenvalues of the Jacobian matrix $J(U)$ of the flux function $f(U)$ are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}.$$

- The Froude number, given by

$$Fr(U) = \frac{|u|}{gh},$$

indicates the flow regime: subcritical ($Fr < 1$), critical ($Fr = 1$) or supercritical ($Fr > 1$).

- The stationary solutions of this system are implicitly given by

$$q = C_1, \quad \frac{1}{2}\frac{q^2}{h^2} + gh - gH = C_2, \tag{2}$$

where C_i, $i = 1, 2$ are arbitrary constants.

- Water at rest equilibria are the 1-parameter family of stationary solutions corresponding to $C_1 = 0$, i.e.

$$q = 0, \quad h - H = \bar{\eta}, \tag{3}$$

where $\bar{\eta}$ is a constant, corresponding to the vertical coordinate of the elevation of the unperturbed surface of the water.

Shallow water equations: stationary solutions

- The eigenvalues of the Jacobian matrix $J(U)$ of the flux function $f(U)$ are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}.$$

- The Froude number, given by

$$Fr(U) = \frac{|u|}{gh},$$

  indicates the flow regime: subcritical ($Fr < 1$), critical ($Fr = 1$) or supercritical ($Fr > 1$).

- The stationary solutions of this system are implicitly given by

$$q = C_1, \quad \frac{1}{2}\frac{q^2}{h^2} + gh - gH = C_2, \tag{2}$$

  where C_i, $i = 1, 2$ are arbitrary constants.

- Water at rest equilibria are the 1-parameter family of stationary solutions corresponding to $C_1 = 0$, i.e.

$$q = 0, \quad h - H = \bar{\eta}, \tag{3}$$

  where $\bar{\eta}$ is a constant, corresponding to the vertical coordinate of the elevation of the unperturbed surface of the water.

Shallow water equations: stationary solutions

- The eigenvalues of the Jacobian matrix $J(U)$ of the flux function $f(U)$ are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}.$$

- The Froude number, given by

$$Fr(U) = \frac{|u|}{gh},$$

  indicates the flow regime: subcritical ($Fr < 1$), critical ($Fr = 1$) or supercritical ($Fr > 1$).
- The stationary solutions of this system are implicitly given by

$$q = C_1, \quad \frac{1}{2}\frac{q^2}{h^2} + gh - gH = C_2, \tag{2}$$

  where C_i, $i = 1, 2$ are arbitrary constants.
- Water at rest equilibria are the 1-parameter family of stationary solutions corresponding to $C_1 = 0$, i.e.

$$q = 0, \quad h - H = \bar{\eta}, \tag{3}$$

  where $\bar{\eta}$ is a constant, corresponding to the vertical coordinate of the elevation of the unperturbed surface of the water.

Shallow water equations: stationary solutions

- The eigenvalues of the Jacobian matrix $J(U)$ of the flux function $f(U)$ are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}.$$

- The Froude number, given by

$$Fr(U) = \frac{|u|}{gh},$$

  indicates the flow regime: subcritical ($Fr < 1$), critical ($Fr = 1$) or supercritical ($Fr > 1$).

- The stationary solutions of this system are implicitly given by

$$q = C_1, \quad \frac{1}{2}\frac{q^2}{h^2} + gh - gH = C_2, \tag{2}$$

  where C_i, $i = 1, 2$ are arbitrary constants.

- Water at rest equilibria are the 1-parameter family of stationary solutions corresponding to $C_1 = 0$, i.e.

$$q = 0, \quad h - H = \bar{\eta}, \tag{3}$$

  where $\bar{\eta}$ is a constant, corresponding to the vertical coordinate of the elevation of the unperturbed surface of the water.

Shallow water equations: well-balanced methods

- Well balanced schemes preserve (in some sense) stationary solutions.
- This property is important when the waves generated by small perturbations of an equilibrium are to be simulated: numerical errors should not break the equilibrium.
- In the context of shallow water system 'well-balanced' has in general two different meanings.

Shallow water equations: well-balanced methods

- Well balanced schemes preserve (in some sense) stationary solutions.
- This property is important when the waves generated by small perturbations of an equilibrium are to be simulated: numerical errors should not break the equilibrium.
- In the context of shallow water system 'well-balanced' has in general two different meanings.

Shallow water equations: well-balanced methods

- Well balanced schemes preserve (in some sense) stationary solutions.
- This property is important when the waves generated by small perturbations of an equilibrium are to be simulated: numerical errors should not break the equilibrium.
- In the context of shallow water system 'well-balanced' has in general two different meanings.

- **Methods that preserve water-at-rest solutions**:
  - Bermúdez and Vázquez-Cendón, 1994: *C-property*.
  - Many different numerical methods that satisfy this property have been introduced in the literature: see Bouchut 2004, chapters of Xing and Castro, Morales, CP in Handbook of Numerical Methods 2017 and their references for a review.
  - Different techniques have been applied: source term upwinding (Bermúdez and Vázquez-Cendón 1994), Hydrostatic Reconstruction technique . Audusse, Bouchut, Bristeau, Klein, Perthame 2004, etc.
  - In the framework of finite difference methods, high-order schemes that satisfy the C-property were introduced in Caselles, Donat, Haro, 2009 and Xing, Shu 2006.

Shallow water equations: stationary solutions

- **Methods that preserve arbitrary stationary solutions**:
  - A first order numerical methods that preserve all the stationary based on a Generalized Hydrostatic Reconstruction technique was presented in Castro, Pardo, CP 2007.
  - High-order well-balanced finite volume methods that preserves all the stationary solutions. A non-exhaustive list:
    - Noelle, Pankratz, Puppo, Natvig, 2006,
    - Noelle, Xing, C.-W. Shu 2007,
    - Russo, Khe 2009 ,
    - Canestrelli, Siviglia, Dumbser, Toro 2009,
    - Bouchut, Morales 2010,
    - Castro, López, CP 2013,
    - Xing, 2014,
    - Berthon, Chalons 2016,
    - Cheng, Chertock, Herty, Kurganov, Wu 2019,
    - ...
- To the best of our knowledge, high order finite difference methods with this enhanced well-balanced property have not been described before

Well-balanced methods for other systems of balance laws

- The design of high-order well-balanced numerical methods for different systems of balance laws is a very front active. Some examples:
  - Variants of the shallow water model (with friction, Coriolis term, RIPA model, shallow water system in spherical coordinates, etc.): Lukácová-Medvid'ová, Noelle, Kraft 2007 , Chertok, Kurganov, Liu 2015, Chertock, Cui, Kurganov, Wu 2015, Sánchez-Linares, Morales, Castro 2016, Castro, Ortega, CP 2017 ...
  - Euler equations with gravity: Käppeli, Mishra 2014, Chandrashekar, Klingenberg 2015, Li, Xing 2016, Chandrashekar, Zenk 2017, Gaburro, Dumbser, Castro 2017, Klingenberg, Puppo, Semplice 2018, Chertock, Cui, Kurganov, Özcan, Tadmor 2018, Gaburro, Castro, Dumbser 2018, Li, Xing 2018, Berberich, Chandrashekar, Klingenberg 2019, Grosheintz-Laval, Käppeli 2019, Castro, CP 2020, Castro, Gómez, CP 2021, . . .
  - Other flow models like blood flow in vessels (Müller, CP, Toro 2013), relativistic fluids on a Schwarzschild background (LeFloch, CP, Pimentel 2020). etc.

Outline

Well-balanced methods: a general framework for 1d problems

- Many different definitions of well-balanced can be found in the literature depending on the problem, on the methods, on what is preserved, etc...
- Following discussions with M.J.Castro, I. Gómez-Bueno, C. Klingenberg, some general definitions will be given here.
- Let us consider hyperbolic 1d systems of balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \tag{4}$$

where $U(x,t)$ takes value in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$).
- The system has nontrivial **stationary solutions** U^* that satisfy the ODE system:

$$F(U^*)_x = S(U^*)H_x.$$

Well-balanced methods: a general framework for 1d problems

- Many different definitions of well-balanced can be found in the literature depending on the problem, on the methods, on what is preserved, etc...
- Following discussions with M.J.Castro, I. Gómez-Bueno, C. Klingenberg, some general definitions will be given here.
- Let us consider hyperbolic 1d systems of balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \tag{4}$$

where $U(x,t)$ takes value in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$).
- The system has nontrivial **stationary solutions** U^* that satisfy the ODE system:

$$F(U^*)_x = S(U^*)H_x.$$

Well-balanced methods: a general framework for 1d problems

- Many different definitions of well-balanced can be found in the literature depending on the problem, on the methods, on what is preserved, etc...
- Following discussions with M.J.Castro, I. Gómez-Bueno, C. Klingenberg, some general definitions will be given here.
- Let us consider hyperbolic 1d systems of balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \tag{4}$$

where $U(x,t)$ takes value in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$).
- The system has nontrivial **stationary solutions** U^* that satisfy the ODE system:

$$F(U^*)_x = S(U^*)H_x.$$

Well-balanced methods: a general framework for 1d problems

- Many different definitions of well-balanced can be found in the literature depending on the problem, on the methods, on what is preserved, etc...
- Following discussions with M.J.Castro, I. Gómez-Bueno, C. Klingenberg, some general definitions will be given here.
- Let us consider hyperbolic 1d systems of balance laws of the form

$$U_t + F(U)_x = S(U)H_x, \tag{4}$$

  where $U(x,t)$ takes value in $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$).
- The system has nontrivial **stationary solutions** U^* that satisfy the ODE system:

$$F(U^*)_x = S(U^*)H_x.$$

Well-balanced methods: a general framework for 1d problems

- Let us consider a consistent semidiscret numerical method to solve (4):

$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j, \tag{5}$$

  where

$$\mathbf{U}(t) = \{U_i(t)\}_{i \in \mathcal{I}}.$$

  Here $U_i(t)$ represents the ith degree of freedom and $\mathcal{I}$, the set of indices.

