Symmetric Hyperbolic Thermodynamically Compatible (SHTC) equations: structure, constraints, asymptotic limits

E. Romenski

Sobolev Institute of Mathematics

Class of
Symmetric Hyperbolic Thermodynamically Compatible
(SHTC) systems

- Each system is hyperbolic and can be transformed to a symmetric hyperbolic system in the sense of Friedrichs
- Solution satisfies thermodynamic laws (conservation of energy and entropy growth)

Many well-posed systems of mathematical physics and continuum mechanics can be written in the form of thermodynamically compatible system.

Examples: gas dynamics, magneto-hydrodynamics, nonlinear and linear elasticity, hyperbolic heat conduction, electrodynamics of moving media, etc.

Symmetric hyperbolic systems (Friedrichs 1954)

$U = (U_1, U_2, \ldots, U_N)^T$ - unknown variables depending on time and spatial coordinates (t, x_i)

$$A(t, x_i)\frac{\partial U}{\partial t} + B_k(t, x_i)\frac{\partial U}{\partial x_k} = S(t, x_i)U$$ - Linear symmetric system

$$A = A^T, \; B_k = B_k^T$$

The system is hyperbolic if $A > 0$

It means that the roots λ of the equation $\det(\lambda A + \xi_k B_k) = 0$ are real

The Cauchy problem for such a system well-posed, i.e. the solution exists and unique

Quasilinear symmetric hyperbolic systems

Governing PDEs of many models of continuum mechanics can be written as a system of conservation laws:

$$\frac{\partial F^0(U)}{\partial t} + \frac{\partial F^k(U)}{\partial x_k} = S(U)$$

and can also be written in a quasilinear form

$$F_U^0 \frac{\partial U}{\partial t} + F_U^k \frac{\partial U}{\partial x_k} = S(U)$$

The question is: how to write the system in a hyperbolic symmetric form?

$$A(q)\frac{\partial q}{\partial t} + B_k(q)\frac{\partial q}{\partial x_k} = S(q)$$

- quasilinear symmetric system

$$A(q) = A^T(q) > 0, \quad B_k(q) = B_k^T(q)$$

Godunov's form (L-q formulation) of gas dynamics equations

$$q_0 = \left(E + \rho E_\rho - S E_S - u_i u_i / 2\right),\ q_i = u_i\left(i = 1,2,3\right),\ q_4 = T = E_S \qquad \rho = L_{q_0},\ \ \rho u_i = L_{q_i},\ \ \rho S = L_{q_4}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0 \qquad \frac{\partial L_{q_0}}{\partial t} + \frac{\partial L^k{}_{q_0}}{\partial x_k} = 0 \qquad \frac{\partial L_{q_0}}{\partial t} + \frac{\partial \left(u_k L\right)_{q_0}}{\partial x_k} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k + p \delta_{ik}\right)}{\partial x_k} = 0 \quad \Rightarrow \quad \frac{\partial L_{q_i}}{\partial t} + \frac{\partial L^k{}_{q_i}}{\partial x_k} = 0 \quad \Rightarrow \quad \frac{\partial L_{q_i}}{\partial t} + \frac{\partial \left(u_k L\right)_{q_i}}{\partial x_k} = 0$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0 \qquad \frac{\partial L_{q_4}}{\partial t} + \frac{\partial L^k{}_{q_4}}{\partial x_k} = 0 \qquad \frac{\partial L_{q_4}}{\partial t} + \frac{\partial \left(u_k L\right)_{q_4}}{\partial x_k} = 0$$

Original Godunov's form.

Four potentials L, L^1, L^2, L^3

Energy conservation law
$$\frac{\partial \rho\left(E + u_i u_i / 2\right)}{\partial t} + \frac{\partial \left(\rho u_k \left(E + u_i u_i / 2\right) + p u_k\right)}{\partial x_k} = 0$$

takes the form
$$\frac{\partial \left(q_0 L_{q_0} + q_i L_{q_i} + q_4 L_{q_4} - L\right)}{\partial t} + \frac{\partial \left(q_0 \left(u_k L\right)_{q_0} + q_i \left(u_k L\right)_{q_i} + q_4 \left(u_k L\right)_{q_4} - \left(u_k L\right)\right)}{\partial x_k} = 0$$

L - generating potential

q_α - generating variables

Symmetric L-q form of gas dynamics equations

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0$$

$$\frac{\partial L_{q_i}}{\partial t} + \frac{\partial (u_k L)_{q_i}}{\partial x_k} = 0$$

$$\frac{\partial L_{q_4}}{\partial t} + \frac{\partial (u_k L)_{q_4}}{\partial x_k} = 0$$

$$\frac{\partial L_{q_m}}{\partial t} + \frac{\partial (u_k L)_{q_m}}{\partial x_k} = 0, \qquad m = 0,1,2,3,4$$

Quasilinear form

$$L_{q_m q_n} \frac{\partial q_n}{\partial t} + (u_k L)_{q_m q_n} \frac{\partial q_n}{\partial x_k} = 0$$

$$A(U)\frac{\partial U}{\partial t} + B_k(U)\frac{\partial U}{\partial x_k} = 0, \quad U = (q_1, \ldots, q_4)^T, \quad A = A^T = \left(L_{q_m q_n}\right), \quad B_k = B^T = \left((u_k L)_{q_m q_n} \right)$$

The system is symmetric and hyperbolic if $A > 0$

In terms of internal energy (Equation of state) $E(V,S), V = 1/\rho$ must be a convex function:

$$\begin{pmatrix} E_{VV} & E_{VS} \\ E_{SV} & E_{SS} \end{pmatrix} > 0$$

General L-q formulation of Symmetric Hyperbolic Thermodynamically Compatible systems

It was first derived as a result of analysis of various models of continuum mechanics (nonlinear elasticity, electrodynamics of a moving medium, superfluid helium, and so on)

$$\frac{\partial L_{r_i}}{\partial t}+\frac{\partial\left[(u_k L)_{r_i}\right]}{\partial x_k}=0$$

$$\frac{\partial L_{u_i}}{\partial t}+\frac{\partial\left[(u_k L)_{v_i}+\alpha_{mk}L_{\alpha_{mi}}-d_i L_{d_k}-b_i L_{b_k}+\eta_k L_{\eta_i}-\delta_{ik}\eta_m L_{\eta_m}\right]}{\partial x_k}=0$$

$$\frac{\partial L_{\alpha_{ik}}}{\partial t}+\frac{\partial\left[(u_m L)_{\alpha_{im}}\right]}{\partial x_k}+u_j\left(\frac{\partial L_{\alpha_{ik}}}{\partial x_j}-\frac{\partial L_{\alpha_{ij}}}{\partial x_k}\right)=0$$

$$\frac{\partial L_{d_i}}{\partial t}+\frac{\partial\left[(u_k L)_{d_i}-v_i L_{d_k}-\varepsilon_{ikl}b_l\right]}{\partial x_k}+u_i\frac{\partial L_{d_k}}{\partial x_k}=0$$

$$\frac{\partial L_{b_i}}{\partial t}+\frac{\partial\left[(u_k L)_{b_i}-u_i L_{b_k}-\varepsilon_{ikl}d_l\right]}{\partial x_k}+u_i\frac{\partial L_{b_k}}{\partial x_k}=0$$

$$\frac{\partial L_{\eta_k}}{\partial t}+\frac{\partial\left[u_m L_{\eta_m}+n\right]}{\partial x_k}+u_l\left(\frac{\partial L_{\eta_k}}{\partial x_l}-\frac{\partial L_{\eta_l}}{\partial x_k}\right)=0$$

$$\frac{\partial L_n}{\partial t}+\frac{\partial\left[(u_k L)_n+\eta_k\right]}{\partial x_k}=0$$

The system can be written in a symmetric form and is symmetric hyperbolic if

L is a convex function

Integrability conditions

$$\frac{\partial L_{\alpha_{ij}}}{\partial x_k}-\frac{\partial L_{\alpha_{ik}}}{\partial x_j}=0$$

$$\frac{\partial L_{d_k}}{\partial x_k}=0,\qquad \frac{\partial L_{b_k}}{\partial x_k}=0$$

$$\frac{\partial L_{\eta_j}}{\partial x_k}-\frac{\partial L_{\eta_k}}{\partial x_j}=0$$

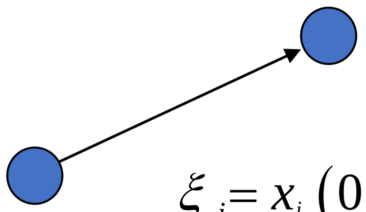
Energy conservation law holds

$$\frac{\partial}{\partial t}\left(r_i L_{r_i}+v_i L_{v_i}+\alpha_{ik}L_{\alpha_{ik}}+d_i L_{d_i}+b_i L_{b_i}+\eta_k L_{\eta_k}+nL_n-L\right)+$$
$$\frac{\partial}{\partial x_k}\left[v_k\left(r_i L_{r_i}+v_i L_{v_i}+\alpha_{ik}L_{\alpha_{ik}}+d_i L_{d_i}+b_i L_{b_i}+\eta_k L_{\eta_k}+nL_n\right)\right]+$$
$$\frac{\partial}{\partial x_k}\left[v_m\left(\alpha_{mk}L_{\alpha_{mi}}-d_i L_{d_k}-b_i L_{b_k}+\eta_k L_{\eta_i}-\delta_{ik}\eta_m L_{\eta_m}\right)\right]$$

No dissipation yet!!

