SPP 2410 Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness (CoScaRa)

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A Structure-Preserving Compact High-Order Method for Multi-Dimensional Hyperbolic Conservation Laws

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Key Science Drivers

- *simulation of turbulent flow — faithful and efficient*
- *genuinely multi-dim. numerical method — no directional bias*

Generalized Active Flux — an Innovative Method

- *multi-dimensional by conception*
- *adapts its stabilizing num. viscosity to multi-dim. structures*
- *reflects directions of information propagation*
- *avoids need for grid refinement*
- *large potential for saving computational cost*

Advantages of This Approach

- *works well in both the super- and subsonic regime*
- *structure preserving for linearized Euler:*
  low Mach, vorticity preserving, stationarity preserving
- *ideally suitable for turbulence even on coarse grids*
- *compact stencil — highly parallelizable*

Current Status of This Approach

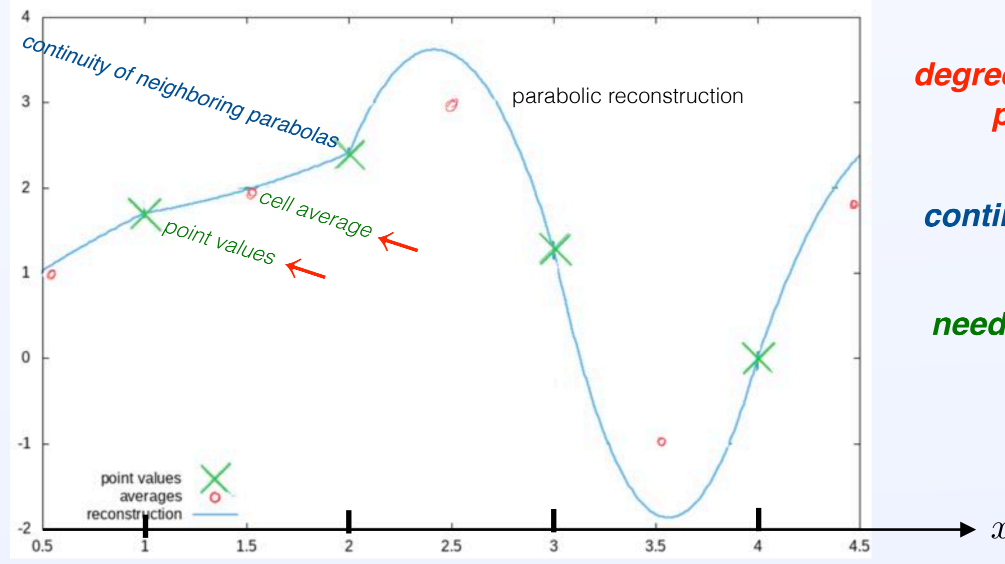
- *in 2-d the method is 3^{rd} order*
- *in 1-d the method is arbitrary order*
- *no proofs for properties of the method for 2-d non-lin. Euler*

Previous Work

T. Eymann and Philip Roe: "Multidimensional active flux schemes", 21st AIAA computational fluid dynamics conference (**2013**)
2-d linearized Euler, 3^{rd} order

W. Barsukow; J. Hohm; C. **Klingenberg**; Philip Roe, "The active flux scheme on Cartesian grids and its low Mach number limit", Journal of Scientific Computing. vol. 81 (**2019**)
2-d linearized Euler, 3^{rd} order — *with proofs*

here we illustrate the above in 1-d:



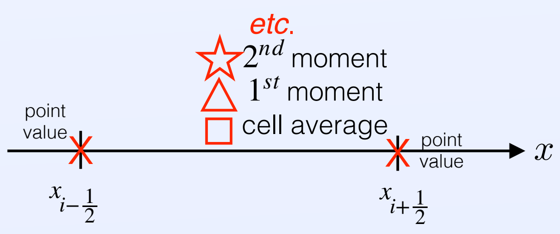
degrees of freedom (→) of Active Flux and its parabolic reconstruction (3^{rd} order)

continuity of neighboring reconstructions — no use of Riemann problems

need to update
- ***point values***
- ***cell averages***

R. Abgrall, W. **Barsukow**: "Extensions of active flux to arbitrary order of accuracy" ESAIM: M2AN (**2023**)
1-d non-linear Euler, arbitrary order

semi-discretization of $q_t + \nabla \cdot f(q) = 0$ *in one space dimension*



$$q_i^{(k)} = \frac{1}{\Delta x^{k+1}} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} x^k q dx$$ the k^{th} moment

