The role of curl-type involution constraints in continuum mechanics

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&
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University of Trento, Italy

Seminar series: Structure preserving methods for hyperbolic PDEs, Oct 28, 2020

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Elasticity in Eulerian coordinates

$$\partial_t(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I} + E_{\boldsymbol{A}}^{\mathrm{T}} \boldsymbol{A}) = 0$$

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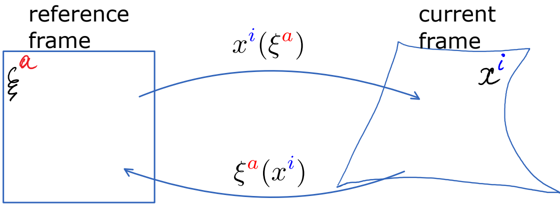

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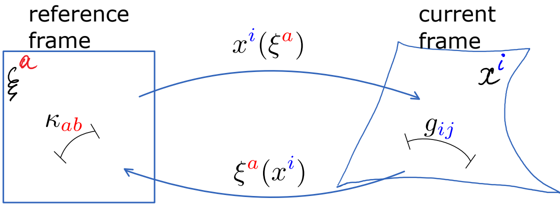

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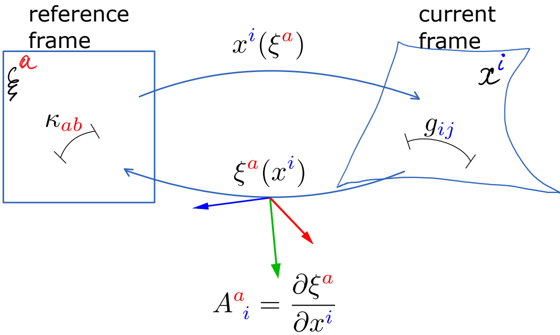

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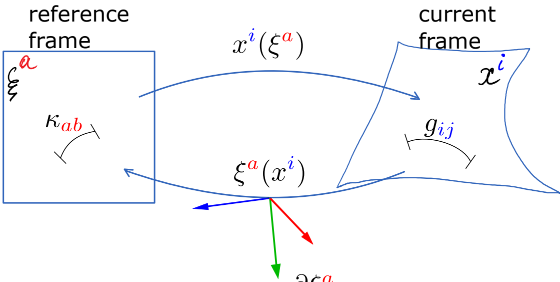

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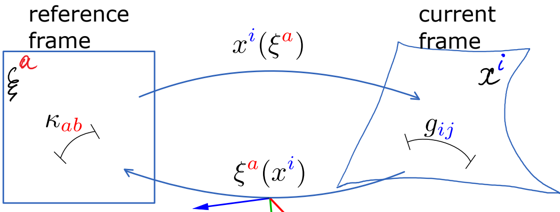
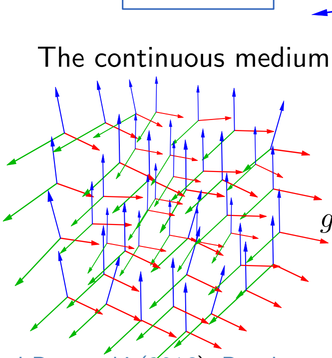


The continuous medium



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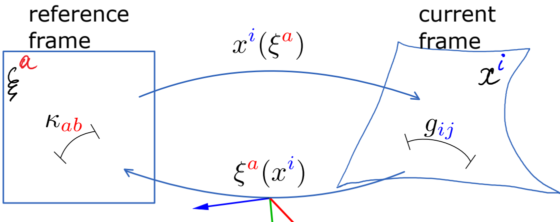

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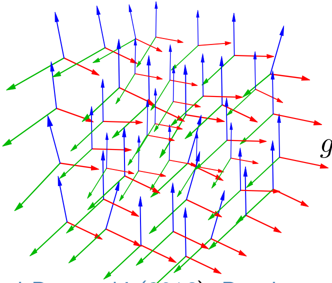

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Curl-involution preserving integration

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here is the video of this lecture:

download slides of the lecture here

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Thursdays at 9:30 am CET, except Tuesday Oct. 27

September	Oktober	November	Dezember
1 Di	1 Do *Castro*	**1 So**	1 Di
2 Mi	2 Fr	2 Mo 45	2 Mi
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4 Fr	**4 So**	4 Mi	4 Fr
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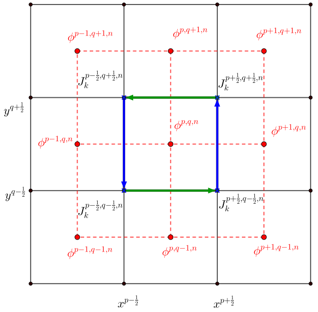

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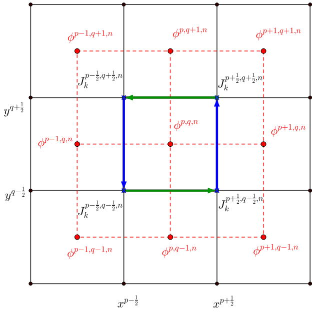