Elasticity equations

Moving medium



$t=0$

$x_i(t)$ - Eulerian coordinates of the particle

$\xi_i = x_i(0)$ - Lagrangian coordinates at t=0

Velocity $$u_i = \frac{d x_i}{d t}$$

Deformation (stretch and rotation) is characterized by the Deformation Gradient

$$F_{ij} = \frac{\partial x_i}{\partial \xi_j}$$

We also use the inverse matrix (Distortion) $$A_{ij} = \frac{\partial \xi_i}{\partial x_j} \qquad \left(A_{ij} F_{jk} = \delta_{ik} \right)$$

Compatibility conditions:

Kinematics: $$\frac{\partial F_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0$$

Steady constraints: $$\frac{\partial F_{ij}}{\partial \xi_k} - \frac{\partial F_{ik}}{\partial \xi_j} = 0$$

Equations of motion can be derived using the variational principle

Elasticity equations derived via variational principle

(H. Goldstein "Classical Mechanics")

Action $$\mathcal{L} = \int \Lambda \, d\xi \, dt$$

Lagrangian $$\Lambda = \Lambda\left(\frac{\partial x_i}{\partial t}, \frac{\partial x_i}{\partial \xi_j}\right) = \rho_0 \frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial t} - \rho_0 E\left(\frac{\partial x_i}{\partial \xi_j}, S\right)$$

ρ_0=const - initial density

E - internal energy

S - entropy

Minimization gives us
Euler-Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial t}\right) - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial\left(\partial x_i / \partial \xi_k\right)}\right) = 0 \qquad ①$$

This equation can be considered as the 2nd order equation for displacement vector $w_i = x_i - \xi_i$ (classical formulation):

$$\frac{\partial^2 w_i}{\partial t^2} - C_{ikjl}\frac{\partial^2 w_j}{\partial \xi_k \partial \xi_l} = 0, \qquad C_{ikjl} = \frac{\partial^2 E}{\partial\left(\partial x_i / \partial \xi_k\right)\partial\left(\partial x_j / \partial \xi_l\right)}$$

Since we interested in the 1st order equations, we write ① in terms of velocity and deformation gradient:

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0$$

Elasticity equations in Lagrangian coordinates

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0 \qquad \times u_i$$

$$E = E\left(F_{ik}, S\right)$$

$$\frac{\partial F_{ik}}{\partial t} - \frac{\partial u_i}{\partial \xi_k} = 0 \qquad \times E_{F_{ik}}$$

$\frac{\partial E}{\partial F_{ik}}$ - Piola-Kirchhoff stress tensor (non-symmetric)

$$\frac{\partial S}{\partial t} = 0 \qquad \times E_S$$

Conservation of energy

$$\frac{\partial}{\partial t}\left(E + \frac{u_i u_i}{2}\right) + \frac{\partial}{\partial \xi_k}\left(-u_i \frac{\partial E}{\partial F_{ik}}\right) = 0$$

For an isotropic media energy depends on three independent invariants of any deformation tensor

We usually take the Finger (or metric) tensor $G = F^{-T} F^{-1},\ G_{ij} = A_{\alpha i} A_{\alpha j},\ A = F^{-1}$

and as its three invariants one can take $tr\,G,\ tr\left(G^2\right),\ tr\left(G^3\right)$

Symmetric form of elasticity equations

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0$$

$$\frac{\partial F_{ik}}{\partial t} - \frac{\partial u_i}{\partial \xi_k} = 0$$

$$\frac{\partial S}{\partial t} = 0$$

It is easy to find a vector of generating variables that are factors in deriving the law of conservation of energy

$$p = \left(p_i, p_{ik}, p_0\right)^T = \left(u_i, E_{F_{ik}}, E_S\right)^T$$

Then we know that $u_i = M_{p_i}, F_{ik} = M_{p_{ik}}, S = M_{p_0}$

One can find the generating potential

$$M\left(p_i, p_{ik}, p_0\right) = \frac{1}{2} u_i u_i + F_{ik} E_{F_{ik}} + S E_S - E$$

$$\frac{\partial M_{p_i}}{\partial t} - \frac{\partial p_{ik}}{\partial \xi_k} = 0$$

$$\frac{\partial M_{p_{ik}}}{\partial t} - \frac{\partial p_i}{\partial \xi_k} = 0$$

$$\frac{\partial M_{p_0}}{\partial t} = 0$$

The system is clearly symmetric.

It is symmetric hyperbolic if the generating potential M is convex.

Elasticity equations in Eulerian coordinates I

Transformation to Eulerian coordinates consists of the transformation of coordinates:

spatial derivatives $\frac{\partial}{\partial \xi_k} \to \frac{\partial x_j}{\partial \xi_k}\frac{\partial}{\partial x_j} = F_{jk}\frac{\partial}{\partial x_j}$ and time derivative $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0$$

$$\frac{\partial F_{ik}}{\partial t} - \frac{\partial u_i}{\partial \xi_k} = 0$$

$$\frac{\partial S}{\partial t} = 0$$

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} - F_{jk}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0$$

$$\frac{\partial F_{ij}}{\partial t} + u_k \frac{\partial F_{ij}}{\partial x_k} - F_{kj}\frac{\partial u_i}{\partial x_k} = 0$$

$$\frac{\partial S}{\partial t} + u_k \frac{\partial S}{\partial x_k} = 0$$

Useful identity $\frac{\partial}{\partial x_j}\left(\frac{F_{jk}}{\det F}\right) = 0, \quad k = 1, 2, 3$

This is a consequence of $\frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} = 0$ where $A_{ij} = \frac{\partial \xi_i}{\partial x_j}$ is the distortion $\left(A = F^{-1}\right)$

Elasticity equations in terms of distortion A

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_k + p\delta_{ik} - \sigma_{ik}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial\left(A_{i\alpha} u_\alpha\right)}{\partial x_k} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0$$

$$p = \rho^2 \frac{\partial E}{\partial \rho}$$ - pressure

$$\sigma_{ik} = -\rho A_{\alpha i} \frac{\partial E}{\partial A_{\alpha k}}$$ - shear stress

$$E = E\left(\rho, A_{ij}, S\right)$$ - internal energy (equation of state)

Involution constraint

$$\frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} = 0$$

Energy conservation

$$\frac{\partial \rho\left(E + u_i u_i / 2\right)}{\partial t} + \frac{\partial\left(\rho u_i\left(E + u_i u_i / 2\right) + u_i\left(p\delta_{ik} - \sigma_{ik}\right)\right)}{\partial x_k} = 0$$

Equation of state

$$E\left(\rho, S, A_{ij}\right) = E_1\left(\rho, S\right) + E_2\left(\rho, S, A_{ij}\right)$$

$E_1\left(\rho, S\right)$ - Hydrodynamic EOS

$$E_2\left(\rho, A, S\right) = \frac{c_{sh}^2}{8}\left(\mathrm{tr}\left(g^2\right) - 3\right)$$ - Shear strain energy (Gavrilyuk et al)

$c_{sh}\left(\rho, S\right)$ - Shear sound velocity

$$g = G / \left(\det G\right)^{1/3}$$ - Normalized Finger tensor

$$G = A^T A$$

Shear stress is trace free

$$\sigma = -\rho \frac{c_{sh}^2}{2}\left(g^2 - \frac{tr\left(g^2\right)}{3} I\right), \qquad tr\left(\sigma\right) = 0$$