- Let p be the order of accuracy of the method, i.e.

$$U_i(t) = \mathcal{U}_i(t) + O(\Delta x^q), \quad \forall i, t,$$

  where $\mathcal{U}_i(t)$ represents the exact value of the ith degree of freedom at time t.

- For finite-difference methods:

$$\mathcal{U}_i(t) = U(x_i, t), \quad \forall i \in \mathcal{I}, \forall t,$$

  while for finite-volume methods:

$$\mathcal{U}_i(t) = \overline{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t)\,dx, \quad \forall i \in \mathcal{I}, \forall t,$$

  with the usual notation for the mesh points x_i, the intercells $x_{i+1/2}$, and the space step Δx (assumed to be constant for simplicity).

Well-balanced methods: a general framework for 1d problems

- Let us consider a consistent semidiscret numerical method to solve (4):

$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j, \tag{5}$$

  where

$$\mathbf{U}(t) = \{U_i(t)\}_{i \in \mathcal{I}}.$$

  Here $U_i(t)$ represents the ith degree of freedom and $\mathcal{I}$, the set of indices.

- Let p be the order of accuracy of the method, i.e.

$$U_i(t) = \mathcal{U}_i(t) + O(\Delta x^q), \quad \forall i, t,$$

  where $\mathcal{U}_i(t)$ represents the exact value of the ith degree of freedom at time t.

- For finite-difference methods:

$$\mathcal{U}_i(t) = U(x_i, t), \quad \forall i \in \mathcal{I}, \forall t,$$

  while for finite-volume methods:

$$\mathcal{U}_i(t) = \overline{U}_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t)\, dx, \quad \forall i \in \mathcal{I}, \forall t,$$

  with the usual notation for the mesh points x_i, the intercells $x_{i+1/2}$, and the space step Δx (assumed to be constant for simplicity).

Well-balanced methods: a general framework for 1d problems

- Let us consider a consistent semidiscret numerical method to solve (4):
$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j, \tag{5}$$
where
$$\mathbf{U}(t) = \{U_i(t)\}_{i \in \mathcal{I}}.$$
Here $U_i(t)$ represents the ith degree of freedom and $\mathcal{I}$, the set of indices.
- Let p be the order of accuracy of the method, i.e.
$$U_i(t) = \mathcal{U}_i(t) + O(\Delta x^q), \quad \forall i, t,$$
where $\mathcal{U}_i(t)$ represents the exact value of the ith degree of freedom at time t.
- For finite-difference methods:
$$\mathcal{U}_i(t) = U(x_i, t), \quad \forall i \in \mathcal{I}, \forall t,$$
while for finite-volume methods:
$$\mathcal{U}_i(t) = \overline{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t)\, dx, \quad \forall i \in \mathcal{I}, \forall t,$$
with the usual notation for the mesh points x_i, the intercells $x_{i+1/2}$, and the space step Δx (assumed to be constant for simplicity).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.
$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\mathcal{U}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\mathcal{U}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.

$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\boldsymbol{\mathcal{U}}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\boldsymbol{\mathcal{U}}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.

$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\mathcal{U}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\mathcal{U}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.

$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\mathcal{U}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\mathcal{U}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.
$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\mathcal{U}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\mathcal{U}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.

$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
    - $U(x,t)$ solution of the system of balance laws (function).
    - $U^*(x)$ stationary solution of the system of balance laws (function).
    - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
    - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
    - $\mathcal{U}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
    - $\mathcal{U}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.
$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\boldsymbol{\mathcal{U}}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\boldsymbol{\mathcal{U}}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

- A **discrete stationary solution** is an equilbrium $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ of the ODE system (5), i.e.
$$\mathcal{H}_j(\Delta x, \mathbf{U}^*) = 0, \quad \forall j.$$

- **Notation summary:** a short guide not to get lost...
  - $U(x,t)$ solution of the system of balance laws (function).
  - $U^*(x)$ stationary solution of the system of balance laws (function).
  - $\mathbf{U}(t) = \{U_i(t)\}_{i\in\mathcal{I}}$ discrete solution (vector).
  - $\mathbf{U}^* = \{U_i^*\}_{i\in\mathcal{I}}$ discrete stationary solution (vector).
  - $\boldsymbol{\mathcal{U}}(t) = \{\mathcal{U}_i(t)\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to a solution U (vector).
  - $\boldsymbol{\mathcal{U}}^* = \{\mathcal{U}_i^*\}_{i\in\mathcal{I}}$ exact values of the degrees of freedom corresponding to the stationary solution U^* (vector).

Well-balanced methods: a general framework for 1d problems

Definition

The numerical method (5) *is said to be* ***well-balanced*** *(WB) for a given stationary solution U^* of* (4) *if there exists a discrete stationary solution $\mathbf{U}^* = \{U_i^*\}$ such that:*

$$U_i^* = \mathcal{U}_i^* + O(\Delta x^q),$$

with $q \geq p$.

Well-balanced methods: a general framework for 1d problems

Definition

The numerical method (5) *is said to be* ***well-balanced*** *(WB) for a given stationary solution* U^* *of* (4) *if there exists a discrete stationary solution* $\mathbf{U}^* = \{U_i^*\}$ *such that:*

$$U_i^* = \mathcal{U}_i^* + O(\Delta x^q),$$

with $q \geq p$*.*

Remarks

- Many well-balanced methods described in the literature fit to this definition.
- Although it is never included in the different definitions of well-balanced, it is implicitly assumed (and numerically checked) that the stability of $\mathbf{U}^*$ for (5) and the stability of U^* for (4) have to be the same.

Definition

The numerical method (5) *is said to be* ***exactly well-balanced*** *(EWB) for a given stationary solution* U^* *of* (4) *if the vector of the exact values of the degrees of freedom*

$$\boldsymbol{\mathcal{U}}^* = \{\mathcal{U}_i^*\}$$

is a discrete stationary solution.

Well-balanced methods: a general framework for 1d problems

Definition

The numerical method (5) *is said to be* ***exactly well-balanced*** *(EWB) for a given stationary solution* U^* *of* (4) *if the vector of the exact values of the degrees of freedom*

$$\boldsymbol{\mathcal{U}}^* = \{\mathcal{U}_i^*\}$$

is a discrete stationary solution.

Remarks

- Most of the papers of the EDANYA group are based on this definition.
- EWB implies WB.

Well-balanced methods: a general framework for 1d problems

Definition

The numerical method (5) *is said to be WB (resp. EWB) for a family of stationary solutions if it is WB (resp. EWB) for any of them. In particular, a numerical method is said to be **fully well-balanced** (FWB) (resp. **fully exactly well-balanced** (FEWB)) if it is WB (resp. EWB) for any stationary solution.*

Well-balanced methods: a general framework for 1d problems

Definition

The numerical method (5) *is said to be WB (resp. EWB) for a family of stationary solutions if it is WB (resp. EWB) for any of them. In particular, a numerical method is said to be* ***fully well-balanced*** *(FWB) (resp.* ***fully exactly well-balanced*** *(FEWB)) if it is WB (resp. EWB) for any stationary solution.*

Remarks

- C-property = EWB for the family of water-at-rest solutions.
- Usually, stationary solutions with stationary shock waves are not considered in the FWB or the EWB definition.

Well-balanced methods: fully discrete schemes

- Usually an ODE solver is applied to the semidiscrete method

$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j. \tag{6}$$

- If, for instance, a standard one-step method is applied:

$$U_i^{n+1} = U_i^n + \Delta t \Phi_i(\mathbf{U}^n, \Delta t), \quad \forall i \in \mathcal{I}, \quad n = 0, 1, \ldots$$

  the well-balanced properties of the semidiscrete method is preserved provided that

$$\Phi_i(\mathbf{U}^*, \Delta t) = 0, \quad \forall i \in \mathcal{I},$$

  for all discrete stationary solution (i.e. any equilibrium of the ODE system (6)).

- RK methods (in particular TVD-RK methods), have this property.

Well-balanced methods: fully discrete schemes

- Usually an ODE solver is applied to the semidiscrete method

$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j. \tag{6}$$

- If, for instance, a standard one-step method is applied:

$$U_i^{n+1} = U_i^n + \Delta t \Phi_i(\mathbf{U}^n, \Delta t), \quad \forall i \in \mathcal{I}, \quad n = 0, 1, \ldots$$

  the well-balanced properties of the semidiscrete method is preserved provided that

$$\Phi_i(\mathbf{U}^*, \Delta t) = 0, \quad \forall i \in \mathcal{I},$$

  for all discrete stationary solution (i.e. any equilibrium of the ODE system (6)).

- RK methods (in particular TVD-RK methods), have this property.

Well-balanced methods: fully discrete schemes

- Usually an ODE solver is applied to the semidiscrete method

$$\frac{dU_j}{dt} = \mathcal{H}_j(\Delta x, \mathbf{U}(t)), \quad \forall j. \tag{6}$$

- If, for instance, a standard one-step method is applied:

$$U_i^{n+1} = U_i^n + \Delta t \Phi_i(\mathbf{U}^n, \Delta t), \quad \forall i \in \mathcal{I}, \quad n = 0, 1, \ldots$$

  the well-balanced properties of the semidiscrete method is preserved provided that

$$\Phi_i(\mathbf{U}^*, \Delta t) = 0, \quad \forall i \in \mathcal{I},$$

  for all discrete stationary solution (i.e. any equilibrium of the ODE system (6)).

- RK methods (in particular TVD-RK methods), have this property.

Well-balanced methods: high-order finite volume methods

- A general methodology to design high-order FEWB methods for systems of conservation laws has been described in Castro, CP 2020 based on state reconstruction operators (see Manuel Castro's talk in this series).
- The key points are the following: given a vector of cell averages $\{\overline{U}_i\}$ to compute the reconstruction at the ith cell
  - Find the stationary solution U_i^* that satisfies:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = \overline{U}_i.$$

  - Apply a standard reconstruction operator to the fluctuations

$$D_j = \overline{U}_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx, \quad j \in S_i,$$

    where S_i is the stencil of the ith cell.