degrees of freedom: point values $\{q_{i+1}\}_{i\in\mathbb{Z}}$ *and moments* $\{q_i^{(k)}\}_{i\in\mathbb{Z}_0} \; k=0,\ldots$

example: point values + up to 2^{nd} moments = **5^{th} order in space**

update of moments:

cell average □ $$\frac{d}{dt}\bar{q}_i = -\frac{f(q_{i+\frac{1}{2}}) - f(q_{i-\frac{1}{2}})}{\Delta x}$$ the method is **conservative** ⟹ convergence by Lax-Wendroff

multiply $0 = q_t + f(q)_x$ by x^k and integrate over the cell

$$0 = \frac{d}{dt}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} x^k q dx + \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} x^k \partial_x f dx$$

$$0 = \frac{d}{dt}q_i^{(k)} - \frac{1}{\Delta x^{k+1}}\left(\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} k x^{k-1} f(q) dx - x^k f(q(x_i + x))\Big|_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\right)$$ integration by parts

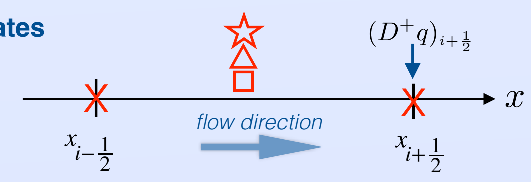
1^{st} moment △
2^{nd} moment ☆
etc.

$$0 = \frac{d}{dt}q_i^{(k)} - \frac{1}{\Delta x^{k+1}}\underbrace{\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} k x^{k-1} f(q) dx}_{\text{use quadrature}} + \frac{f(q_{i+\frac{1}{2}}) - (-1)^k f(q_{i-\frac{1}{2}})}{2^k \Delta x}$$ update of higher moments

update of point values: $$\frac{d}{dt}q_{i+\frac{1}{2}} = -f'(q_{i+\frac{1}{2}})^+ (D^+q)_{i+\frac{1}{2}} - f'(q_{i+\frac{1}{2}})^- (D^-q)_{i+\frac{1}{2}}$$

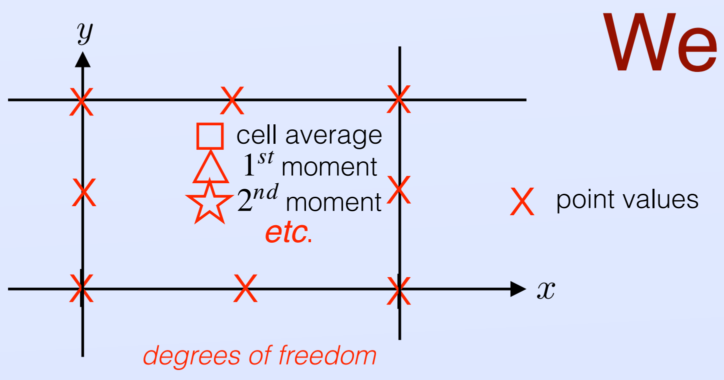
use **upwind** discretization involving both point values and moments, e.g. $(D^+q)_{i+\frac{1}{2}} = \left(15\bar{q}_i - 15q_i^{(1)} - 35q_i^{(2)} + 4q_{i-\frac{1}{2}} + 16q_{i+\frac{1}{2}}\right)/\Delta x$

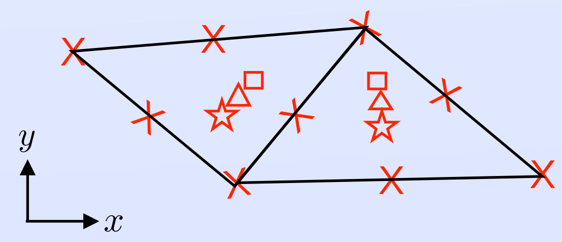
stabilization of this scheme is via upwinding of the point updates



We Shall Extend This to 2 and 3 Space Dimensions

2-dim:





$$q_i^{(k)} = A_{rs} \int\int_{cell} x^r y^s q dx dy \qquad r+s=k \quad r,s \in \mathbb{N}_0$$

k^{th} moments on a cartesian grid in 2-d

The Work Packages

- *show structure preservation for linear and non-linear Euler*
- *comparison with other high order compact methods*
- *show entropy inequality, positivity preservation*
- *generalize to unstructured grids*
- *generalize to other systems of conservation laws*

Novelty of this Project

- *semi-discrete — free to choose time integrator* NEW
- *arbitrary order on compact stencil (high order moments)* NEW
- *applicable for systems of cons. laws (not just Euler)* NEW