Unified model with strain relaxation

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k + p\delta_{ik} - \sigma_{ik}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial \left(A_{i\alpha} u_\alpha\right)}{\partial x_k} + u_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = -\frac{\psi_{ik}}{\theta(\tau)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\psi_{ik} = \frac{\partial E}{\partial A_{ik}}$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = \frac{\rho}{T\,\theta(\tau)} \psi_{ik} \psi_{ik} \geq 0$$

Involution constraint is not valid

$$\frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} \neq 0$$

$p = \rho^2 \dfrac{\partial E}{\partial \rho}$ - pressure

$\sigma_{ik} = -\rho A_{\alpha i} \dfrac{\partial E}{\partial A_{\alpha k}}$ - shear stress

T - temperature

$E = E\left(\rho, A_{ij}, S\right)$ - internal energy (equation of state)

Energy conservation

$$\frac{\partial \rho\left(E + u_i u_i / 2\right)}{\partial t} + \frac{\partial \left(\rho u_i \left(E + u_i u_i / 2\right) + u_i \left(p\delta_{ik} - \sigma_{ik}\right)\right)}{\partial x_k} = 0$$

Equation of state

$$E\left(\rho, S, A_{ij}\right) = E_1(\rho, S) + E_2\left(\rho, S, A_{ij}\right)$$

$E_1(\rho, S)$ - Hydrodynamic EOS

$E_2(\rho, A, S) = \dfrac{c_{sh}^2}{8}\left(\mathrm{tr}\left(g^2\right) - 3\right)$ - Shear strain energy (Gavrilyuk et al)

$c_{sh}(\rho, S)$ - Shear sound velocity

$g = G / \left(\det G\right)^{1/3}$ - Normalized Finger tensor

$G = A^T A$

Shear stress is trace free

$$\sigma = -\rho \frac{c_{sh}^2}{2}\left(g^2 - \frac{tr\left(g^2\right)}{3} I\right), \qquad tr(\sigma) = 0$$

L-q formulation of elasticity equations (symmetric form) I

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k + p\delta_{ik} - \sigma_{ik}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial \left(A_{i\alpha} u_\alpha\right)}{\partial x_k} + u_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = 0$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0$$

$$\frac{\partial L_r}{\partial t} + \frac{\partial \left(u_k L\right)_r}{\partial x_k} = 0$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left(\left(u_k L\right)_{u_i} + \alpha_{mk} L_{\alpha_{mi}} - \delta_{ik} \alpha_{mn} L_{\alpha_{mn}}\right)}{\partial x_k} = 0$$

$$\frac{\partial L_{\alpha_{ik}}}{\partial t} + \frac{\partial \left(u_m L\right)_{\alpha_{im}}}{\partial x_k} + u_m \left(\frac{\partial L_{\alpha_{ik}}}{\partial x_m} - \frac{\partial L_{\alpha_{im}}}{\partial x_k}\right) = 0$$

$$\frac{\partial L_\theta}{\partial t} + \frac{\partial \left(u_k L\right)_\theta}{\partial x_k} = 0$$

Multipliers – generating variables

$$r = E - V E_V - S E_S, \; u_i, \; \alpha_{ik} = \rho E_{\alpha_{ik}}, \; E_S$$

Generating potential

$L = -E_V = p$ - pressure

L-q formulation of elasticity equations (symmetric form)

$$\frac{\partial L_r}{\partial t}+\frac{\partial (u_k L)_r}{\partial x_k}=0$$

$$\frac{\partial L_{u_i}}{\partial t}+\frac{\partial \left((u_k L)_{u_i}+\alpha_{mk}L_{\alpha_{mi}}-\delta_{ik}\alpha_{mn}L_{\alpha_{mn}}\right)}{\partial x_k}=0$$

$$\frac{\partial L_{\alpha_{ik}}}{\partial t}+\frac{\partial (u_m L)_{\alpha_{im}}}{\partial x_k}+u_m\left(\frac{\partial L_{\alpha_{ik}}}{\partial x_m}-\frac{\partial L_{\alpha_{im}}}{\partial x_k}\right)=0$$

$$\frac{\partial L_\theta}{\partial t}+\frac{\partial (u_k L)_\theta}{\partial x_k}=0$$

multipliers

$$r=E-VE_V-SE_S,\ u_i,\ \alpha_{ik}=\rho E_{\alpha_{ik}},\ E_S$$

generating potential

$L=-E_V=p$ - pressure

energy conservation law

$$\frac{\partial\left(rL_r+u_iL_{u_i}+\alpha_{ij}L_{\alpha_{ij}}+\theta L_\theta-L\right)}{\partial t}+\frac{\partial\left(u_k\left(rL_r+u_iL_{u_i}+\alpha_{ij}L_{\alpha_{ij}}+\theta L_\theta-L\right)+u_i\left(L-\alpha_{mn}L_{\alpha_{mn}}\right)\delta_{ik}+\alpha_{nk}L_{\alpha_{ni}}\right)}{\partial x_k}=0$$

$$\frac{\partial L_r}{\partial t}+\frac{\partial (u_k L)_r}{\partial x_k}=0$$

$$\frac{\partial L_{u_i}}{\partial t}+\frac{\partial (u_k L)_{u_i}}{\partial x_k}+L_{\alpha_{im}}\frac{\partial \alpha_{km}}{\partial x_k}-L_{\alpha_{mk}}\frac{\partial \alpha_{mk}}{\partial x_i}=0$$

$$\frac{\partial L_{\alpha_{il}}}{\partial t}+\frac{\partial (u_k L)_{\alpha_{il}}}{\partial x_k}+L_{\alpha_{mi}}\frac{\partial u_m}{\partial x_i}-L_{\alpha_{ml}}\frac{\partial u_k}{\partial x_k}=0$$

$$\frac{\partial L_\theta}{\partial t}+\frac{\partial (u_k L)_\theta}{\partial x_k}=0$$

- symmetric form

Equations of elastic heat conductive medium derived from the variational principle

Action $\mathcal{L} = \int \Lambda \, d\xi \, dt$ Λ - Lagrangian

$$\Lambda = \Lambda\left(\frac{\partial x_i}{\partial t}, \frac{\partial x_i}{\partial \xi_j}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial \xi_j}, S\right) = \rho_0 \frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial t} - \rho_0 E\left(\frac{\partial x_i}{\partial \xi_j}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial \xi_j}, S\right)$$

scalar potential φ for substance flow through the element of the medium is introduced

Minimization gives us the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial(\partial x_i/\partial t)}\right) + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial(\partial x_i/\partial \xi_k)}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial(\partial \varphi/\partial t)}\right) + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial(\partial \varphi/\partial \xi_k)}\right) = 0$$

Since we interested in the 1st order equations, we define as a variables

$u_i = \frac{\partial x_i}{\partial t}$ - velocity, $F_{ij} = \frac{\partial x_i}{\partial \xi_j}$ - deformation gradient, $\frac{\partial \varphi}{\partial t} = n$ - substance density, $\frac{\partial \varphi}{\partial \xi_j} = \eta_j$ - substance flux

Then Euler-Lagrange equations read as

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial u_i}\right) + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial F_{ik}}\right) = 0 \qquad \frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial n}\right) + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial \eta_k}\right) = 0$$

Lagrangian equations of elastic heat conductive medium as a first order system

$$\left.\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial u_i}\right)+\frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial F_{ik}}\right)=0\\ \frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial n}\right)+\frac{\partial}{\partial \xi_k}\left(\frac{\partial \Lambda}{\partial \eta_k}\right)=0\end{aligned}\right\}$$

Euler – Lagrange equations

$$\left.\begin{aligned}\frac{\partial F_{ij}}{\partial t}-\frac{\partial u_i}{\partial \xi_j}=0\\ \frac{\partial \eta_j}{\partial t}-\frac{\partial n}{\partial \xi_j}=0\end{aligned}\right\}$$

Integrability conditions

Lagrangian is a difference of kinetic energy and potential energy: $\Lambda=\frac{1}{2}u_iu_i-U\left(F_{ij},n,\eta_k\right)$