Well-balanced methods: high-order finite volume methods

- A general methodology to design high-order FEWB methods for systems of conservation laws has been described in Castro, CP 2020 based on state reconstruction operators (see Manuel Castro's talk in this series).
- The key points are the following: given a vector of cell averages $\{\overline{U}_i\}$ to compute the reconstruction at the ith cell
  - Find the stationary solution U_i^* that satisfies:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = \overline{U}_i.$$

  - Apply a standard reconstruction operator to the fluctuations

$$D_j = \overline{U}_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx, \quad j \in \mathcal{S}_i,$$

    where $\mathcal{S}_i$ is the stencil of the ith cell.

Well-balanced methods: high-order finite volume methods

- A general methodology to design high-order FEWB methods for systems of conservation laws has been described in Castro, CP 2020 based on state reconstruction operators (see Manuel Castro's talk in this series).
- The key points are the following: given a vector of cell averages $\{\overline{U}_i\}$ to compute the reconstruction at the ith cell
  - Find the stationary solution U_i^* that satisfies:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = \overline{U}_i.$$

  - Apply a standard reconstruction operator to the fluctuations

$$D_j = \overline{U}_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx, \quad j \in \mathcal{S}_i,$$

    where $\mathcal{S}_i$ is the stencil of the ith cell.

Well-balanced methods: high-order finite volume methods

- A general methodology to design high-order FEWB methods for systems of conservation laws has been described in Castro, CP 2020 based on state reconstruction operators (see Manuel Castro's talk in this series).
- The key points are the following: given a vector of cell averages $\{\overline{U}_i\}$ to compute the reconstruction at the ith cell
  - Find the stationary solution U_i^* that satisfies:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = \overline{U}_i.$$

  - Apply a standard reconstruction operator to the fluctuations

$$D_j = \overline{U}_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx, \quad j \in \mathcal{S}_i,$$

  where $\mathcal{S}_i$ is the stencil of the ith cell.

- The problem to be solved consists in finding a solution of an ODE system with given average.
- If this problem can be exactly solved, FEWB methods are derived.
- If the problem is solved numerically, FWB methods are derived. Two different techniques have been considered so far:
  - control techniques: Castro, Gómez-Bueno, CP 2021;
  - collocation RK methods: Castro, Gómez-Bueno, CP, Russo in preparation.

Well-balanced methods: high-order finite volume methods

- The problem to be solved consists in finding a solution of an ODE system with given average.
- If this problem can be exactly solved, FEWB methods are derived.
- If the problem is solved numerically, FWB methods are derived. Two different techniques have been considered so far:
  - control techniques: Castro, Gómez-Bueno, CP 2021;
  - collocation RK methods: Castro, Gómez-Bueno, CP, Russo in preparation.

Well-balanced methods: high-order finite volume methods

- The problem to be solved consists in finding a solution of an ODE system with given average.
- If this problem can be exactly solved, FEWB methods are derived.
- If the problem is solved numerically, FWB methods are derived. Two different techniques have been considered so far:
  - control techniques: Castro, Gómez-Bueno, CP 2021;
  - collocation RK methods: Castro, Gómez-Bueno, CP, Russo in preparation.

Well-balanced methods: high-order finite volume methods

- The problem to be solved consists in finding a solution of an ODE system with given average.
- If this problem can be exactly solved, FEWB methods are derived.
- If the problem is solved numerically, FWB methods are derived. Two different techniques have been considered so far:
  - control techniques: Castro, Gómez-Bueno, CP 2021;
  - collocation RK methods: Castro, Gómez-Bueno, CP, Russo in preparation.

Well-balanced methods: high-order finite volume methods

- The problem to be solved consists in finding a solution of an ODE system with given average.
- If this problem can be exactly solved, FEWB methods are derived.
- If the problem is solved numerically, FWB methods are derived. Two different techniques have been considered so far:
  - control techniques: Castro, Gómez-Bueno, CP 2021;
  - collocation RK methods: Castro, Gómez-Bueno, CP, Russo in preparation.

Outline

FEWB finite difference methods: main idea

- The goal in CP, Parés-Pulido 2020 was to use a similar idea for finite-difference methods based on flux reconsturctions.
- The idea is easy: let us consider a high-order flux reconstruction operator that, given a set of flux values $\{F(U_j)\}_{j\in\mathcal{I}}$, provides reconstructions of the flux at the intercells

$$\widehat{F}_{i+1/2} = \mathcal{R}(F(U_{i-r}), \ldots, F(U_{i+s})),$$

where $\mathcal{S}_{i+1/2} = \{x_{i-s}, \ldots, x_{i+r}\}$ is the stencil associated to the intercell, in such a way that, if there exists a smooth function U satisfying

$$U_i = U(x_i), \quad \forall i \in \mathcal{I},$$

then,

$$F(U)_x(x_i) = \frac{\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}}{\Delta x} + O(\Delta x^p).$$

ENO or WENO conservative reconstructions are examples of such operators: see Shu 88, Jiang, Shu 1996, Shu 98.

FEWB finite difference methods: main idea

- The goal in CP, Parés-Pulido 2020 was to use a similar idea for finite-difference methods based on flux reconsturctions.
- The idea is easy: let us consider a high-order flux reconstruction operator that, given a set of flux values $\{F(U_j)\}_{j\in\mathcal{I}}$, provides reconstructions of the flux at the intercells

$$\widehat{F}_{i+1/2} = \mathcal{R}(F(U_{i-r}), \dots, F(U_{i+s})),$$

where $\mathcal{S}_{i+1/2} = \{x_{i-s}, \dots, x_{i+r}\}$ is the stencil associated to the intercell, in such a way that, if there exists a smooth function U satisfying

$$U_i = U(x_i), \quad \forall i \in \mathcal{I},$$

then,

$$F(U)_x(x_i) = \frac{\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}}{\Delta x} + O(\Delta x^p).$$

ENO or WENO conservative reconstructions are examples of such operators: see Shu 88, Jiang, Shu 1996, Shu 98.

FEWB finite difference methods: idea

- Let U_i^* be the stationary solution satisfying the Cauchy problem:

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U(x_i, t), \end{cases} \tag{7}$$

  where $U(x_i, t)$ is the point-value of the sought solution.
- Then one has trivially

$$S(U(x_i, t))H_x(x_i) = S(U_i^*(x_i))H_x(x_i) = F(U_i^*)_x(x_i),$$

  so that (4) can be rewritten at (x_i, t) as follows:

$$U_t + (F(U) - F(U_i^*))_x = 0. \tag{8}$$

- Now, the flux reconstruction can be applied to the set $\{F(U_j) - F(U_i^*(x_j))\}_{j \in \mathcal{I}}$ to approximate its derivative at x_i at time t.

FEWB finite difference methods: idea

- Let U_i^* be the stationary solution satisfying the Cauchy problem:

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U(x_i, t), \end{cases} \tag{7}$$

  where $U(x_i, t)$ is the point-value of the sought solution.
- Then one has trivially

$$S(U(x_i, t))H_x(x_i) = S(U_i^*(x_i))H_x(x_i) = F(U_i^*)_x(x_i),$$

  so that (4) can be rewritten at (x_i, t) as follows:

$$U_t + (F(U) - F(U_i^*))_x = 0. \tag{8}$$

- Now, the flux reconstruction can be applied to the set $\{F(U_j) - F(U_i^*(x_j))\}_{j \in \mathcal{I}}$ to approximate its derivative at x_i at time t.

FEWB finite difference methods: idea

- Let U_i^* be the stationary solution satisfying the Cauchy problem:

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U(x_i,t), \end{cases} \tag{7}$$

  where $U(x_i,t)$ is the point-value of the sought solution.
- Then one has trivially

$$S(U(x_i,t))H_x(x_i) = S(U_i^*(x_i))H_x(x_i) = F(U_i^*)_x(x_i),$$

  so that (4) can be rewritten at (x_i,t) as follows:

$$U_t + (F(U) - F(U_i^*))_x = 0. \tag{8}$$

- Now, the flux reconstruction can be applied to the set $\{F(U_j) - F(U_i^*(x_j))\}_{j\in\mathcal{I}}$ to approximate its derivative at x_i at time t.

FEWB finite difference methods: definition

- More precisely, the following numerical method is proposed to solve the system of balance laws (4):

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = 0, \tag{9}$$

where the "numerical fluxes" $\widehat{\mathcal{F}}_{i,i\pm 1/2}$ are computed as follows:

1. Look for the solution $U_i^*(x)$ of the Cauchy problem

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{10}$$

2. Define

$$\mathcal{F}_j = F(U_j) - F(U_i^*(x_j)), \quad j = i-1-r, \dots, i+s$$

3. Compute

$$\begin{aligned} \widehat{\mathcal{F}}_{i,i+1/2} &= \mathcal{R}(\mathcal{F}_{i-r}, \dots, \mathcal{F}_{i+s}), \\ \widehat{\mathcal{F}}_{i,i-1/2} &= \mathcal{R}(\mathcal{F}_{i-1-r}, \dots, \mathcal{F}_{i-1+s}). \end{aligned}$$

FEWB finite difference methods: definition

- More precisely, the following numerical method is proposed to solve the system of balance laws (4):

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = 0, \tag{9}$$

where the "numerical fluxes" $\widehat{\mathcal{F}}_{i,i\pm 1/2}$ are computed as follows:

1. Look for the solution $U_i^*(x)$ of the Cauchy problem

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{10}$$

2. Define

$$\mathcal{F}_j = F(U_j) - F(U_i^*(x_j)), \quad j = i-1-r, \dots, i+s$$

3. Compute

$$\begin{aligned} \widehat{\mathcal{F}}_{i,i+1/2} &= \mathcal{R}(\mathcal{F}_{i-r}, \dots, \mathcal{F}_{i+s}), \\ \widehat{\mathcal{F}}_{i,i-1/2} &= \mathcal{R}(\mathcal{F}_{i-1-r}, \dots, \mathcal{F}_{i-1+s}). \end{aligned}$$

FEWB finite difference methods: definition

- More precisely, the following numerical method is proposed to solve the system of balance laws (4):