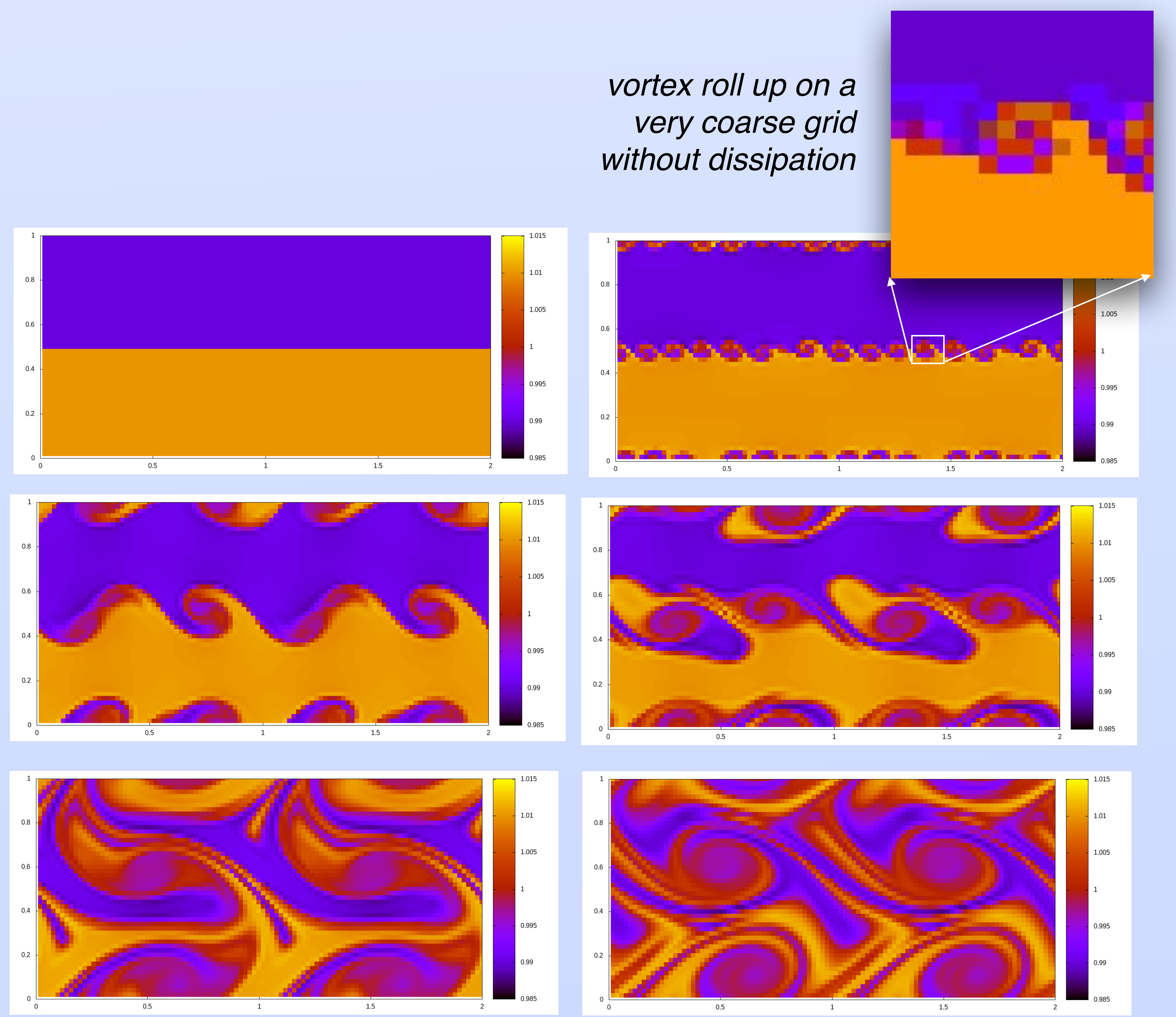
The Project's Research in the Context of SPP 2410

This numerical approach is ideally suited to simulate turbulent flow

We plan to cooperate within SPP 2410 with

- *Project-Helzel/Lukacova (no. 15), the low Mach property*
- *Project-Abgrall/Gassner (no. 11), the extension to unstructured grids*
- *Project-Krause/Mishra (no. 21), providing high order reference solutions in the low Mach limit*
- *Project-Fantuzzi (no. 7), a priori estimate to validate 2D flow simulations*

vortex roll up on a very coarse grid without dissipation



preliminary result of our proposed method (3^{rd} order):
evolution of a Kelvin-Helmholtz instability on a coarse 100 x 50 Cartesian grid

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