It is convenient to introduce a Generalized Internal Energy $E=U-nU_n$

and new variable $\theta=U_n$ Then $U_{F_{ij}}=E_{F_{ij}},\ n=-E_\theta$

$$\left.\begin{aligned}\frac{\partial u_i}{\partial t}-\frac{\partial}{\partial \xi_j}\left(\frac{\partial E}{\partial F_{ij}}\right)=0\\ \frac{\partial \theta}{\partial t}+\frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \eta_m}\right)=0\end{aligned}\right\}$$

Euler – Lagrange equations

$$\left.\begin{aligned}\frac{\partial F_{ij}}{\partial t}-\frac{\partial u_i}{\partial \xi_j}=0\\ \frac{\partial \eta_m}{\partial t}+\frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \theta}\right)=0\end{aligned}\right\}$$

Integrability conditions

Arbitrary number of equations $\frac{\partial q_i}{\partial t}=0$ can be added to the system

The system in terms of generalized internal energy

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_j}\left(\frac{\partial E}{\partial F_{ij}}\right) = 0 \qquad \times u_i$$

$$\frac{\partial F_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0 \qquad \times \frac{\partial E}{\partial F_{ij}}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \eta_m}\right) = 0 \qquad \times \frac{\partial E}{\partial \theta}$$

$$\frac{\partial \eta_m}{\partial t} + \frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \theta}\right) = 0 \qquad \times \frac{\partial E}{\partial \eta_m}$$

$$\frac{\partial q_i}{\partial t} = 0 \qquad \times \frac{\partial E}{\partial q_i}$$

integrability conditions – involution constraints

$$\frac{\partial F_{ij}}{\partial \xi_k} - \frac{\partial F_{ik}}{\partial \xi_j} = 0$$

$$\frac{\partial \eta_m}{\partial \xi_k} - \frac{\partial \eta_k}{\partial \xi_m} = 0$$

Additional energy conservation law holds

$$\frac{\partial}{\partial t}\left(E + \frac{u_i u_i}{2}\right) + \frac{\partial}{\partial \xi_j}\left(-u_i \frac{\partial E}{\partial F_{ij}} + \frac{\partial E}{\partial \theta}\frac{\partial E}{\partial \eta_m}\right) = 0$$

The above system is appropriate for physical consideration, but for the proof of symmetric hyperbolicity the L-q formulation is preferable.

The system in terms of generating potential L

$$\frac{\partial L_{u_i}}{\partial t} - \frac{\partial p_{ij}}{\partial \xi_j} = 0 \qquad \times\, u_i$$

$$\frac{\partial L_{p_{ij}}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0 \qquad \times\, p_{ij} = \frac{\partial E}{\partial F_{ij}}$$

$$\frac{\partial L_n}{\partial t} + \frac{\partial j_m}{\partial \xi_m} = 0 \qquad \times\, n = \frac{\partial E}{\partial \theta}$$

$$\frac{\partial L_{j_m}}{\partial t} + \frac{\partial n}{\partial \xi_m} = 0 \qquad \times\, j_m = \frac{\partial E}{\partial \eta_m}$$

$$\frac{\partial L_{s_i}}{\partial t} = 0 \qquad \times\, s_i = \frac{\partial E}{\partial q_i}$$

The system is obviously symmetric

Additional energy conservation law is fulfilled

$$\frac{\partial}{\partial t}\left(L - u_i L_{u_i} - p_{ij} L_{p_{ij}} - n L_n - j_m L_{j_m} - s_i L_{s_i}\right) + \frac{\partial}{\partial \xi_k}\left(-u_i p_{ik} + n j_k\right) = 0$$

Integrabilty conditions (involution constraints):

$$\frac{\partial L_{p_{ij}}}{\partial \xi_k} - \frac{\partial L_{p_{ik}}}{\partial \xi_j} = 0 \qquad \frac{\partial L_{j_m}}{\partial \xi_k} - \frac{\partial L_{j_k}}{\partial \xi_m} = 0$$

One can prove that if these equalities hold at t=0, then they hold for t>0

Elastic heat conductive medium equations in Eulerian coordinates I

Transformation to Eulerian coordinates consists of the transformation of coordinates:

spatial derivatives $\frac{\partial}{\partial \xi_k} \to \frac{\partial x_j}{\partial \xi_k}\frac{\partial}{\partial x_j} = F_{jk}\frac{\partial}{\partial x_j}$ and time derivative $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + u_k\frac{\partial}{\partial x_k}$

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial \xi_k}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0 \quad \Longrightarrow \quad \frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} - F_{jk}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial F_{ik}}\right) = 0 \quad \Longrightarrow \quad \frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_k - \rho F_{jl} E_{F_{ik}}\right)}{\partial x_k} = 0$$

$$\frac{\partial F_{ik}}{\partial t} - \frac{\partial u_i}{\partial \xi_k} = 0 \quad \Longrightarrow \quad \frac{\partial F_{ij}}{\partial t} + u_k\frac{\partial F_{ij}}{\partial x_k} - F_{kj}\frac{\partial u_i}{\partial x_k} = 0$$

$$\frac{\partial S}{\partial t} = 0 \quad \Longrightarrow \quad \frac{\partial S}{\partial t} + u_k\frac{\partial S}{\partial x_k} = 0 \quad \Longrightarrow \quad \frac{\partial \rho S}{\partial t} + \frac{\partial \rho u_k S}{\partial x_k} = 0$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \eta_m}\right) = 0 \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} + u_k\frac{\partial \theta}{\partial x_k} + F_{jm}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial \eta_m}\right) = 0$$

$$\frac{\partial \eta_m}{\partial t} + \frac{\partial}{\partial \xi_m}\left(\frac{\partial E}{\partial \theta}\right) = 0 \quad \Longrightarrow \quad \frac{\partial \eta_m}{\partial t} + u_k\frac{\partial \eta_m}{\partial x_k} + F_{jm}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial \theta}\right) = 0$$

We know that the first three equations can be written in a conservative form with the use of identity $\frac{\partial}{\partial x_j}\left(\frac{F_{jk}}{\det F}\right) = 0$

and continuity equation $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \qquad \rho = \frac{\rho_0}{\det F}$

Elastic heat conductive medium equations in Eulerian coordinates II

$$\frac{\partial\theta}{\partial t}+u_k\frac{\partial\theta}{\partial x_k}+F_{jm}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial\eta_m}\right)=0 \quad\longrightarrow\quad \frac{\rho_0}{\det F}\left(\frac{\partial\theta}{\partial t}+u_k\frac{\partial\theta}{\partial x_k}\right)+\frac{\rho_0 F_{jm}}{\det F}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial\eta_m}\right)=0 \quad\longrightarrow$$

$$\left(\frac{\partial\rho\theta}{\partial t}+\frac{\partial\rho u_k\theta}{\partial x_k}\right)+\frac{\partial}{\partial x_j}\left(\rho F_{jm}\frac{\partial E}{\partial\eta_m}\right)=0$$

$$\frac{\partial\eta_m}{\partial t}+u_k\frac{\partial\eta_m}{\partial x_k}+F_{jm}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial\theta}\right)=0$$

Let us introduce ↓ new variable $w_k=\eta_m A_{mk}$ and use equation for A_{ik} : $\frac{\partial A_{ik}}{\partial t}+u_\alpha\frac{\partial A_{ik}}{\partial x_\alpha}+A_{i\alpha}\frac{\partial u_\alpha}{\partial x_k}=0$

$$A_{mk}\left(\frac{\partial\eta_m}{\partial t}+u_k\frac{\partial\eta_m}{\partial x_k}\right)+A_{mk}F_{jm}\frac{\partial}{\partial x_j}\left(\frac{\partial E}{\partial\theta}\right)=0 \quad\longrightarrow$$

$$\frac{\partial w_k}{\partial t}+u_l\frac{\partial w_k}{\partial x_l}-\eta_m\left(\frac{\partial A_{ik}}{\partial t}+u_l\frac{\partial A_{ik}}{\partial x_l}\right)+\frac{\partial}{\partial x_k}\left(\frac{\partial E}{\partial\theta}\right)=0 \quad\longrightarrow$$

$$\frac{\partial w_k}{\partial t}+u_l\frac{\partial w_k}{\partial x_l}+w_l\frac{\partial u_l}{\partial x_k}+\frac{\partial}{\partial x_k}\left(\frac{\partial E}{\partial\theta}\right)=0 \quad\longrightarrow$$

$$\frac{\partial w_k}{\partial t}+\frac{\partial w_l u_l}{\partial x_k}+\frac{\partial}{\partial x_k}\left(\frac{\partial E}{\partial\theta}\right)+u_l\left(\frac{\partial w_k}{\partial x_l}-\frac{\partial w_l}{\partial x_k}\right)=0$$