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = 0, \tag{9}$$

where the “numerical fluxes” $\widehat{\mathcal{F}}_{i,i\pm 1/2}$ are computed as follows:

1. Look for the solution $U_i^*(x)$ of the Cauchy problem

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{10}$$

2. Define

$$\mathcal{F}_j = F(U_j) - F(U_i^*(x_j)), \quad j = i-1-r, \dots, i+s$$

3. Compute

$$\begin{aligned} \widehat{\mathcal{F}}_{i,i+1/2} &= \mathcal{R}(\mathcal{F}_{i-r}, \dots, \mathcal{F}_{i+s}), \\ \widehat{\mathcal{F}}_{i,i-1/2} &= \mathcal{R}(\mathcal{F}_{i-1-r}, \dots, \mathcal{F}_{i-1+s}). \end{aligned}$$

FEWB finite difference methods: definition

- More precisely, the following numerical method is proposed to solve the system of balance laws (4):

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = 0, \tag{9}$$

where the "numerical fluxes" $\widehat{\mathcal{F}}_{i,i\pm 1/2}$ are computed as follows:

1. Look for the solution $U_i^*(x)$ of the Cauchy problem

$$\begin{cases} F(U_i^*)_x = S(U_i^*)H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{10}$$

2. Define

$$\mathcal{F}_j = F(U_j) - F(U_i^*(x_j)), \quad j = i-1-r, \dots, i+s$$

3. Compute

$$\begin{aligned} \widehat{\mathcal{F}}_{i,i+1/2} &= \mathcal{R}(\mathcal{F}_{i-r}, \dots, \mathcal{F}_{i+s}), \\ \widehat{\mathcal{F}}_{i,i-1/2} &= \mathcal{R}(\mathcal{F}_{i-1-r}, \dots, \mathcal{F}_{i-1+s}). \end{aligned}$$

FEWB finite difference methods: definition

- The method is well-defined if the Cauchy problem (10) has a unique solution whose interval of definition contains the extended stencil

$$\mathcal{S}_i = \{x_{i-r-1}, \dots, x_{i+s}\}$$

  associated to the *i*th cell. In that case, we have proved that:
  - The numerical method is consistent and its order of accuracy is the one of the reconstruction operator.
  - It is fully exactly well-balanced.

FEWB finite difference methods: definition

- The method is well-defined if the Cauchy problem (10) has a unique solution whose interval of definition contains the extended stencil
$$\mathcal{S}_i = \{x_{i-r-1}, \ldots, x_{i+s}\}$$
associated to the ith cell. In that case, we have proved that:
  - The numerical method is consistent and its order of accuracy is the one of the reconstruction operator.
  - It is fully exactly well-balanced.

Remarks:

- The reconstruction operator is not applied to $\{F(U_j)\}$ and to $\{F(U_i^*(x_j))\}$ but to their differences $\{F(U_j) - F(U_i^*(x_j))\}$: subtraction and reconstruction **are not commutative**!
- In the notation $\widehat{\mathcal{F}}_{i,i+1/2}$ the index $i + 1/2$ corresponds to the intercell and the index i to the center of the cell where the initial condition of (7) is imposed.
- In general $\widehat{\mathcal{F}}_{i,i+1/2} \neq \widehat{\mathcal{F}}_{i+1,i+1/2}$ as one can expect due to the non conservative nature of the system of equations.
- Notice that two reconstructions have to be computed at every stencil $\mathcal{S}_{i+1/2}$: $\widehat{\mathcal{F}}_{i,i+1/2}$ and $\widehat{\mathcal{F}}_{i+1,i+1/2}$.

FEWB finite difference methods: definition

- Observe that, if the eigenvalues of $J(U)$ do not vanish, the Cauchy problems (10) to be solved can be written normal form:

$$\begin{cases} \dfrac{dU_i^*}{dx} = J(U_i^*)^{-1} S(U_i^*) H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{11}$$

In this case, there is always a unique maximal solution (under the adequate smoothness assumptions).

- If the eigenvalues may vanish (as it happens in practice...), (10) may have no solution or to have more than one.
- If (10) has no solution or if its solution cannot be defined in the whole stencil, the values at the points of the cells cannot be the point-values of a stationary solution. In this case, only the fluxes are reconstructed and a standard treatment of the source term is applied:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}\right) = S(U_i) H_x(x_i), \tag{12}$$

- If there is more than one solution, a criterion is needed to choose one of them: see CP, Parés-Pulido 2020 for details for the shallow-water system.

FEWB finite difference methods: definition

- Observe that, if the eigenvalues of $J(U)$ do not vanish, the Cauchy problems (10) to be solved can be written normal form:

$$\begin{cases} \dfrac{dU_i^*}{dx} = J(U_i^*)^{-1} S(U_i^*) H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{11}$$

  In this case, there is always a unique maximal solution (under the adequate smoothness assumptions).

- If the eigenvalues may vanish (as it happens in practice...), (10) may have no solution or to have more than one.
- If (10) has no solution or if its solution cannot be defined in the whole stencil, the values at the points of the cells cannot be the point-values of a stationary solution. In this case, only the fluxes are reconstructed and a standard treatment of the source term is applied:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}\right) = S(U_i)H_x(x_i), \tag{12}$$

- If there is more than one solution, a criterion is needed to choose one of them: see CP, Parés-Pulido 2020 for details for the shallow-water system.

FEWB finite difference methods: definition

- Observe that, if the eigenvalues of $J(U)$ do not vanish, the Cauchy problems (10) to be solved can be written normal form:

$$\begin{cases} \dfrac{dU_i^*}{dx} = J(U_i^*)^{-1} S(U_i^*) H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{11}$$

  In this case, there is always a unique maximal solution (under the adequate smoothness assumptions).

- If the eigenvalues may vanish (as it happens in practice...), (10) may have no solution or to have more than one.
- If (10) has no solution or if its solution cannot be defined in the whole stencil, the values at the points of the cells cannot be the point-values of a stationary solution. In this case, only the fluxes are reconstructed and a standard treatment of the source term is applied:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}\right) = S(U_i) H_x(x_i), \tag{12}$$

- If there is more than one solution, a criterion is needed to choose one of them: see CP, Parés-Pulido 2020 for details for the shallow-water system.

FEWB finite difference methods: definition

- Observe that, if the eigenvalues of $J(U)$ do not vanish, the Cauchy problems (10) to be solved can be written normal form:

$$\begin{cases} \dfrac{dU_i^*}{dx} = J(U_i^*)^{-1}S(U_i^*)H_x, \\ U_i^*(x_i) = U_i. \end{cases} \tag{11}$$

  In this case, there is always a unique maximal solution (under the adequate smoothness assumptions).
- If the eigenvalues may vanish (as it happens in practice...), (10) may have no solution or to have more than one.
- If (10) has no solution or if its solution cannot be defined in the whole stencil, the values at the points of the cells cannot be the point-values of a stationary solution. In this case, only the fluxes are reconstructed and a standard treatment of the source term is applied:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}\right) = S(U_i)H_x(x_i), \tag{12}$$

- If there is more than one solution, a criterion is needed to choose one of them: see CP, Parés-Pulido 2020 for details for the shallow-water system.

WB finite difference methods for a family of stationary solutions

- Let us suppose that k-parameter family of stationary solutions

$$U^*(x; C_1, \dots, C_k),$$

with $k < N$, where N is the number of unknowns. In this case the Cauchy problem is replaced by solving a nonlinear system

$$u^*_{j_l}(x_i; C^i_1, \dots, C^i_k) = u_{i,j_l}, \quad l = 1, \dots, k, \tag{13}$$

where u^*_j, $u_{i,j}$ denote respectively the jth component of U^* and U_i and $\{j_1, \dots, j_k\}$ is a predetermined set of k indices.

- The expression of the method is in this case as follows:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = (S(U_i) - S(U^*_i(x_i)))H_x(x_i), \tag{14}$$

- High-order finite-difference methods that preserve water-at-rest solutions are easily derived using this procedure.

WB finite difference methods for a family of stationary solutions

- Let us suppose that k-parameter family of stationary solutions

$$U^*(x; C_1, \dots, C_k),$$

with $k < N$, where N is the number of unknowns. In this case the Cauchy problem is replaced by solving a nonlinear system

$$u^*_{j_l}(x_i; C^i_1, \dots, C^i_k) = u_{i,j_l}, \quad l = 1, \dots, k, \tag{13}$$

where u^*_j, $u_{i,j}$ denote respectively the jth component of U^* and U_i and $\{j_1, \dots, j_k\}$ is a predetermined set of k indices.

- The expression of the method is in this case as follows:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = (S(U_i) - S(U^*_i(x_i)))H_x(x_i), \tag{14}$$

- High-order finite-difference methods that preserve water-at-rest solutions are easily derived using this procedure.

WB finite difference methods for a family of stationary solutions

- Let us suppose that k-parameter family of stationary solutions

$$U^*(x; C_1, \dots, C_k),$$

with $k < N$, where N is the number of unknowns. In this case the Cauchy problem is replaced by solving a nonlinear system

$$u^*_{j_l}(x_i; C^i_1, \dots, C^i_k) = u_{i,j_l}, \quad l = 1, \dots, k, \tag{13}$$

where u^*_j, $u_{i,j}$ denote respectively the jth component of U^* and U_i and $\{j_1, \dots, j_k\}$ is a predetermined set of k indices.

- The expression of the method is in this case as follows:

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{\mathcal{F}}_{i,i+1/2} - \widehat{\mathcal{F}}_{i,i-1/2}\right) = (S(U_i) - S(U^*_i(x_i)))H_x(x_i), \tag{14}$$

- High-order finite-difference methods that preserve water-at-rest solutions are easily derived using this procedure.

FEWB finite difference vs. FEWB finite volume

- Main **advantage** of the FEWB finite difference methods: Cauchy problems for ODE's have to be solved at the reconstruction procedure instead of ODE systems with prescribed averages, what is much easier and less costly if the solutions are to be numerically computed.

