# REGULARITY OF VISCOUS SOLUTIONS FOR A DEGENERATE NON-LINEAR CAUCHY PROBLEM

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Abstract. We consider the Cauchy problem for a class of nonlinear degenerate parabolic equation with forcing. By using the vanishing viscosity method we obtain generalized solutions. We prove some regularity results about this generalized solutions.

## 1. Introduction

We consider the Cauchy problem for the following nonlinear degenerate parabolic equation with forcing

$$u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u),\ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1}$$

$$u(x,0) = u_0(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{2}$$

where $\gamma$ is a non-negative constant. Equation (1) arisen in severals applications of biology and phisycs, see [15], [12]. Equation (1) is of degenerate parabolic type: parabolicity it is loss at points where $u = 0$, see [15], [1] for a most datailed description. In [11] a weak solution for the homogeneous equation (1) is constructed by using the vanishing viscosity method, this method was introduced by Lions and Crandall [10], when they studied the existence of solutions to Hamilton-Jacobi equations

$$u_t + H(x,t,u,Du) = 0$$

and consists in view the equation (1) as the limit for $\epsilon \to 0$ of the equation

$$u_t = \epsilon\Delta u + u\Delta u - \gamma|\nabla u|^2 + f(t,u), \tag{3}$$

where $\epsilon$ is a small positive number. The reguarity of the weak solutions for the homogeneous Cauchy problem (1),(2) was studied by the author in [9].In this paper we extend the above results for the inhomogeneous case, this extension is interesting, from physical viewpoint, since the equation (1) is related with non-equilibrium process in poros media due to external forces. We obtain the following main theorem,

**Theorem 1.1.** *If $\gamma \geq \sqrt{2N} - 1$, $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, where $M$ is a positive constant such as*

$$\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \leq 0,$$

*then the viscosity solutions of the Cauchy problem (1), (2) satisfies*

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M. \tag{4}$$

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## 2. Preliminaries

**Definition 2.1.** A function $u \in L^\infty(\Omega) \bigcap L^2_{Loc}([0,+\infty); H^1_{Loc}(\mathbb{R}^N))$, is called a weak solution of (1),(2) if it satisfies the following conditions:

(i) $u(x,t) \geq 0$ , $a.e$ in $\Omega$.

(ii) $u(x,t)$ satisfies the following relation

$$\int_{\mathbb{R}^N} u_0 \psi(x,0)\, dx + \iint_\Omega (u\psi_t - u\nabla u\cdot\nabla\psi - (1+\gamma)|\nabla u|^2\psi - f(t,u)\psi)\, dx\, dt = 0, \tag{5}$$

for any $\psi \in C^{1,1}(\overline{\Omega})$ with compact support in $\overline{\Omega}$.

For the construction of a weak solution to the Cauchy problem (1),(2), we use the viscosity method: we add the term $\epsilon\Delta u$ in the equation (1) and we consider the following Cauchy problem

$$u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u) + \epsilon\Delta u, \;\; u \in \Omega, \tag{6}$$

$$u(x,0) = u_0(x), x \in \mathbb{R}^N \tag{7}$$

where $\gamma \geq 0$, the existence of solutions is garanteed by the Maximum principle and then we investigate the convergence of the solutions when $\epsilon \to 0$, in fact, we will show that when $\epsilon \to 0$, $u^\epsilon$ converges to the weak solution of (1),(2), but to cost of the loss of the uniqueness.

**Definition 2.2.** The weak solution for the Cauchy problem (1),(2) constructed by the vanishing viscosity method is called viscosity solution.

## 3. Estimates of Hölder

In this section we begin by collecting some a priori estimates for the function $u$.