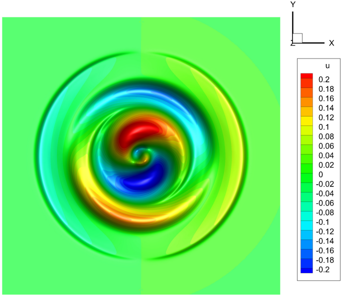

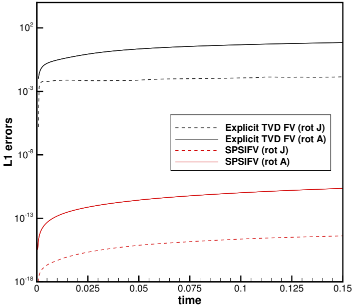

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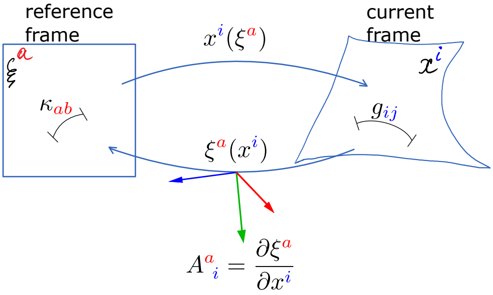

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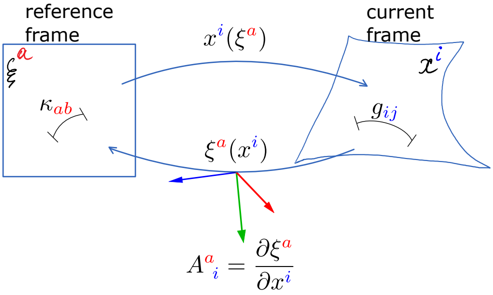

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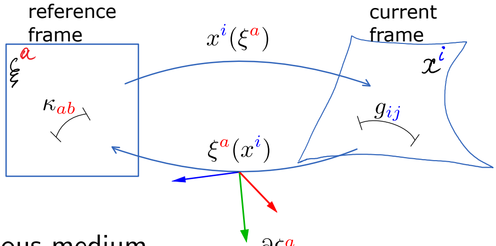

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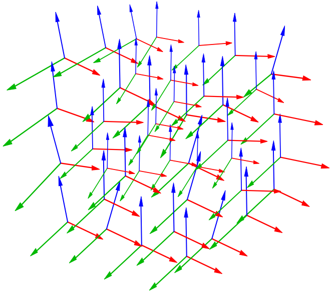

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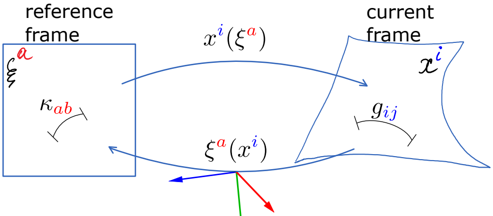

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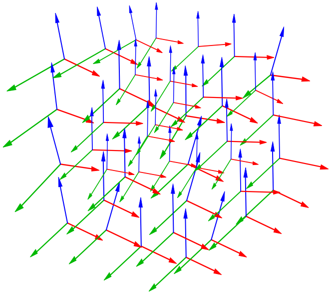

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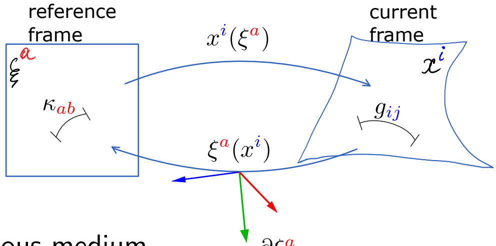

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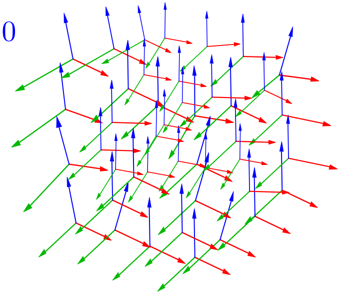

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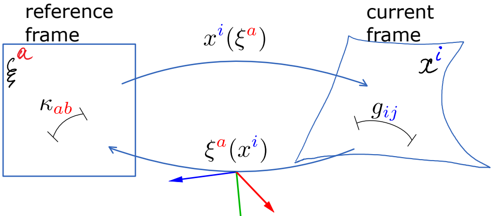

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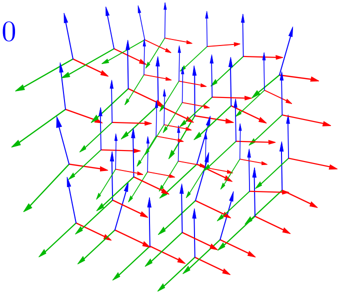

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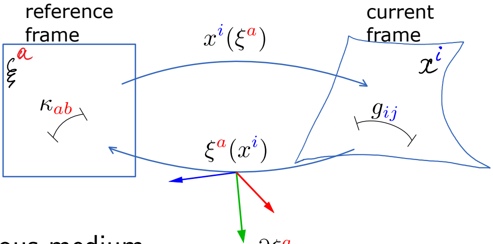

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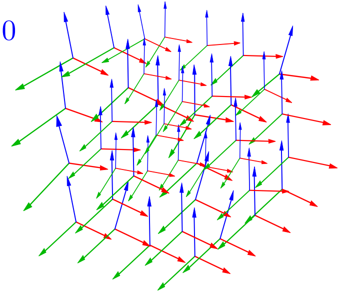