- Nevertheless, for first and second order finite volume methods the problems to be solved are Cauchy problems as well if the midpoint rule is used to compute cell-averages. Nevertheless, the numerical methods are not equivalent in this case.

FEWB finite difference vs. FEWB finite volume

- Main **advantage** of the FEWB finite difference methods: Cauchy problems for ODE's have to be solved at the reconstruction procedure instead of ODE systems with prescribed averages, what is much easier and less costly if the solutions are to be numerically computed.

- Nevertheless, for first and second order finite volume methods the problems to be solved are Cauchy problems as well if the midpoint rule is used to compute cell-averages. Nevertheless, the numerical methods are not equivalent in this case.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

FEWB finite difference vs. FEWB finite volume

- Main **drawback** of the FEWB finite differences: if the system includes pure conservation laws (as the mass equation for the shallow water equations) the methods introduced may not be conservative for them: there may be a conservation error that tends to 0 with Δx. This is not the case for finite volume methods.
- Nevertheless, there are some exceptions to this:
  - Methods that preserve only one stationary solution are conservative for the conservation laws included in the system.
  - Methods that preserve water-at-rest solutions for the shallow water equation are mass conservative.
  - We have obtained methods for the shallow water system that are FEWB and mass-conservative using a flux splitting whose viscosity vanishes for stationary solutions, but that they are oscillatory for supercritical regimes.
- The design of FWEB mass-conservative stable methods for the shallow water equations seems to be a challenging problem.

Outline

Numerical tests

- 3d and 5th order WENO conservative reconstruction based on the (global) Lax-Friedrichs flux-splitting have been applied: see Jiang, Shu 1996, Shu 98.
- Standard high-order methods

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\widehat{F}_{i+1/2} - \widehat{F}_{i-1/2}\right) = S(U_i)H_x(x_i),$$

  have been considered to compare the numerical results.
- The third order TVD-RK3 method is applied for the time discretization: see Gottlieb, Shu 1998.
- In all cases, explicit or implicit expressions of the solutions of the Cauchy problems (10) are available.

A linear problem: order test

- We consider the linear scalar problem
$$u_t + u_x = u.$$
- The stationary solutions are:
$$u^*(x) = Ce^x, \quad C \in \mathbb{R}. \tag{15}$$
- We consider the initial condition:
$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ p(x) & \text{if } 0 \leq x \leq 1, \\ 1 & \text{otherwise}, \end{cases} \tag{16}$$
where p is the 11th degree polynomial
$$p(x) = x^6 \left(\sum_{k=0}^{5} (-1)^k \left(\begin{array}{c} 5+k \\ k \end{array} \right) (x-1)^k \right)$$
such that
$$p(0) = 0, \quad p(1) = 1, \quad p^k(0) = p^k(1) = 0, \quad k = 1, \ldots, 5$$
see Figure 1.

A linear problem: order test

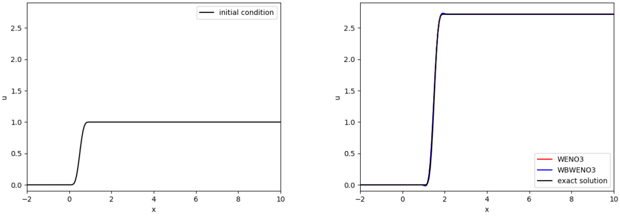


Figure: Test 5.1.1: initial condition (left). Exact solution and numerical solution obtained with WBWENO3 and WBWENO5 at time $t = 1$ using a mesh of 200 cells

	WBWENO3		WBWENO5	
Cells	Error	Order	Error	Order
100	1.023E-1	-	4.0910E-2	-
200	2.084E-2	2.29	2.4407E-3	4.06
400	3.019E-3	2.78	9.1315E-5	4.74
800	3.867E-4	2.96	3.0121E-6	4.92
1600	4.855E-5	2.99	9.4857E-8	4.98

Table: Errors in L^1 norm and convergence rates for WBWENOp, $p = 3, 5$ at time $t = 1$.

Burgers' equation with source term: preservation of a stationary solution

- We consider next the scalar equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = u^2 \, H_x.$$

- The stationary solutions are given by

$$u^*(x) = Ce^{H(x)}, \quad C \in \mathbb{R}.$$

- We consider first $H(x) = x$ and we take the stationary solution

$$u(x) = e^x$$

as initial condition.

Burgers' equation with source term: preservation of a stationary solution

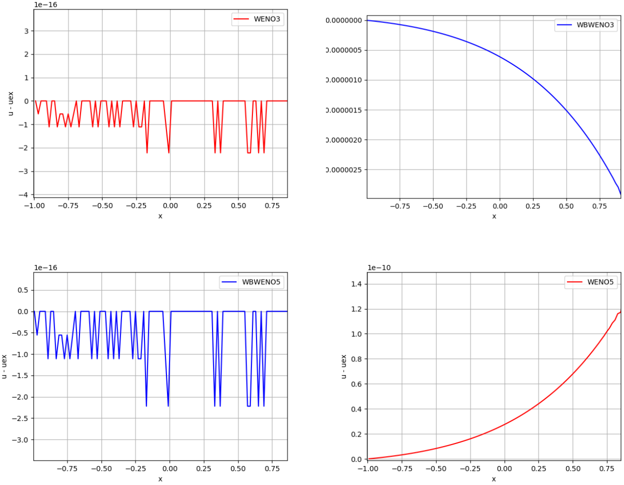


Figure: Differences between the numerical solutions at time $t = 8$ and the stationary solution using a 200-cell mesh. Up-left: WBWENO3. Up-right: WENO3. Down-left: WB3. WENO5. Down-right: WENO5

Burgers' equation with source term: preservation of a stationary solution

Cells	WENO3 Error	WENO3 Order	WBWENO3 Error
100	1.9044E-06	-	8.9928E-17
200	2.4762E-07	2.94	1.4543E-16
400	3.1550E-08	2.97	1.5304E-14
800	3.9817E-09	2.98	1.6560E-14

Table: Errors in L^1 norm and convergence rates for WB3 and WBWENO3 at time $t = 8$.

Cells	WENO5 Error	WENO5 Order	WBWENO5 Error
20	7.7695E-07	-	2.2759e-16
40	3.5170E-09	7.78	1.5543e-16
80	2.0005E-10	4.13	1.1657e-16
160	1.0352E-11	4.27	2.9559e-16

Table: Errors in L^1 norm and convergence rates for WENO5 and WBWENO5 at time $t = 8$.

Burgers' equation with source term: preservation of a stationary solution with oscillatory smooth H

- Let us consider now:

$$H(x) = x + 0.1\sin(100x).$$

- The stationary solution

$$u(x) = e^{x+0.1\sin(100x)},$$

is taken as initial condition.

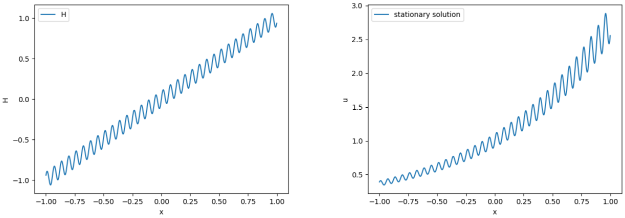
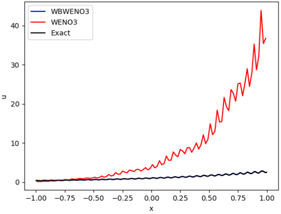


Figure: Graph of the function H (left) and stationary solution (right)

Burgers' equation with source term: preservation of a stationary solution with oscillatory smooth H



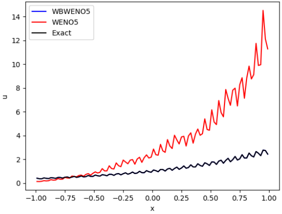
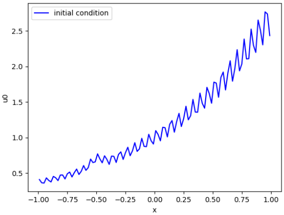


Figure: Exact solution and numerical solutions obtained at time $t = 1$ and a mesh of 100 cells. Left: WBWENO3, WENO3. Right: WBWENO5, WENO5

Burgers' equation with source term: Perturbation of a stationary solution with oscillatory smooth H

- We consider now the initial condition.

$$u_0(x) = e^{x+0.1\sin(100x)} + 0.1e^{-200(x+5)^2},$$



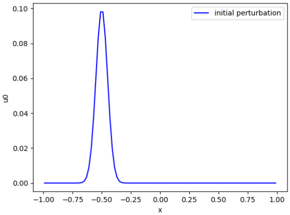


Figure: Initial condition. Left: graph. Right: difference with the stationary solution

Burgers' equation with source term: Perturbation of a stationary solution with oscillatory smooth H

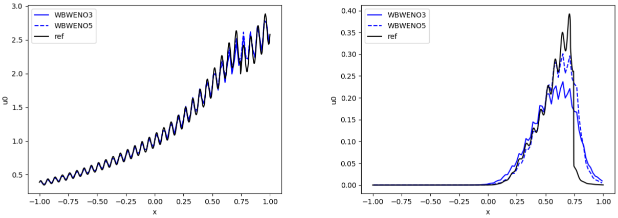


Figure: Reference and numerical solutions obtained with WBWENO3 and WBWENO5 at time $t = 1$ and a mesh of 100 cells. Left: graphs. Right: difference with the stationary solutions

Shallow water equations

- Three different numerical methods are considered for the shallow water system:
  - WENO*p*: standard WENO reconstruction and standard treatment of the source term.
  - WBWENO*p*: FEWB WENO method.
  - WBWARWENO*p*: methods that preserve water at rest stationary solutions
- In one test case two other versions are considered:
  - WB1WENO*p*: method that only preserves one given stationary solution.
  - WBMCWENO*p*: methods that preserve every stationary solutions and the total mass.