One can prove that $\frac{\partial w_k}{\partial x_l}-\frac{\partial w_l}{\partial x_k}=0$

Finally we have $\left(\frac{\partial\rho\theta}{\partial t}+\frac{\partial\rho u_k\theta}{\partial x_k}\right)+\frac{\partial}{\partial x_j}\left(\rho F_{jm}\frac{\partial E}{\partial\eta_m}\right)=0 \quad\longrightarrow\quad \frac{\partial\rho\theta}{\partial t}+\frac{\partial\rho u_k\theta}{\partial x_k}+\frac{\partial}{\partial x_k}\left(\rho\frac{\partial E}{\partial w_k}\right)=0$

Elastic heat conductive medium equations in Eulerian coordinates III

Consider equations in terms of distortion A

Note, that since we do the change of state variables $w_k = \eta_m A_{mk},\ E(\rho, A_{ij}, S, \theta, \eta_m) \to E(\rho, A_{ij}, S, \theta, w_k)$
the derivative of energy with respect to distortion changes:

$$\frac{\partial E}{\partial A_{\alpha k}} \to \frac{\partial E}{\partial A_{\alpha k}} + \frac{\partial E}{\partial w_k}\eta_\alpha$$

Momentum equation reads as

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_k + p\delta_{ik} + \rho A_{\alpha i} E_{A_{\alpha k}} + \rho w_i E_{w_k}\right)}{\partial x_k} = 0$$

Elastic heat conductive medium equations in Eulerian coordinates IV

Final formulation applicable for design of heat conduction in the elastic medium

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k + p\delta_{ik} + \rho A_{\alpha i} E_{A_{\alpha k}} + \rho w_i E_{w_k}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial \left(A_{i\alpha} u_\alpha\right)}{\partial x_k} + u_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial w_k}{\partial t} + \frac{\partial \left(w_l u_l + E_\theta\right)}{\partial x_k} + u_l \left(\frac{\partial w_k}{\partial x_l} - \frac{\partial w_l}{\partial x_k}\right) = 0$$

$$\frac{\partial \rho \theta}{\partial t} + \frac{\partial \left(\rho u_k \theta + \rho E_{w_k}\right)}{\partial x_k} = 0$$

Variables θ, w_k should be identified with physical variables and the generalized internal energy E should be defined as a function of state variables

This system can be applied for the design of two-phase compressible flow.
In this case the entropy equation should be added

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho u_k S}{\partial x_k} = 0$$

Elastic heat conductive medium equations in Eulerian coordinates V

For the heat conductive medium we take $\theta = S$ - entropy, $w_k = J_k$ - thermal impulse

Generalized energy can be taken as

$$E\left(\rho, S, A_{ij}\right) = E_1\left(\rho, S\right) + E_2\left(\rho, S, A_{ij}\right) + \frac{c_h^2}{2} J_i J_i$$

$E_1(\rho, S)$ - Hydrodynamic EOS

$$E_2\left(\rho, A, S\right) = \frac{c_{sh}^2}{8}\left(\mathrm{tr}\left(g^2\right) - 3\right)$$ - Shear strain energy

$c_{sh}(\rho, S)$ - Shear sound velocity

$g = G / \left(\det G\right)^{1/3}$ - Normalized Finger tensor

$G = A^T A$

c_h relates to heat wave propagation

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_k + p\delta_{ik} + \rho A_{\alpha i} E_{A_{\alpha k}} + \rho J_i E_{J_k}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial\left(A_{i\alpha} u_\alpha\right)}{\partial x_k} + u_j\left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial J_k}{\partial t} + \frac{\partial\left(J_l u_l + E_S\right)}{\partial x_k} + u_l\left(\frac{\partial J_k}{\partial x_l} - \frac{\partial J_l}{\partial x_k}\right) = 0$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial\left(\rho u_k S + \rho E_{J_k}\right)}{\partial x_k} = 0$$

E_S - temperature

$$\frac{\partial \rho\left(E + u_i u_i / 2\right)}{\partial t} + \frac{\partial\left(\rho u_k\left(E + u_i u_i / 2\right) + u_i\left(p\delta_{ik} + \rho A_{\alpha i} E_{A_{\alpha k}} + J_i E_{J_k}\right) + E_S E_{J_k}\right)}{\partial x_k} = 0$$

- Conservation of energy

Elastic heat conductive medium, introduction of dissipative source terms

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k + p \delta_{ik} - \sigma_{ik} + \rho J_i E_{J_k} \right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial \left(A_{i\alpha} u_\alpha \right)}{\partial x_k} + u_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} \right) = - \frac{\psi_{ik}}{\theta_1(\tau_1)}$$

Strain relaxation (shear stress relaxation)

$$\frac{\partial J_k}{\partial t} + \frac{\partial \left(J_l u_l + E_S \right)}{\partial x_k} + u_l \left(\frac{\partial J_k}{\partial x_l} - \frac{\partial J_l}{\partial x_k} \right) = - \frac{H_k}{\theta_2(\tau_2)}$$

Heat flux relaxation

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \left(\rho u_k S + \rho E_{J_k} \right)}{\partial x_k} = \frac{\rho}{T \theta_1(\tau_1)} \psi_{ik} \psi_{ik} + \frac{\rho}{T \theta_2(\tau_2)} H_{ik} H_i \geq 0$$

Entropy production
(2nd law of thermodynamics)

$$\psi_{ik} = \frac{\partial E}{\partial A_{ik}}, \quad \theta_1(\tau_1) = \tau_1 \frac{2 c_s^2}{\rho \left(\det G \right)^{1/3}}$$

$$H_k = \frac{\partial E}{\partial J_k}, \quad \theta_2(\tau_2) = \frac{1}{3} \tau_2 \frac{c_h^2}{\rho T}$$

Energy conservation law remains unchanged

Unified model of continuum mechanics, asymptotic limits

$\tau_1 = \infty$ corresponds to elastic medium

$\tau_1 \to 0$ formal asymptotic expansion $G = G^0 + \tau_1 G^1 + \dots$ G - the Finger tensor

gives us the Navier-Stokes equations for compressible viscous flow with the viscosity $\mu = \tau_1 c_s^2$:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_k + p\delta_{ik} - \sigma_{ij}\right)}{\partial x_k} = 0 \qquad \sigma_{ij} = \tau_1 c_s^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} - \frac{2}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)\right)$$

$0 < \tau(\sigma, T) < \infty$ allows one to model strain-rate and temperature dependent inelastic deformations