Unified formulation for Fluids and Solids

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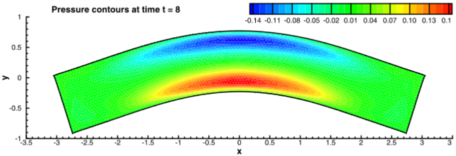
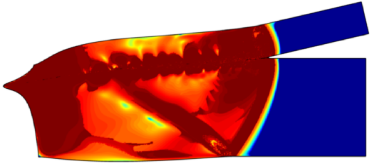
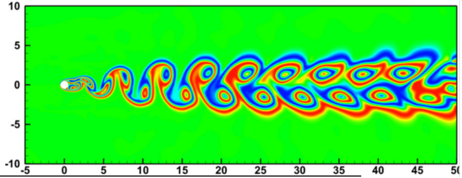


$\tau = \infty,$ elastic solids



$\tau \sim \left(\frac{\sigma_Y}{\sigma}\right)^n,$ inelastic solids



$0 < \tau \ll 1,$ viscous fluids

Peshkov and Romenski (2016); Dumbser et al. (2016, 2018); Peshkov et al. (2019a)

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Navier-Stokes limit

Navier-Stokes limit: $\tau \ll$ **observation time**

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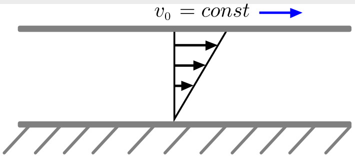

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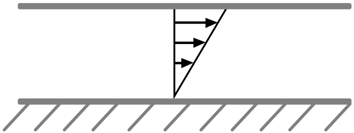

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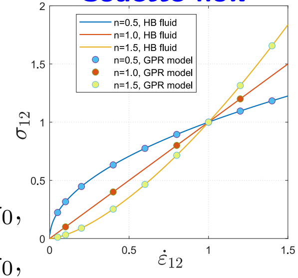
Couette flow



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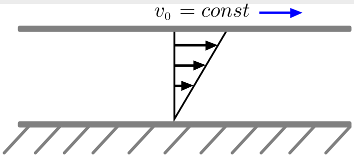

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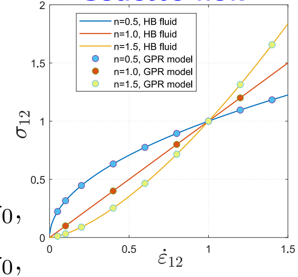
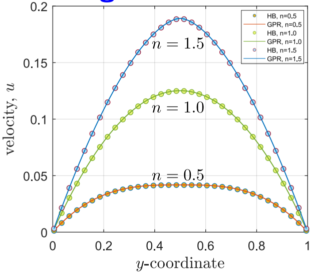


Hagen–Poiseuille



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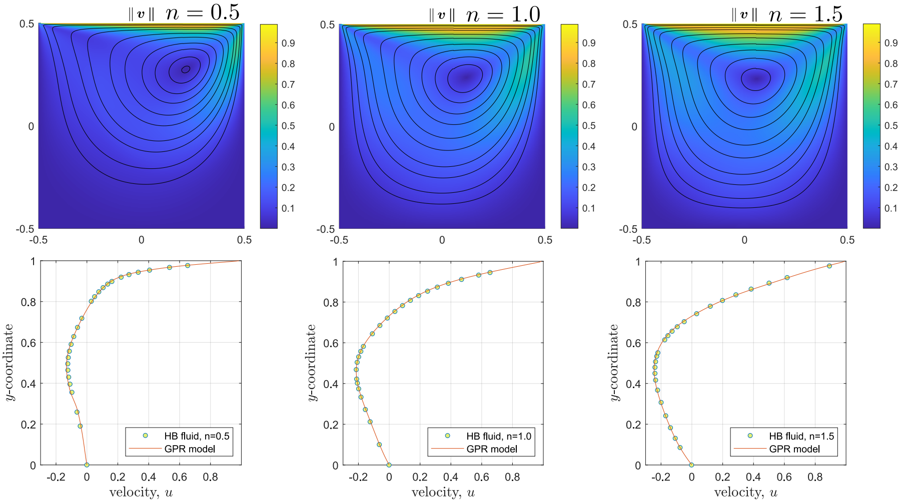
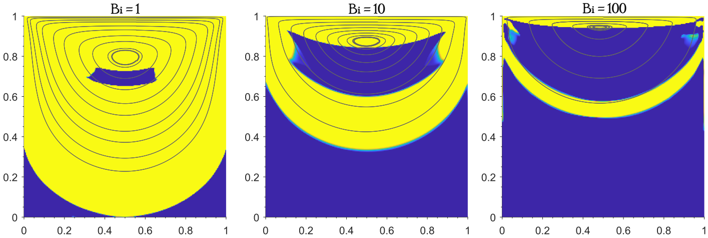
Navier-Stokes limit

Lid-driven cavity flow of **Hershel-Bulkley** fluid, $\sigma_0 = 0$, Re $= 100$:



Solid-fluid transition

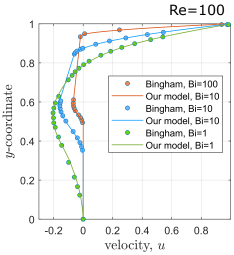
Lid-driven cavity flow of **Hershel-Bulkley** fluid, $\sigma_0 > 0$, Re $= 1$:



$$\partial_t \boldsymbol{A} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} + \boldsymbol{A} \nabla \boldsymbol{v} = -\frac{1}{\tau} E_{\boldsymbol{A}},$$

$$\tau = \left(\frac{c}{\tau_s} + \frac{1-c}{\tau_l} \right)^{-1}$$

$$\tau_l \ll 1, \qquad 1 \gg \tau_s$$



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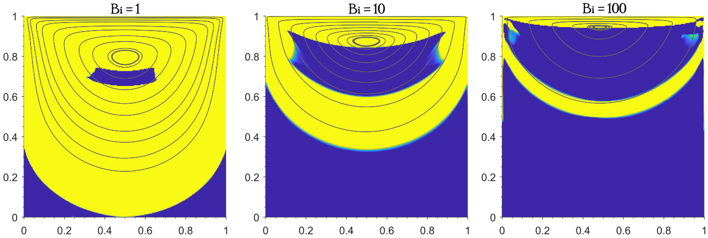