Shallow water equations: preservation of a subcritical stationary solution

We consider the shallow water system with the bottom depth given by

$$H(x) = \begin{cases} -0.25(1+\cos(5\pi x)) & \text{if } -0.2 \leq x \leq 0.2; \\ 0 & \text{otherwise;} \end{cases} \tag{17}$$

and we take as initial condition the subcritical stationary solution (h^*, q^*) characterized by

$$q^* = 2.5. \quad h^*(-3) = 2.$$

Shallow water equations: preservation of a subcritical stationary solution

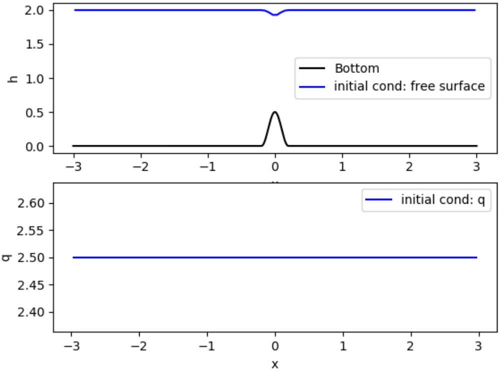


Figure: Subcritical stationary solution: surface elevation (top) and mass-flow (bottom)

Shallow water equations: preservation of a subcritical stationary solution

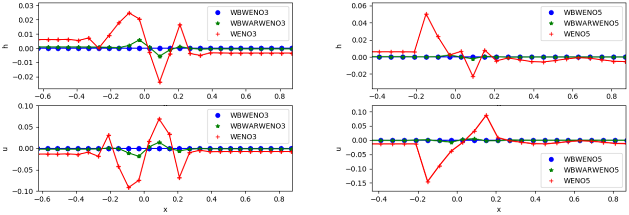


Figure: Zoom of the differences between the numerical solutions obtained at time $t = 4.$ with WBWENOp, WBWARWENp, and WENO3p , $p = 3$ (left) and $p = 5$ (right), using a mesh of 100 cells and the exact solution: surface elevation (top) and discharge (bottom)

Shallow water equations: preservation of a subcritical stationary solution

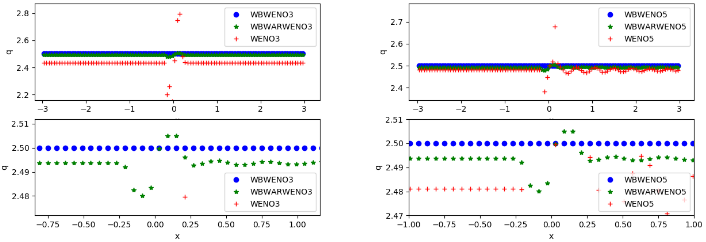


Figure: Numerical results for the variable q at $t = 4.$ using a mesh of 100 cells with WBWENOp, WBWARWENp, and WENOp, $p = 3$ (left) and $p = 5$ (right): general view (top) and zoom close to $x = 0$ (bottom).

Shallow water equations: preservation of a subcritical stationary solution

	WBWENO3	WBWARWENO3		WENO3	
Cells	Error	Error	Order	Error	Order
50	0	4.9069E-2	-	3.6778E-1	-
100	2.8110E-15	2.3981E-2	1.03	9.3955E-2	1.968
200	2.6378E-15	4.3491E-3	2.46	1.3430E-2	2.806
400	4.6629E-17	5.9130E-4	2.8787	1.7931E-3	2.904

Table: Errors in L^1 norm and convergence rates for WBWENO3, WBWARWENO3, and WENO3 at time $t = 4$.

	WBWENO5	WBWARWENO5		WENO5	
Cells	Error	Error	Order	Error	Order
50	0	3.3234E-2	-	4.1777E-1	-
100	2.6645E-17	7.5930E-3	2.129	5.1077E-2	3.031
200	4.6629E-17	4.7013E-4	4.013	3.8702E-3	3.722
400	4.6629E-17	2.5026E-05	4.231	4.18112E-4	3.210

Table: Errors in L^1 norm and convergence rates for WB1WENO5, WBWENO5, WBMCWENO5, WBWARWENO5, and WENO5 at time $t = 4$.

Shallow water equations: perturbation of a subcritical stationary solution

In this test case, a perturbation of size $\Delta h = 0.02$ is added to the thickness h in the interval $[-0.4, -0.3]$ at $t = 0$.

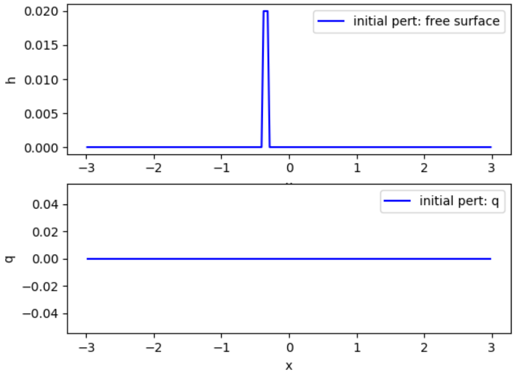


Figure: Initial perturbation: surface elevation (top) and mass-flow (bottom)

Shallow water equations: perturbation of a subcritical stationary solution

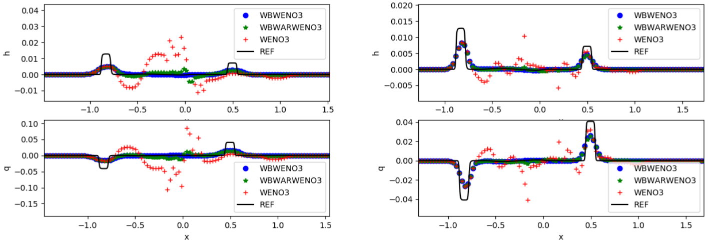


Figure: Zoom of the differences between the numerical solutions obtained at time $t = 0.15$ using a mesh of 200 points and the stationary solution. WBWENOp, WBWARWENOp, WENOp, $p = 3$ (left) and $p = 5$ (right); surface elevation (top) and discharge (down).

Shallow water equations: preservation of a transcritical stationary solution over a discontinuous bottom

- We consider now a discontinuous topography given by the depth function

$$H(x) = \begin{cases} -0.25(1+\cos(5\pi(x+1.2))) & \text{if } -1.4 \leq x \leq -1, \\ 1 & \text{if } x > 0; \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

- We take now as initial condition the transcritical admissible stationary solution characterized by:

$$q^* = 2.5, \quad h^*(0) = \frac{(2.5)^{2/3}}{g^{1/3}}$$

that is subcritical at the left of $x = 0$ and supercritical at its right.

WBWENO3	WENO3	WBWENO5	WENO5
7.9602E-16	1.3178	7.9602e-16	0.6229

Table: Errors in L^1 norm for WB1WENOp, WBWENOp, and WENOp, $p = 3, 5$ at time $t = 4$.

Shallow water equations: preservation of a transcritical stationary solution over a discontinuous bottom

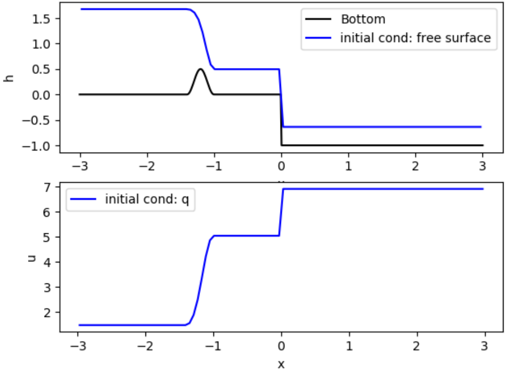


Figure: Transcritical stationary solution: surface elevation (top) and velocity (bottom)

Shallow water equations: preservation of a transcritical stationary solution over a discontinuous bottom

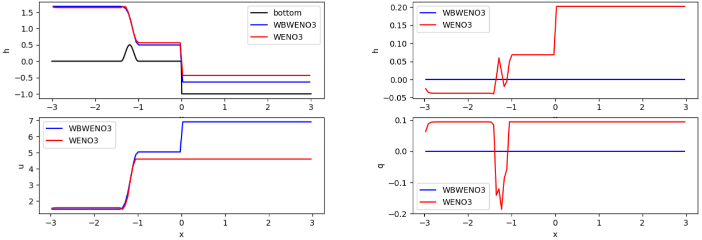


Figure: Numerical solutions obtained at time $t = 4.$ with WBWENO3, and WENO3 using a mesh of 100 cells: surface elevation (top) and velocity (bottom). Right: Difference between the numerical solutions and the stationary solution: surface elevation (top) and mass-flow (bottom)

Shallow water equations: preservation of a transcritical stationary solution over a discontinuous bottom

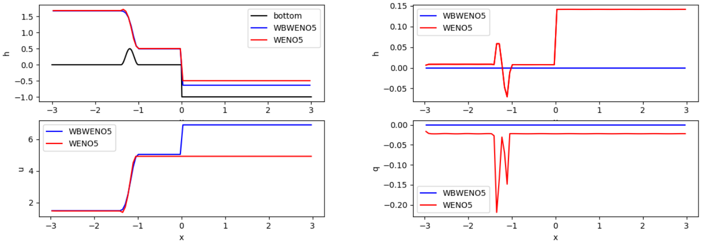


Figure: Numerical solutions obtained at time $t = 4.$ with WBWENO5, and WENO5 using a mesh of 100 cells: surface elevation (top) and velocity (bottom). Right: Difference between the numerical solutions and the stationary solution: surface elevation (top) and mass-flow (bottom)

Shallow water equations: perturbation of a transcritical stationary solution over a discontinuous bottom

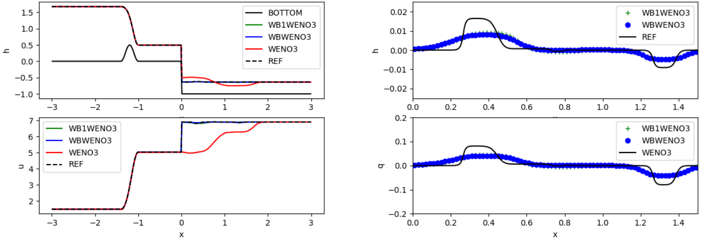


Figure: Numerical solutions obtained at time $t = 0.2$ with WB1WENO3, WBWENO3, and WENO3 using a mesh of 300 cells: surface elevation (top) and velocity (bottom). Right: Difference between the numerical solutions obtained with WB1WENO3 and WBWENO3 and the stationary solution: surface elevation (top) and mass-flow (bottom)