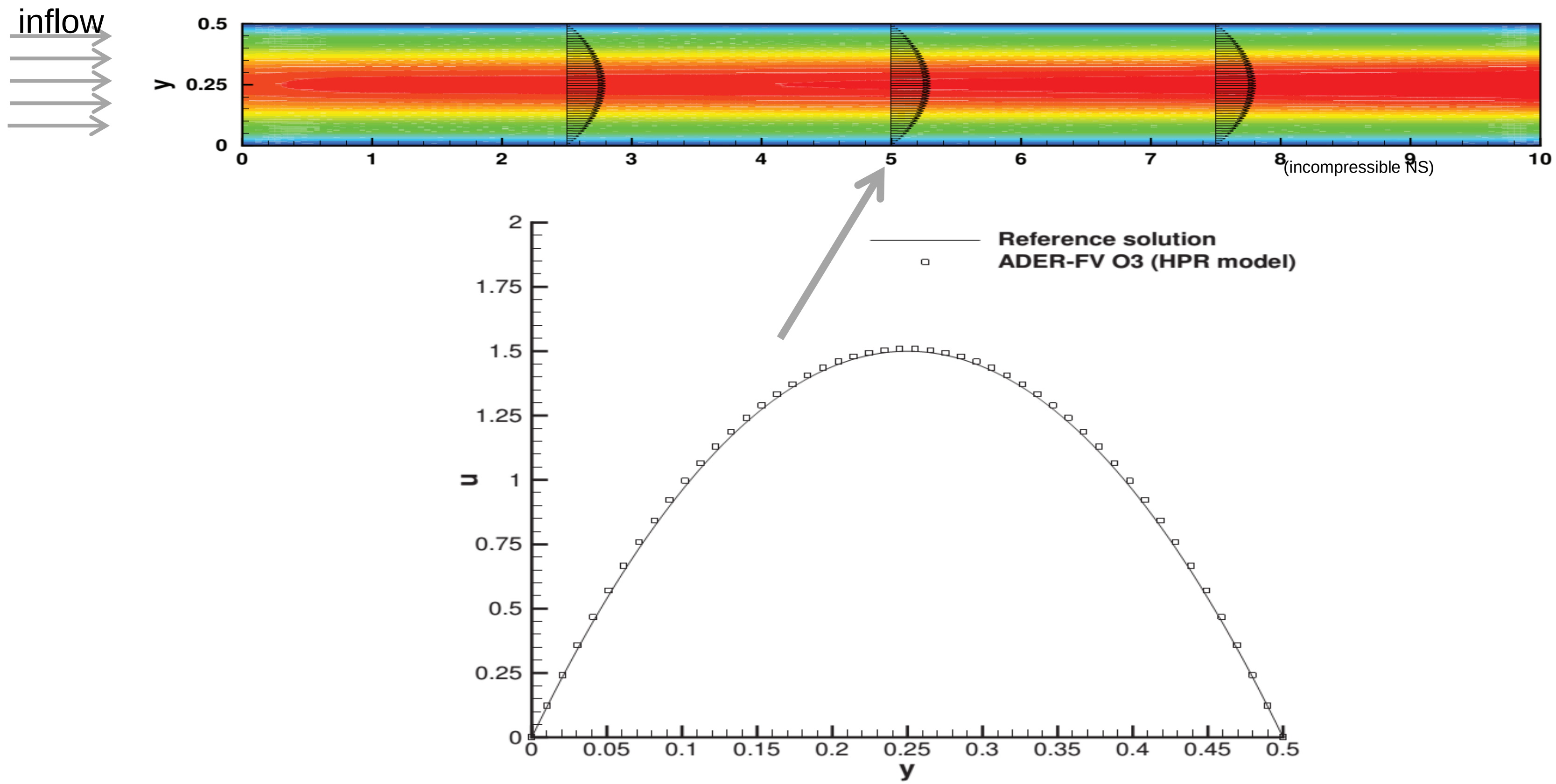
---

$\tau_2 \to 0$ gives us the Fourier heat conduction law

$$\frac{\partial \rho\left(E + u_i u_i / 2\right)}{\partial t} + \frac{\partial\left(\rho u_k \left(E + u_i u_i / 2\right) + u_i\left(p\delta_{ik} + \rho A_{\alpha i} E_{A_{\alpha k}} + J_i E_{J_k}\right) + q_k\right)}{\partial x_k} = 0$$

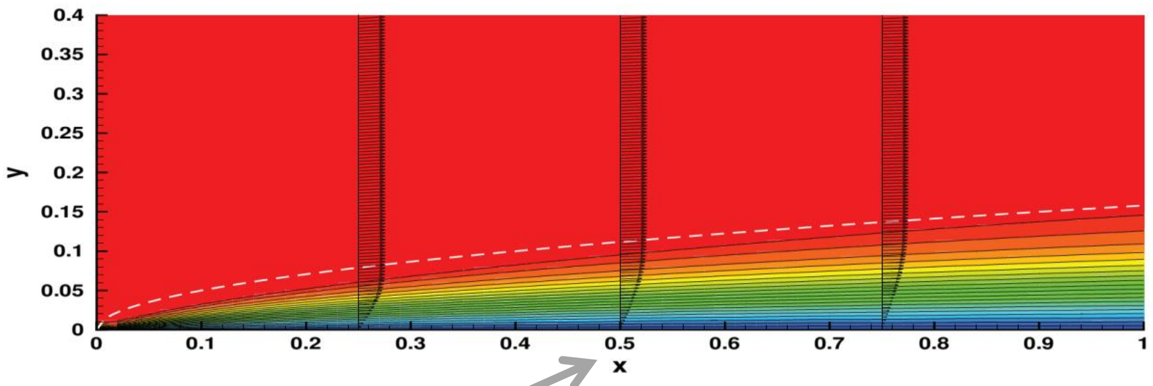
$$q_k = E_S E_{J_k} = c_h^2 \tau_2 \frac{\partial T}{\partial x_k}$$