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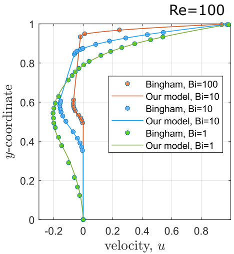

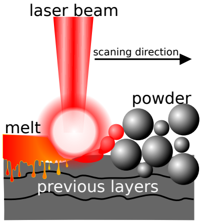

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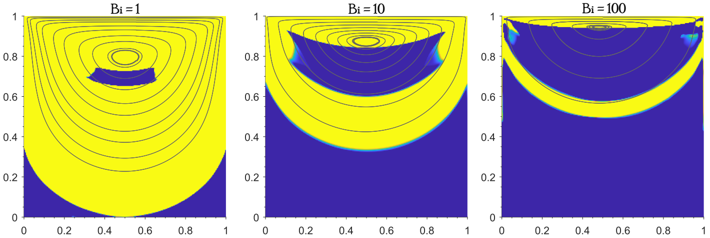

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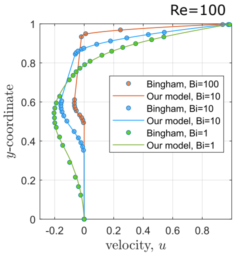

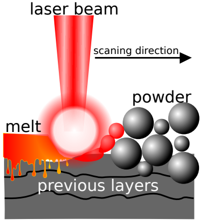

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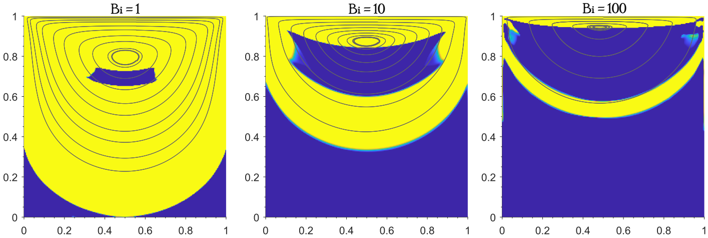

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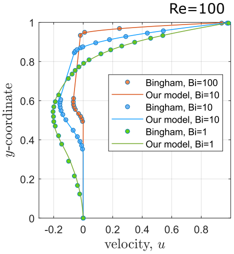

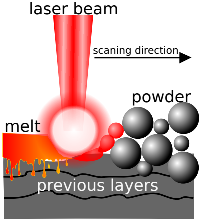

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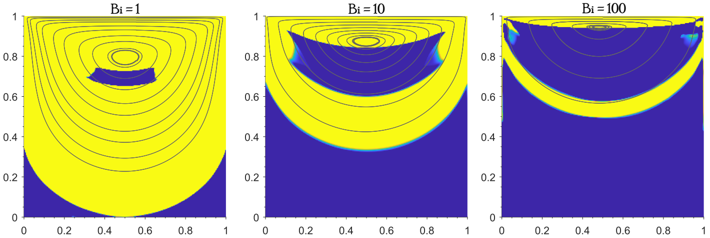

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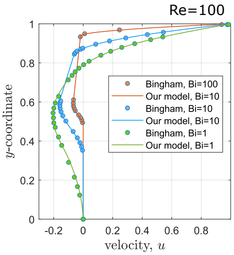

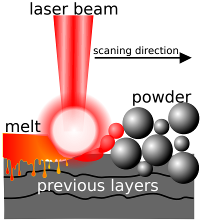

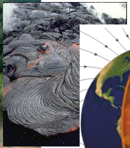

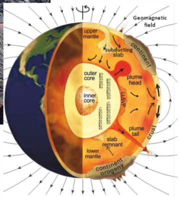

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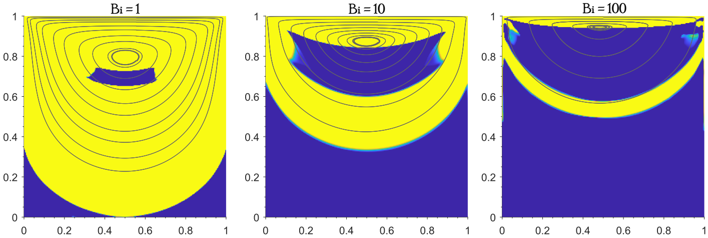

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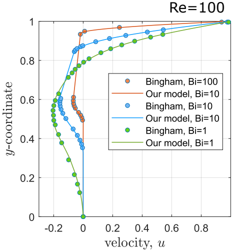