Shallow water equations: perturbation of a transcritical stationary solution over a discontinuous bottom

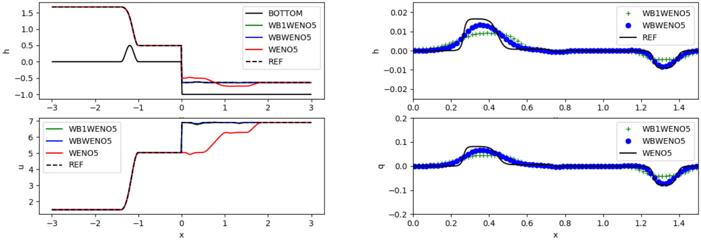


Figure: Left: Numerical solutions obtained at time $t = 0.2$ with WB1WENO5, WBWENO5, and WENO5 using a mesh of 300 cells: surface elevation (top) and velocity (bottom). Right: Difference between the numerical solutions obtained with WB1WENO5 and WBWENO5 and the stationary solution: surface elevation (top) and mass-flow (bottom)

Shallow water equations: mass conservation and computational cost

In order to measure the mass conservation properties of the different methods and compare the computational cost, we consider now the depth function

$$H(x) = \begin{cases} 0.13 + 0.05(x-10)^2 & \text{if } 8 \leq x \leq 12; \\ 0.33 & \text{otherwise.} \end{cases}$$

and the initial condition

$$h_0(x) = h^*(x) + 0.5\chi_{[5,7]}, \quad q_0(x) = 1,$$

where $h^*(x)$ is the thickness corresponding to the stationary solution characterized by

$$q^* = 1, \quad h^*(10) = 1,$$

and $\chi_{[a,b]}$ denotes the characteristic function of an interval $[a, b]$.

Shallow water equations: mass conservation and computational cost

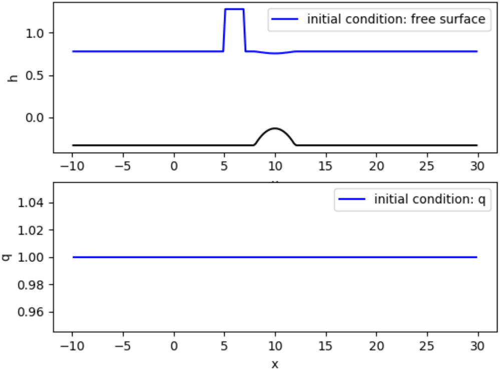


Figure: Initial condition: surface elevation (top) and mass-flow (bottom)

Shallow water equations: mass conservation and computational cost

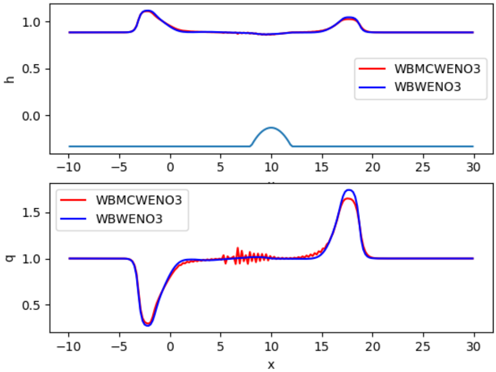


Figure: Numerical solutions obtained at time $t = 2.5$ with WBWENO3 and WBMCWENO3 using a mesh of 200 cells: surface elevation (top) and mass-flow (bottom)

Shallow water equations: mass conservation and computational cost

WENO3	WBWENO3	WB1WENO3	WBWARWENO3	WBMCWENO3
8.1062E-15	9.5985E-06	7.3825E-15	8.54056E-15	7.3825E-15

Table: Maximum relative deviation of the total mass.

Shallow water equations: mass conservation and computational cost

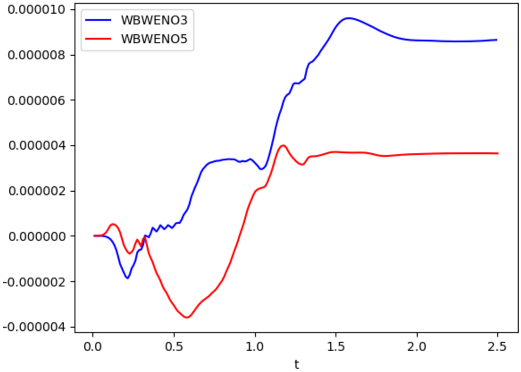


Figure: Evolution of the relative deviation of the total mass with time for WBWENO3 and WBWENO5

Shallow water equations: mass conservation and computational cost

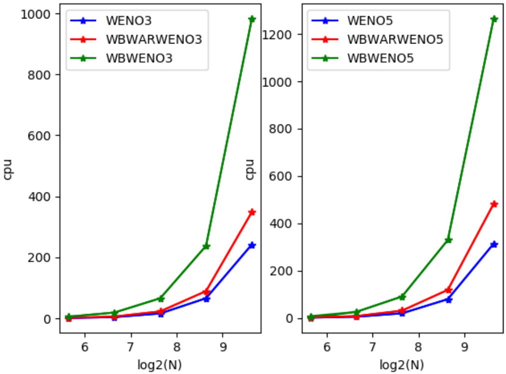


Figure: CPU times as a function of $\log_2(N)$ for WENOP, WBWARWENOp and WBWENOp, $p = 3, 5$

FWB finite difference methods: comments

- **Singular source terms:** If the function H has jump discontinuities, the source term becomes a nonconservative product whose definition is ambiguous. Nevertheless, in this context there is a natural definition of admissible weak solutions. The methods can be easily extended to this case: see CP, Parés-Pulido 2020 for details.
- **FWB methods:** if the exact solutions of the Cauchy problems (10) are not known or if their computation is difficult or costly, a numerical solver can be applied to compute the numerical fluxes, i.e. the flux reconstruction operator is applied to

$$\mathcal{F}_j = F(U_j) - F(U^*_{i,j}), \quad j = i-1-r, \dots, i+s,$$

where $U^*_{i,j}$ are the approximation of the solution U^*_i of (10) at the stencil points. The resulting numerical method is EWB if the ODE solver is adequately chosen.

FWB finite difference methods: comments

- **Singular source terms:** If the function H has jump discontinuities, the source term becomes a nonconservative product whose definition is ambiguous. Nevertheless, in this context there is a natural definition of admissible weak solutions. The methods can be easily extended to this case: see CP, Parés-Pulido 2020 for details.
- **FWB methods:** if the exact solutions of the Cauchy problems (10) are not known or if their computation is difficult or costly, a numerical solver can be applied to compute the numerical fluxes, i.e. the flux reconstruction operator is applied to

$$\mathcal{F}_j = F(U_j) - F(U^*_{i,j}), \quad j = i-1-r, \dots, i+s,$$

where $U^*_{i,j}$ are the approximation of the solution U^*_i of (10) at the stencil points. The resulting numerical method is EWB if the ODE solver is adequately chosen.

FWB finite difference methods: comments

- **Multidimensional problems:**Although the same principle can be used to discretize the source terms for multidimensional problmes, the main difficulty comes from the fact that now the problem to be solved for finding U^* is a PDE system, what is much more difficult to solve either exactly or numerically than an ODE system.
- Moreover the condition
$$U^*(x_{i,j}) = U_{i,j},$$
does not determine a stationary solution: there may exist infinitely many stationary solutions satisfying this equality.
- On the other hand, the extension to 2d problems of the numerical methods that preserve a given family of known stationary solution is straightforward.
- In particular, in the case of the shallow water model, the extension of the numerical methods that preserve water-at-rest solutions to 2d is straightforward.

FWB finite difference methods: comments

- **Multidimensional problems:**Although the same principle can be used to discretize the source terms for multidimensional problmes, the main difficulty comes from the fact that now the problem to be solved for finding U^* is a PDE system, what is much more difficult to solve either exactly or numerically than an ODE system.
- Moreover the condition
$$U^*(x_{i,j}) = U_{i,j},$$
does not determine a stationary solution: there may exist infinitely many stationary solutions satisfying this equality.
- On the other hand, the extension to 2d problems of the numerical methods that preserve a given family of known stationary solution is straightforward.
- In particular, in the case of the shallow water model, the extension of the numerical methods that preserve water-at-rest solutions to 2d is straightforward.

FWB finite difference methods: comments

- **Multidimensional problems:**Although the same principle can be used to discretize the source terms for multidimensional problmes, the main difficulty comes from the fact that now the problem to be solved for finding U^* is a PDE system, what is much more difficult to solve either exactly or numerically than an ODE system.
- Moreover the condition
$$U^*(x_{i,j}) = U_{i,j},$$
does not determine a stationary solution: there may exist infinitely many stationary solutions satisfying this equality.
- On the other hand, the extension to 2d problems of the numerical methods that preserve a given family of known stationary solution is straightforward.
- In particular, in the case of the shallow water model, the extension of the numerical methods that preserve water-at-rest solutions to 2d is straightforward.

FWB finite difference methods: comments

- **Multidimensional problems:**Although the same principle can be used to discretize the source terms for multidimensional problmes, the main difficulty comes from the fact that now the problem to be solved for finding U^* is a PDE system, what is much more difficult to solve either exactly or numerically than an ODE system.
- Moreover the condition
$$U^*(x_{i,j}) = U_{i,j},$$
does not determine a stationary solution: there may exist infinitely many stationary solutions satisfying this equality.
- On the other hand, the extension to 2d problems of the numerical methods that preserve a given family of known stationary solution is straightforward.
- In particular, in the case of the shallow water model, the extension of the numerical methods that preserve water-at-rest solutions to 2d is straightforward.