Steady laminar Hagen-Poiseuille flow, *Re*=50



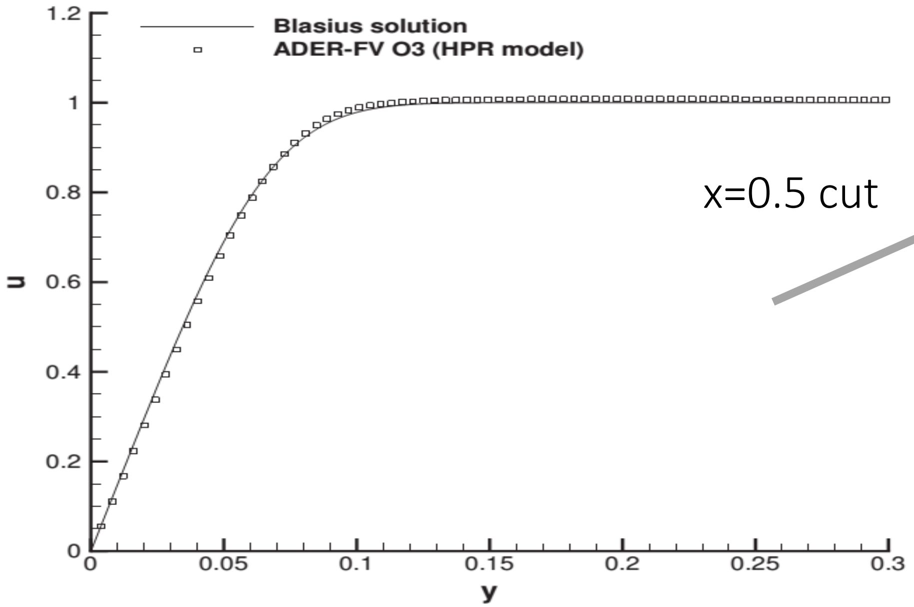
Blasius boundary layer, *Re*=1000

Velocity contours

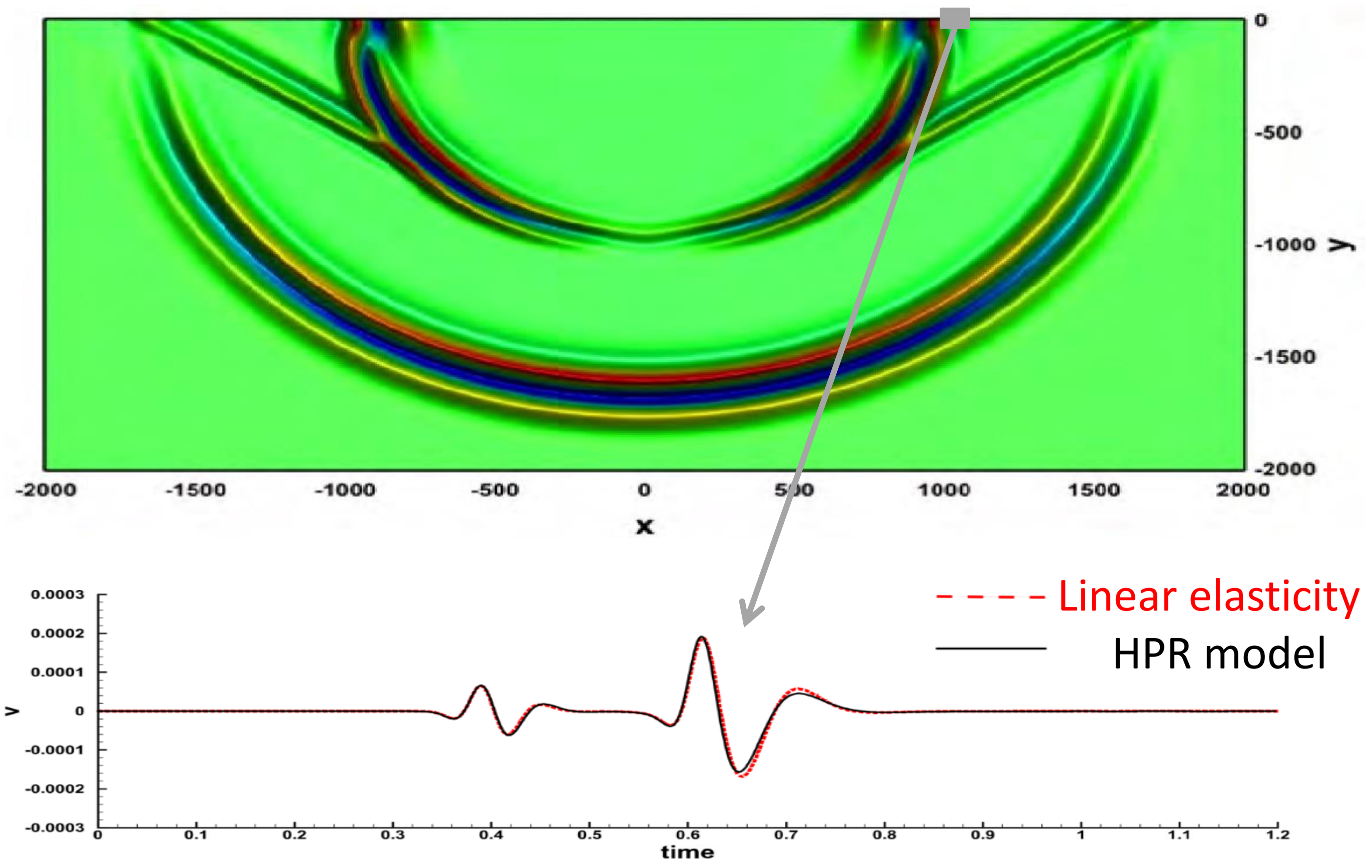


Inflow
u=1

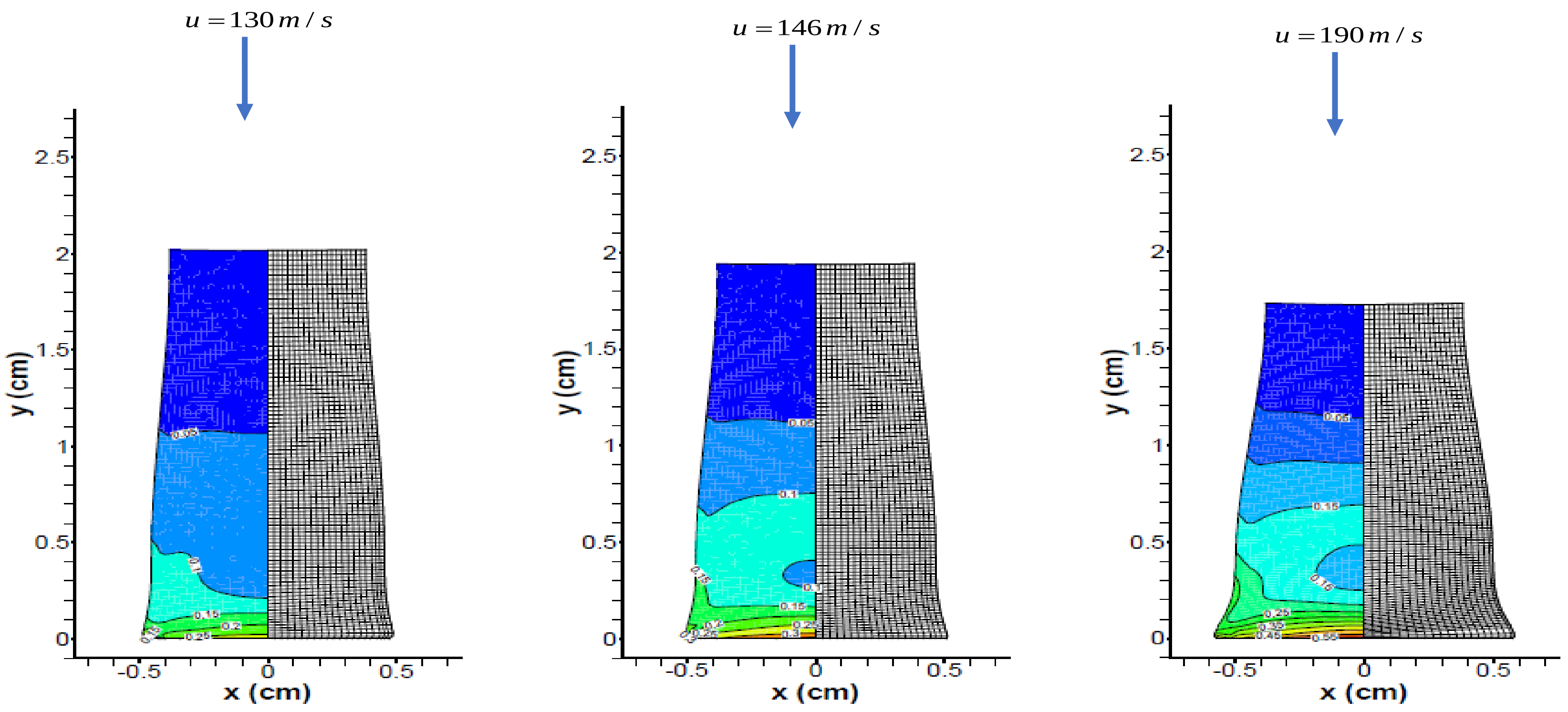
x=0.5 cut



Elasticity: Seismic wave propagation



Elastic-plastic deformation of material with hardening. Taylor test problem



Equations of elastic heat conductive medium in the presence of electromagnetic field

Action $\mathcal{L} = \int \Lambda \, d\xi \, dt$ with Lagrangian

$$\Lambda = \Lambda\left(\frac{\partial x_i}{\partial t}, \frac{\partial x_i}{\partial \xi_j}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial \xi_j}, S\right) = \rho_0 \frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial t} - \rho_0 E\left(\frac{\partial x_i}{\partial \xi_j}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial \xi_j}, -\frac{\partial a_i}{\partial t} - \frac{\partial \psi}{\partial \xi_j}, \varepsilon_{ijk}\frac{\partial a_k}{\partial \xi_j}, S\right)$$

If to introduce variables defined by the potentials as:

x_i, ξ_j - Eulerian and Lagrangian coordinates, a_i, φ, ϑ - potentials

$$\frac{\partial x_i}{\partial \xi_j} = F_{ij}, \quad d_i = -\frac{\partial a_i}{\partial t} - \frac{\partial \psi}{\partial \xi_j}, \quad h_i = \varepsilon_{ijk}\frac{\partial a_k}{\partial \xi_j}, \quad \theta = \frac{\partial \varphi}{\partial t}, \quad \eta_j = \frac{\partial \varphi}{\partial \xi_j}$$

and introduce the energy potential as $U = v_i \Lambda_{v_i} + d_i \Lambda_{d_i} - \Lambda; \; u_i = \Lambda_{v_i}, \; e_i = \Lambda_{d_i}$

then the Euler-Lagrange equations can be formulated as the first order system supplemented by the integrability conditions

Lagrangian thermodynamically compatible 1st order system equivalent to Euler-Lagrange equations

Euler-Lagrange equations can be reformulated as the first order system supplemented by the integrability conditions (involution costraints)

$$\frac{\partial u_i}{\partial t} - \frac{\partial U_{F_{ij}}}{\partial \xi_j} = 0$$

$$\frac{\partial F_{ij}}{\partial t} - \frac{\partial U_{u_i}}{\partial \xi_j} = 0$$

$$\frac{\partial e_i}{\partial t} - \varepsilon_{ijk}\frac{\partial U_{h_k}}{\partial \xi_j} = 0$$

$$\frac{\partial h_i}{\partial t} + \varepsilon_{ijk}\frac{\partial U_{e_k}}{\partial \xi_j} = 0$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial U_{\eta_j}}{\partial \xi_j} = 0$$

$$\frac{\partial \eta_j}{\partial t} - \frac{\partial U_{\theta}}{\partial \xi_j} = 0$$

$$\frac{\partial F_{ij}}{\partial \xi_k} - \frac{\partial F_{ik}}{\partial \xi_j} = 0$$

$$\frac{\partial e_j}{\partial \xi_j} = 0$$

$$\frac{\partial h_j}{\partial \xi_j} = 0$$

$$\frac{\partial \eta_j}{\partial \xi_k} - \frac{\partial \eta_k}{\partial \xi_j} = 0$$