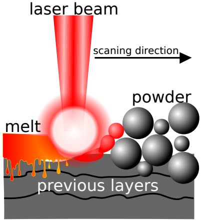

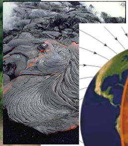

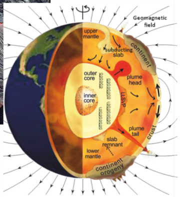

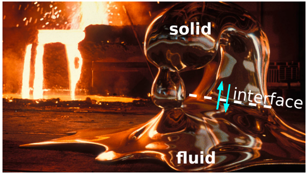

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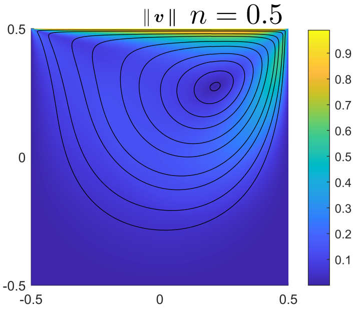
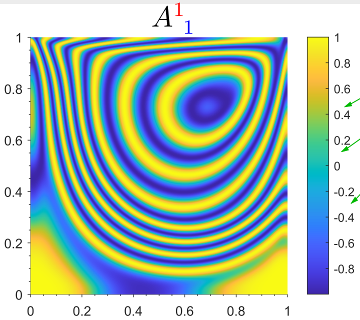
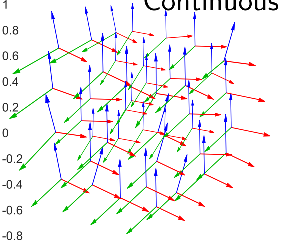
Distortion spin

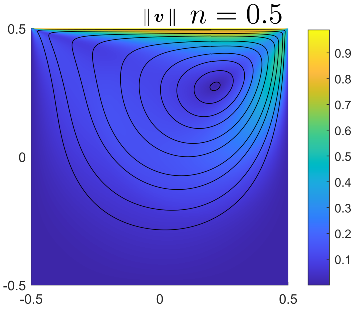


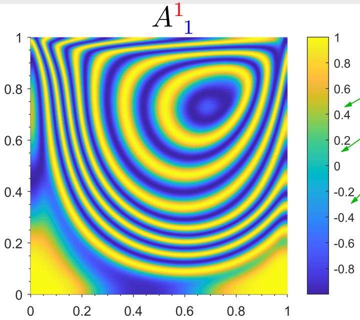


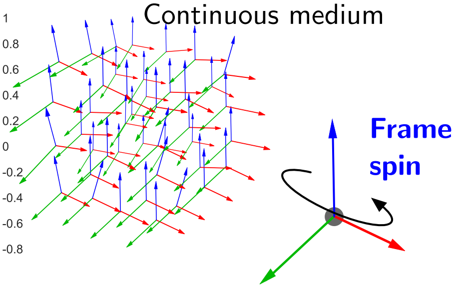
Continuous medium



Distortion spin







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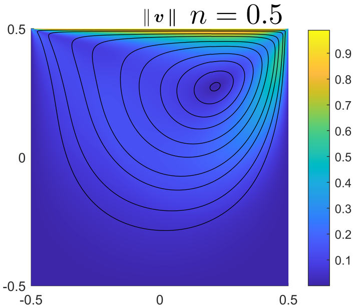

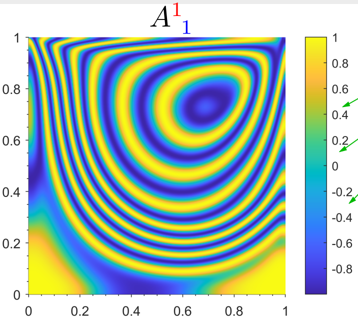

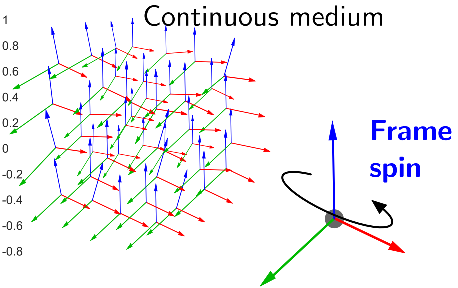

$$\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} + \boldsymbol{A} \nabla \boldsymbol{v} = -\frac{1}{\tau} E_{\boldsymbol{A}}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \neq 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v} + \tau^{-1} E_{\boldsymbol{A}}) + \boldsymbol{v} \nabla \cdot \boldsymbol{B} = 0$$

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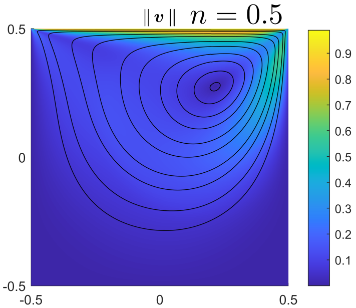

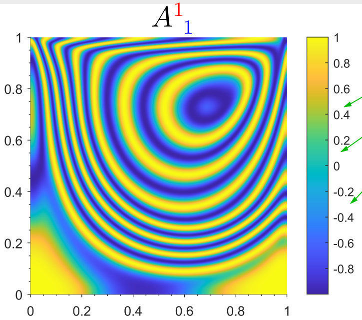

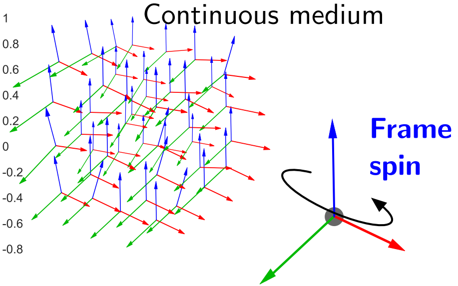

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Coupling via energy (Lagrangian)

$$E = E(\rho, \rho\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{B}, \ldots)$$