References 1

- E. Audusse, F. Bouchut, M. O. Bristeau, R. Klein, B. Perthame.
  A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows.
  SIAM Journal on Scientific Computing, 25: 2050— 2065 (2004).
- J. P. Berberich, P. Chandrashekar, C. Klingenberg.
  High order well-balanced finite volume methods for multi-dimensional systems of hyperbolic balance laws.
  arXiv:1903.05154 (2019)
- A. Bermúdez and M. E. Vázquez.
  Upwind methods for hyperbolic conservation laws with source terms.
  Computers & Fluids, 23(8):1049–1071 (1994).
- C. Berthon, C. Chalons.
  A fully well-balanced, positive and entropy-satisfying Godunov-type method for the shallow-water equations.
  Mathematics of Computation, 85: 1281-1307, (2016).

References 2

F. Bouchut.
Non-linear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources.
Frontiers in Mathematics, Birkhauser, 2004.

F. Bouchut, T. Morales, A subsonic-well-balanced reconstruction scheme for shallow water flows.
SIAM Journal on Numerical Analysis, 48:1733–1758, (2010).

A. Canestrelli, A. Siviglia, M. Dumbser, E. F. Toro.
Well-balanced high-order centred schemes for non-conservative hyperbolic systems. Applications to shallow water equations with fixed and mobile bed.
Advances in Water Resources, 32(6): 834–844 (2009).

V. Caselles, R. Donat, G. Haro.
Flux-gradient and source-term balancing for certain high resolution shock-capturing schemes.
Computers & Fluids, 38:16–36 (2009).

References 3

M. J. Castro, J.M. Gallardo J. López, and C. Parés.
Well-balanced high order extensions of Godunov method for linear balance laws.
SIAM Journal on Numerical Analysis, 46:1012–1039 (2008)

M. J. Castro, I. Gómez-Bueno, C. Parés.
High-order well-balanced methods for systems of balance laws: a control-based approach.
To appear in *Applied Mathematics and Computations*.

M. J. Castro, J. A. López-García, and C. Parés.
High order exactly well-balanced numerical methods for shallow water systems.
Journal of Computational Physics, 246:242–264 (2013).

M. J. Castro, T. Morales, C. Parés.
Well-balanced schemes and path-conservative numerical methods.
Hand book of Numerical Analysis, 18: 131 – 175 (2017).

M. J. Castro, S. Ortega, C. Parés.

Well-balanced methods for the shallow water equations in spherical coordinates

Computers & Fluids, 157: 196–207 (2017).

M. J. Castro, A. Pardo, C. Parés.

Well-balanced numerical schemes based on a generalized hydrostatic reconstruction technique.

Mathematical Models and Methods in Applied Sciences, 17: 2065—2113(2007).

M.J. Castro, C. Parés.

Well-balanced high-order finite volume methods for systems of balance laws

Journal of Scientific Computing, 82, Article number: 48 (2020).

References 5

P. Chandrashekar and C. Klingenberg.

A second order well-balanced finite volume scheme for Euler equations with gravity.

SIAM J. Sci. Comput.,37(3):B382–B402 (2015).

P. Chandrashekar and M. Zenk.

Well-balanced nodal discontinuous Galerkin method for Euler equations with gravity.

J. Sci. Comput.,71(3):1062–1093 (2017).

Y. Cheng, A. Chertock, M. Herty, A. Kurganov, T. Wu.

A new approach for designing moving-water equilibria preserving schemes for the shallow water equations.

Journal of Scientific Computing, 80: 538–554, (2019).

A. Chertock, S. Cui, A. Kurganov, T. Wu.

Well-balanced positivity preserving central-upwind scheme for the shallow water system with friction terms.

International Journal for Numerical Methods in Fluids, 78:355— 383 (2015).

A. Chertock, S. Cui, A. Kurganov, S.Özcan, E. Tadmor.

Well-balanced schemes for the Euler equations with gravitation: Conservative formulation using global fluxes.

J. Comput. Phys., 358:36–52 (2018).

A. Chertock, A. Kurganov, Y. Liu.

Central-upwind schemes for the system of shallow water equations with horizontal temperature gradients.

Numerische Mathematik, 127: 595—639 (2014).

G. Dal Maso, P. G. Lefloch, and F. Murat.

Definition and weak stability of nonconservative products.

Journal de Mathématiques Pures et Appliquées, 74(6):483–548 (1995).

References 7

E. Gaburro, M. Dumbser, M. J. Castro.

Direct Arbitrary-Lagrangian-Eulerian finite volume schemes on moving nonconforming unstructured meshes.

Computers & Fluids, 159: 254–275 (2017).

E. Gaburro, M. J. Castro, M. Dumbser.

Well-balanced Arbitrary-Lagrangian-Eulerian finite volume schemes on moving nonconforming meshes for the Euler equations of gas dynamics with gravity.

Monthly Notices of the Royal Astronomical Society, 477(2): 2251– 2275 (2018).

E. Gaburro, M. J. Castro, M. Dumbser.

A well balanced diffuse interface method for complex nonhydrostatic free surface flows,

Computers & Fluids, 175: 180–198 (2018).

S. Gottlieb and C.-W. Shu.

Total variation diminishing Runge-Kutta schemes.

Mathematics of Computation of the American Mathematical Society, 67(221): 73–85 (1998).

L. Grosheintz-Laval, R. Käppeli,

High-order well-balanced finite volume schemes for the Euler equations with gravitation.

Journal of Computational Physics, 378: 324–343 (2019).

G. Jiang, C.W.-Shu.

Efficient implementation of weighted ENO schemes.

Journal of Computationas Physcs, 126: 202—228 (1996).

R. Käppeli and S. Mishra.

Well-balanced schemes for the Euler equations with gravitation.

J. Comput. Phys., 259:199–219 (2014).

References 9

C. Klingenberg, G. Puppo and M. Semplice.

Arbitrary order finite volume well-balanced schemes for the Euler equations with gravity.

https://arxiv.org/abs/1807.02341 (2018).

Ph.G. LeFloch, C. Parés, E. Pimentel.

Well-balanced algorithms for relativistic fluids on a Schwarzschild background.

arXiv preprint arXiv:2011.07587 (2020)

G. Li and Y. Xing.

High order finite volume WENO schemes for the Euler equations under gravitational fields.

J. Comput. Phys., 316:145–163 (2016).

G. Li and Y. Xing.

Well-balanced discontinuous Galerkin methods with hydrostatic reconstruction for the Euler equations with gravitation.

J.Comput. Phys., 352:445–462 (2018).

References 10

M. Lukávcová-Medvid'ová, S. Noelle, and M. Kraft.
Well-balanced finite volume evolution Galerkin methods for the shallow water equations.
Journal of Computational Physics, 221(1):122–147, (2007).

L. O. Müller, C. Parés, and E. F. Toro.
Well-balanced High-order Numerical Schemes for One-dimensional Blood Flow in Vessels with Varying Mechanical Properties.
Journal of Computational Physics, 242:53–85, (2013).

S. Noelle, N. Pankratz, G. Puppo, and J. R. Natvig.
Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows.
Journal of Computational Physics, 213(2):474–499 (2006).

S. Noelle, Y. Xing, and C.-W. Shu.
High-order well-balanced finite volume WENO schemes for shallow water equation with moving water.
Journal of Computational Physics, 226(1):29–58 (2007).

References 11

C. Parés, C. Parés-Pulido.

Well-balanced high-order finite differences methods for systems of balance laws.

arXiv:2001.10074, 2020.

G. Russo and A. Khe.

High order well balanced schemes for systems of balance laws.

In *Hyperbolic problems: theory, numerics and applications*, volume 67 of *Proc. Sympos. Appl. Math.*, pages 919–928. Amer. Math. Soc., Providence, RI (2009).

C. Sánchez-Linares, T. Morales de Luna, M.J. Castro.

A HLLC scheme for Ripa model.

Applied Mathematics and Computation, 272: 369–384, (2016).

C.-W. Shu., S. Osher.

Efficient implementation of essentially non-oscillatory shock-capturing schemes.

Journal of Computational Physics. 77: 439–471 (1988).

C.-W. Shu., S. Osher.

Efficient implementation of essentially non-oscillatory shock-capturing schemes, II.

Journal of Computational Physics. 83: 32–78 (1989).

C.-W. Shu.

Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws.

In Cockburn, B., Johnson, C., Shu, C.-W., Tadmor, E., and Quarteroni, A. (eds.), *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, Lecture Notes in Mathematics, Vol. 1697, Springer, Berlin, pp. 325–432, 1998.

A. Thomann, M. Zenk, and C. Klingenberg.

A second order positivity preserving well-balanced finite volume scheme for Euler equations with gravity for arbitrary hydrostatic equilibria.

International Journal for numerical methods in fluids. 89(11): 465–482 (2019).

References 13

D. Varma and P. Chandrashekar.
A second order well-balanced finite volume scheme for Euler equations with gravity.
Computers & Fluids, 181: 292–313 (2019)

Y. Xing.
Numerical Methods for the Nonlinear Shallow Water Equations.
In *Handbook of Numerical Methods* V. 18: 361–384, Springer 2017.

Y. Xing and C.W.-Shu.
High-Order well-balanced finite difference WENO schemes with the exact conservation property for the shallow water equations.
Journal of Computational Physics, 208: 206–227 (2006)

Y. Xing and C.W.-Shu.
High-Order well-balanced finite difference WENO schemes for a class of hyperbolic systems with source terms.
Journal of Scientific Computing, 27: 477–494 (2006)

References 14

Y. Xing and C.-W. Shu.

High order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms.

Journal of Computational Physics, 214(2):567–598 (2006).

Y. Xing and C.-W. Shu.

High order well-balanced WENO scheme for the gas dynamics equations under gravitational fields.

Journal of Scientific Computing, 54(2-3):645–662 (2013).

Y. Xing.

Exactly well-balanced discontinuous Galerkin methods for the shallow water equations with moving water equilibrium.

Journal of Computational Physics, 257: 536–553 (2014).