Conservation of energy

$$\frac{\partial U}{\partial t} - \frac{\partial\left(U_{u_i}U_{F_{ij}} + \varepsilon_{ijk}U_{e_i}U_{h_k} - U_{\theta}U_{\eta_j}\right)}{\partial \xi_j} = 0$$

System is symmetric and hyperbolic if U is a convex function

Eulerian SHTC system can be obtained from the Lagrangian one by the cumbersome transformations of coordinates and variables

After passing to Euler coordinates, changing of unknowns and Legendre transformations of potential we arrive to L-q formulation of SHTC system (next slide)

L-q formulation of general Symmetric Hyperbolic Thermodynamically Compatible System

It was first derived as a result of analysis of various models of continuum mechanics (nonlinear elasticity, electrodynamics of a moving medium, superfluid helium, and so on)

$$\frac{\partial L_{r_i}}{\partial t}+\frac{\partial\left[(u_k L)_{r_i}\right]}{\partial x_k}=0$$

$$\frac{\partial L_{u_i}}{\partial t}+\frac{\partial\left[(u_k L)_{v_i}+\alpha_{mk}L_{\alpha_{mi}}-d_i L_{d_k}-b_i L_{b_k}+\eta_k L_{\eta_i}-\delta_{ik}\eta_m L_{\eta_m}\right]}{\partial x_k}=0$$

$$\frac{\partial L_{\alpha_{ik}}}{\partial t}+\frac{\partial\left[(u_m L)_{\alpha_{im}}\right]}{\partial x_k}+u_j\left(\frac{\partial L_{\alpha_{ik}}}{\partial x_j}-\frac{\partial L_{\alpha_{ij}}}{\partial x_k}\right)=0$$

$$\frac{\partial L_{d_i}}{\partial t}+\frac{\partial\left[(u_k L)_{d_i}-v_i L_{d_k}-\varepsilon_{ikl}b_l\right]}{\partial x_k}+u_i\frac{\partial L_{d_k}}{\partial x_k}=0$$

$$\frac{\partial L_{b_i}}{\partial t}+\frac{\partial\left[(u_k L)_{b_i}-u_i L_{b_k}-\varepsilon_{ikl}d_l\right]}{\partial x_k}+u_i\frac{\partial L_{b_k}}{\partial x_k}=0$$

$$\frac{\partial L_{\eta_k}}{\partial t}+\frac{\partial\left[u_m L_{\eta_m}+n\right]}{\partial x_k}+u_l\left(\frac{\partial L_{\eta_k}}{\partial x_l}-\frac{\partial L_{\eta_l}}{\partial x_k}\right)=0$$

$$\frac{\partial L_n}{\partial t}+\frac{\partial\left[(u_k L)_n+\eta_k\right]}{\partial x_k}=0$$

Involution constraints

$$\frac{\partial L_{\alpha_{ij}}}{\partial x_k}-\frac{\partial L_{\alpha_{ik}}}{\partial x_j}=0$$

$$\frac{\partial L_{d_k}}{\partial x_k}=0,\quad \frac{\partial L_{b_k}}{\partial x_k}=0$$

$$\frac{\partial L_{\eta_j}}{\partial x_k}-\frac{\partial L_{\eta_k}}{\partial x_j}=0$$

Energy conservation law holds

$$\frac{\partial}{\partial t}\left(r_i L_{r_i}+v_i L_{v_i}+\alpha_{ik}L_{\alpha_{ik}}+d_i L_{d_i}+b_i L_{b_i}+\eta_k L_{\eta_k}+nL_n-L\right)+$$
$$\frac{\partial}{\partial x_k}\left[v_k\left(r_i L_{r_i}+v_i L_{v_i}+\alpha_{ik}L_{\alpha_{ik}}+d_i L_{d_i}+b_i L_{b_i}+\eta_k L_{\eta_k}+nL_n\right)\right]+$$
$$\frac{\partial}{\partial x_k}\left[v_m\left(\alpha_{mk}L_{\alpha_{mi}}-d_i L_{d_k}-b_i L_{b_k}+\eta_k L_{\eta_i}-\delta_{ik}\eta_m L_{\eta_m}\right)\right]=0$$

The system can be written in a symmetric form and is symmetric hyperbolic if L is a convex function

Legendre transformation $dL=L_{r_i}dr_i+....+L_n dn=d\left(L_{r_i}r_i+....+L_n n\right)-r_i\, d\,L_{r_i}-...-n\, d\, L_n=d\left(L_{r_i}r_i+....+L_n n\right)-L_{q_i}\,d\,q_i-....-L_\theta\, d\,\theta$

can be used for the definition of the generalized energy potential $d\,E=d\left(L_{r_i}r_i+....+L_n n-L\right)=r_i\,d\,L_{r_i}+...+n\,d\,L_n$

Symmetric hyperbolic thermodynamically compatible system *in terms of generalized energy*

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

$$\frac{\partial \rho m_i}{\partial t} + \frac{\partial\left(\rho m_i u_k + \rho^2 E_\rho \delta_{ik} + \rho A_{lk} E_{A_{li}} + \rho J_i E_{J_k} - \rho e_k E_{e_i} - \rho h_k E_{h_i} \rho A_{mi} E_{A_{mk}} + \rho J_i E_{J_k}\right)}{\partial x_k} = 0$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial\left(A_{im} u_m\right)}{\partial x_k} + u_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k}\right) = - \frac{E_{A_{ik}}}{\theta_1 \left(\tau_1\right)}$$

$$\frac{\partial e_i}{\partial t} + \frac{\partial\left(u_i e_i - u_i e_k - \varepsilon_{ikl} E_{h_l}\right)}{\partial x_k} + u_i \frac{\partial e_k}{\partial x_k} = - \frac{E_{e_i}}{\eta}$$

$$\frac{\partial h_i}{\partial t} + \frac{\partial\left(u_i h_i - u_i h_k + \varepsilon_{ikl} E_{e_l}\right)}{\partial x_k} + u_i \frac{\partial h_k}{\partial x_k} = 0$$

$$\frac{\partial J_k}{\partial t} + \frac{\partial\left[u_m J_m + \rho E_\theta\right]}{\partial x_k} + u_j \left(\frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k}\right) = - \frac{E_{J_k}}{\theta_2 \left(\tau_2\right)}$$

$$\frac{\partial \rho \theta}{\partial t} + \frac{\partial\left[\rho u_k \theta + \rho E_{J_k}\right]}{\partial x_k} = 0$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = Q \geq 0$$

E - generalized energy, ρ - density, u_i - velocity

m_i - momentum

A_{ik} - distortion

e_i, h_i are associated with the electromagnetic field

J_k, θ are associated with the flow of any substance through an element of the medium

S - entropy

Involution constraints

$$\frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} = 0 \qquad \frac{\partial e_k}{\partial x_k} = 0, \quad \frac{\partial h_k}{\partial x_k} = 0 \qquad \frac{\partial J_i}{\partial x_k} - \frac{\partial J_k}{\partial x_j} = 0$$

Conservation of energy

$$\frac{\partial}{\partial t}\mathbf{E} + \frac{\partial}{\partial x_k}\left[u_k \mathbf{E} + \rho u_i \delta_{ik}\left(\rho E_\rho + e_m E_{e_m} + h_m E_{h_m}\right) + u_i\left(A_{mi} E_{A_{mk}} - e_k E_{e_i} - h_k E_{h_i}\right) + \varepsilon_{ijk} E_{e_i} E_{h_j} + \rho E_\theta E_{J_k}\right] = 0$$

$\mathbf{E} = \rho\left(E + u_i u_i / 2\right)$ - total energy

No heat flux

More pictures and theory can be found in
M. Dumbser, I. Peshkov, E. Romenski, O. Zanotti, *Journal of Computational Physics*, 2016, 2017
I. Peshkov, M. Pavelka, E. Romenski, M. Grmela, *Continuum Mechanics and Thermodynamics*, 2018
and references therein

Summary

- Class of hyperbolic thermodynamically compatible systems with involution constraints can be formulated from the first principles
- Many well-known equations of continuum mechanics belong to this class
- New well-posed models of complex physical processes can be formulated with the use of SHTC theory by the proper choice of equations, variables and thermodynamic potential such as: *unified model of continuum with hyperbolic heat conduction, multiphase compressible flow, Including flow of immisible fluids flow with surface tension…..*