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Variational formulation in 4D

4-distortion ${A^a}_\mu$, $a, \mu = 0, 1, 2, 3$

$$\boldsymbol{A} = \begin{pmatrix} -u_0 & -u_1 & -u_2 & -u_3 \\ {A^1}_0 & {A^1}_1 & {A^1}_2 & {A^1}_3 \\ {A^2}_0 & {A^2}_1 & {A^2}_2 & {A^2}_3 \\ {A^3}_0 & {A^3}_1 & {A^3}_2 & {A^3}_3 \end{pmatrix}$$

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Distortion $\partial_\mu A^{a}{}_{\nu} - \partial_\nu A^{a}{}_{\mu} = T^{a}{}_{\mu\nu}$

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Back to 3D

4D equations

Eul.-Lagr. $\partial_\lambda \left(\varepsilon^{\mu\nu\eta\lambda} L_{\overset{\star}{T}^{a}{}_{\nu\eta}} \right) = -L_{A^a{}_\mu}$

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3D equations (3+1)

$\alpha \sim length^{-1}$

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Back to 3D

4D equations

Eul.-Lagr. $\partial_\lambda \left(\varepsilon^{\mu\nu\eta\lambda} L_{\overset{\star}{T}^{a\nu\eta}} \right) = -L_{A^a{}_\mu}$

Identity $\partial_\lambda \overset{\star}{T}^{a\mu\lambda} = 0$

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Distortion $\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu = T^a{}_{\mu\nu}$

3D equations (3+1)

$$\partial_t \boldsymbol{D} + \nabla \times (\boldsymbol{D} \times \boldsymbol{v} - E_{\boldsymbol{B}}) = \frac{1}{\alpha} E_{\boldsymbol{A}}$$

$$\alpha \sim length^{-1}$$

$$\boldsymbol{D} \sim \partial_t \boldsymbol{A} + \boldsymbol{v} \cdot \nabla \boldsymbol{A}$$

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Back to 3D

4D equations

Eul.-Lagr. $\partial_\lambda \left(\varepsilon^{\mu\nu\eta\lambda} L_{\overset{\star}{T}{}^{a\nu\eta}} \right) = -L_{A^a{}_\mu}$

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3D equations (3+1)

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$$\partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v} + E_{\boldsymbol{D}}) = 0$$

$$\alpha \sim length^{-1}$$

$$\boldsymbol{D} \sim \partial_t \boldsymbol{A} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} \qquad \boldsymbol{B} \sim \nabla \times \boldsymbol{A}$$

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Back to 3D

4D equations

Eul.-Lagr. $\partial_\lambda \left(\varepsilon^{\mu\nu\eta\lambda} L_{\overset{\star}{T}^{a}{}_{\nu\eta}} \right) = -L_{A^a{}_\mu}$

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3D equations (3+1)

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$$\alpha \sim length^{-1}$$

$$\boldsymbol{D} \sim \partial_t \boldsymbol{A} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} \qquad \boldsymbol{B} \sim \nabla \times \boldsymbol{A}$$

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Back to 3D

4D equations

Eul.-Lagr. $\partial_\lambda \left(\varepsilon^{\mu\nu\eta\lambda} L_{\overset{\star}{T}^{a\nu\eta}} \right) = -L_{A^a{}_\mu}$

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Acoustic metamaterials (other micromorphic solids)

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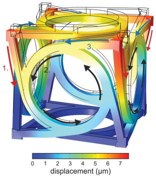

Meta-atom, reproduced from[1]

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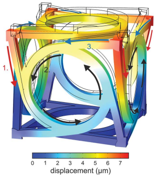

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Energy due to torsion

$$E = \frac{1}{2}\left(\frac{1}{\epsilon}\|\boldsymbol{D}\|^2 + \frac{1}{\mu}\|\boldsymbol{B}\|^2\right) - \frac{c_{\mathrm{sp}}}{2} \boldsymbol{I}^a \cdot (\boldsymbol{D}_a \times \boldsymbol{B}^a)$$

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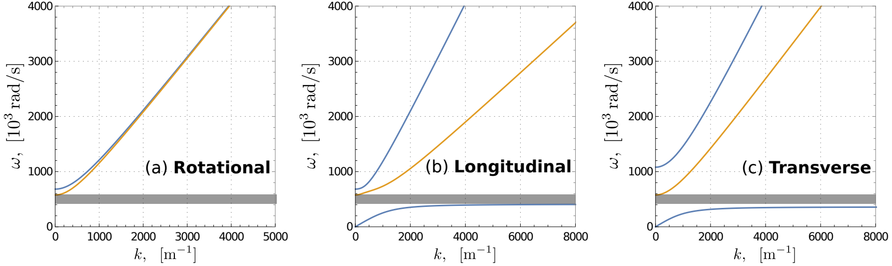
Acoustic band gap

$$E = \frac{1}{2}\left(\frac{1}{\epsilon}\|\boldsymbol{D}\|^2 + \frac{1}{\mu}\|\boldsymbol{B}\|^2\right) - \frac{c_{\text{sp}}}{2}\boldsymbol{I}^a \cdot (\boldsymbol{D}_a \times \boldsymbol{B}^a)$$



Complete band gap (gray rectangles) for $c_{\text{sp}} > 0$.

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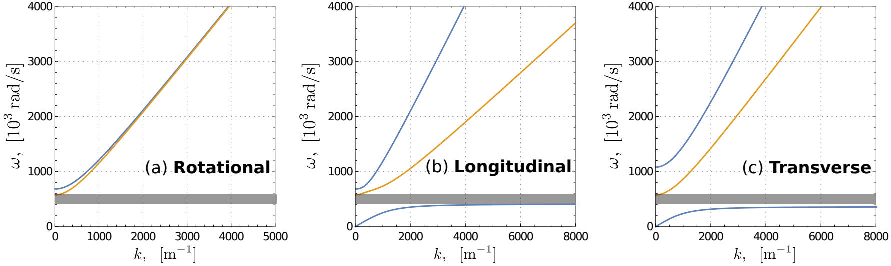

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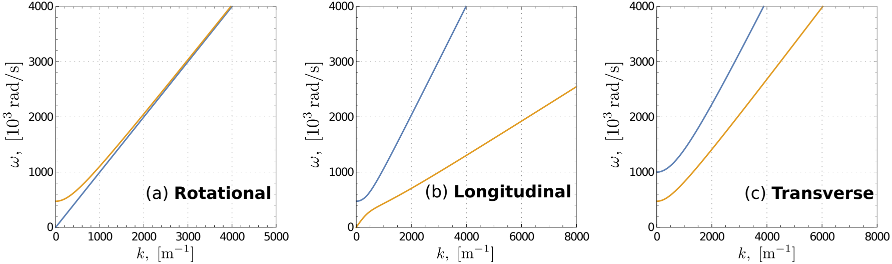

No band gap
for $c_{\text{sp}} = 0$.

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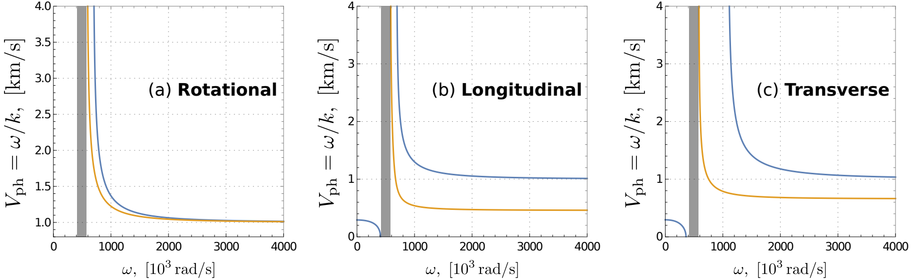

Phase velocities

$$V_{ph} = \frac{\omega}{k}.$$

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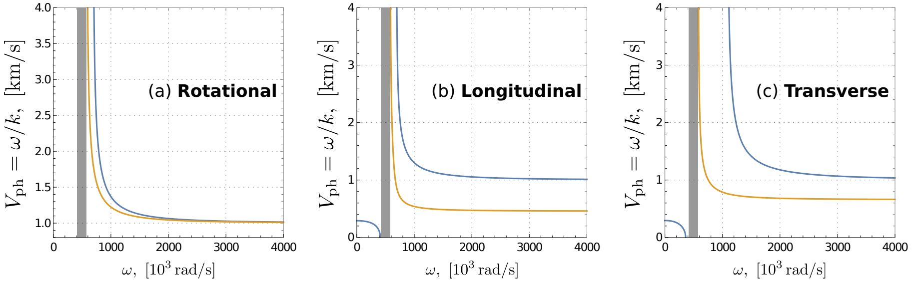

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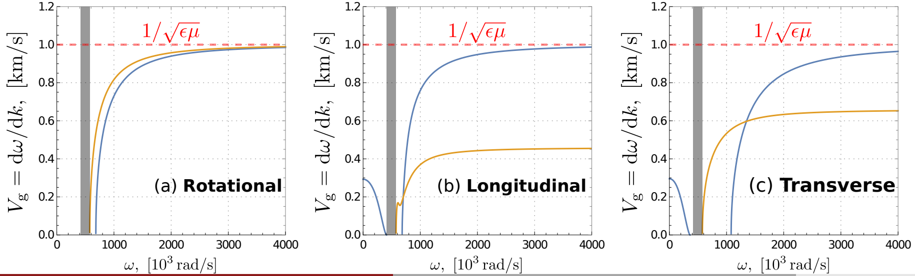

Group velocities

$$V_{\text{g}} = \frac{d\omega}{dk} < 1/\sqrt{\epsilon\mu}.$$

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What about turbulence?

$$\text{Re} = \frac{\rho v L}{\eta}$$

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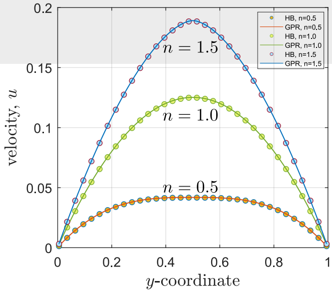
Hagen–Poiseuille flow



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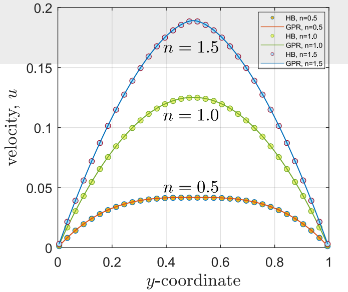

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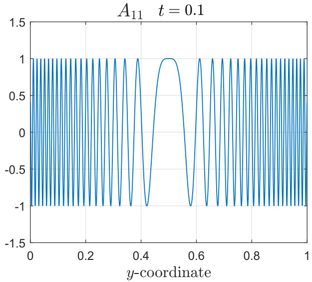

Rotation of the frame field

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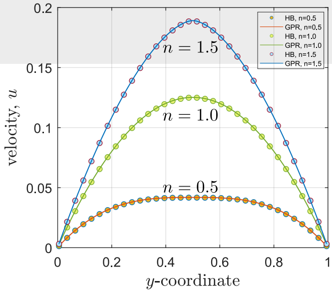

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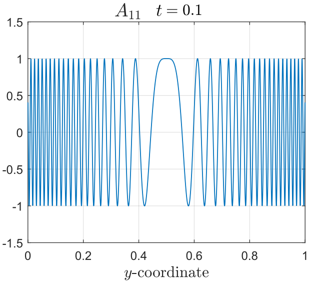

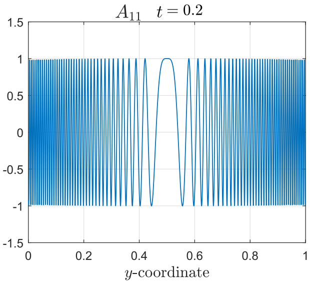

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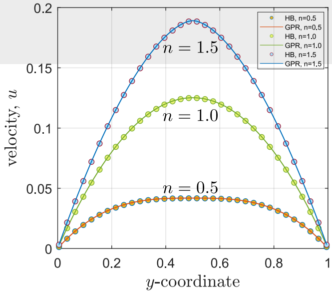

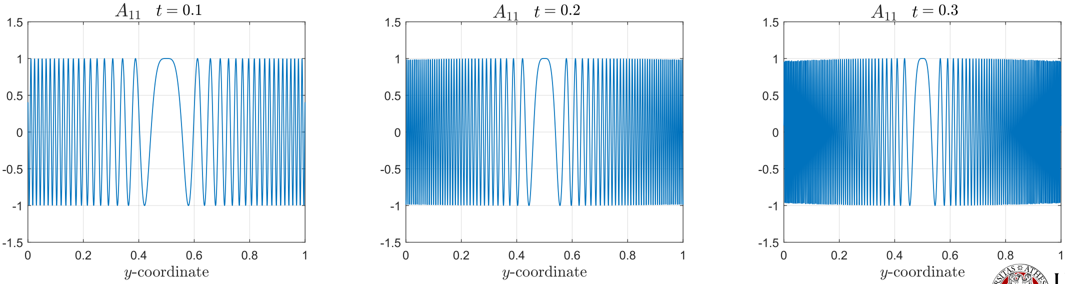

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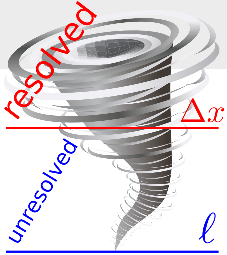
$$\text{Re} = \frac{\Delta x}{\ell}$$



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$$\partial_t \boldsymbol{M} + \nabla \cdot (\boldsymbol{M} \otimes \boldsymbol{v} + P\boldsymbol{I} + E_{\boldsymbol{A}}^{\mathsf{T}} \boldsymbol{A} - E_{\boldsymbol{D}}^{\mathsf{T}} \boldsymbol{D} - E_{\boldsymbol{B}}^{\mathsf{T}} \boldsymbol{B}) = 0 \qquad \boldsymbol{M} = \rho \boldsymbol{v} + \boldsymbol{D}_a \times \boldsymbol{B}^a$$

$$\partial_t \boldsymbol{A} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} + \boldsymbol{A} \nabla \boldsymbol{v} = -\frac{1}{\alpha} E_{\boldsymbol{D}}$$

Peshkov et al. (2019b)

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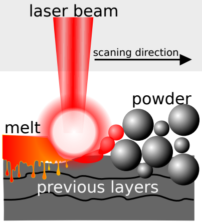
Other applications

Aldrovandi and Pereira (2013); Cai et al. (2016)

Other applications



1. Solidification (other growing interfaces, tissue growth)

$$(\nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu) v^\alpha = R^\alpha{}_{\mu\nu\sigma} v^\sigma - 2T^\sigma{}_{\mu\nu} \nabla_\sigma v^\alpha$$

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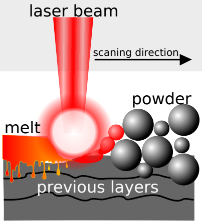
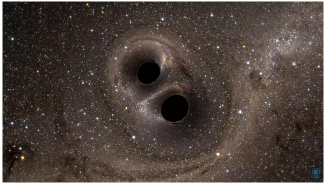


2. Teleparallelism (gravity theory via torsion)

$$g_{ij} = \kappa_{ab} A^a{}_i A^b{}_j$$



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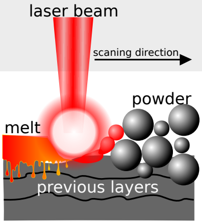

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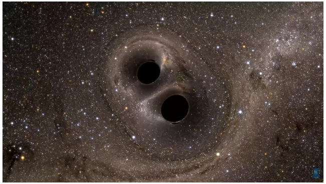

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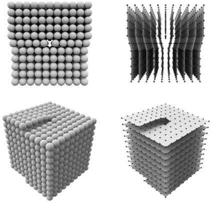
3. Dislocations



Aldrovandi and Pereira (2013); Cai et al. (2016)

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