**Theorem 3.1.** *If $\gamma \geq \sqrt{2N} - 1$, the initial data (2) satisifes $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, where $M$ is a positive constant, $\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \leq 0$ and $f \in C^1(\mathbb{R}^+ \times \mathbb{R})$ satisfies, $f \geq 0$, $f_u \leq 0$ , then the viscosity solution $u(x,t)$ of Cauchy problem (1),(2) satisfies*

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M, \;\; in\ \overline{\Omega}.$$

*Proof.* Let

$$w = \frac{1}{2}\sum_{i=1}^N u_{x_i}^2. \tag{8}$$

Deriving with respect $t$ in (8) and replacing in (1) we have

$$w_t = \sum_{i=1}^N u_{x_i}\left[u_{x_i}\Delta u + u\left(\sum_{j=1}^N u_{x_ix_jx_j}\right) - 2\gamma w_{x_i} + f_u u_{x_i}\right].$$

By other hand

$$\begin{aligned}
\Delta w &= \frac{1}{2}\sum_{j=1}^N\left(\sum_{i=1}^N u_{x_i}^2\right)_{x_jx_j} \\
&= \frac{1}{2}\left[\sum_{j=1}^N (2u_{x_1}u_{x_1x_j})_{x_j} + \sum_{j=1}^N (2u_{x_2}u_{x_2x_j})_{x_j} + \cdots + \sum_{j=1}^N (2u_{x_N}u_{x_Nx_j})_{x_j}\right]
\end{aligned}$$

$$\Delta w = \sum_{i,j=1}^N u_{x_ix_j}^2 + \sum_{i,j=1}^N u_{x_i}u_{x_ix_jx_j}, \tag{9}$$

thereby,

$$w_t = 2w\Delta u + u\Delta w - u\sum_{i,j=1}^N u_{x_ix_j}^2 - 2\gamma\sum_{i=1}^N u_{x_i}w_{x_i} + 2f_u w. \tag{10}$$

Set,

$$z = g(u)w. \tag{11}$$

Deriving two times with respect $x_i$ in (11) we have

$$w_{x_i} = (g^{-1})_{x_i} z + g^{-1} z_{x_i} \tag{12}$$

$$w_{x_i x_i} = (g^{-1})_{x_i x_i} z + 2(g^{-1})_{x_i} z_{x_i} + g^{-1} z_{x_i x_i}. \tag{13}$$

From equations (9),(12), (13) we have that,

$$\Delta w = \sum_{i=1}^{N} w_{x_i x_i} = \sum_{i=1}^{N} \Big[ (g^{-1})_{x_i x_i} z + 2(g^{-1})_{x_i} z_{x_i} + g^{-1} z_{x_i x_i} \Big],$$

Deriving two times with respect $x_i$ in (11) we have

$$(g^{-1}(u))_{x_i} = -g^{-2} g^{'} u_{x_i} \tag{14}$$

$$(g^{-1}(u))_{x_i x_i} = \left( \frac{2g^{'2} - gg^{''}}{g_4} \right) g u_{x_i}^2 - \frac{g^{'}}{g^2} u_{x_i x_i}, \tag{15}$$

then,

$$\begin{aligned} \Delta w &= \left( \frac{2g^{'2} - gg^{''}}{g^4} \right) g \sum_{i=1}^{N} u_{x_i}^2 z - \frac{g^{'}}{g^2} \sum_{i=1}^{N} u_{x_i x_i} z - 2g^{-2} g^{'} \sum_{i=1}^{N} u_{x_i} z_{x_i} + g^{-1} \sum_{i=1}^{N} z_{x_i x_i} \\ &= g^{-1} \sum_{i=1}^{N} z_{x_i x_i} - 2g^{-2} g^{'} \sum_{i=1}^{N} u_{x_i} z_{x_i} + 2 \left( \frac{2g^{'2} - gg^{''}}{g^4} \right) gwz - \frac{g^{'}}{g^2} z \sum_{i=1}^{N} u_{x_i x_i} \end{aligned}$$

$$\Delta w = g^{-1} \Delta z - 2g^{-2} g^{'} \sum_{i=1}^{N} u_{x_i} z_{x_i} + 2 \left( \frac{2g^{'2} - gg^{''}}{g^4} \right) z^2 - \frac{g^{'}}{g^2} z \Delta u. \tag{16}$$

From (10), (11), (12), (16), we obtain

$$\begin{aligned} z_t =& u\Delta z - (2g^{-1} u g^{'} + 2\gamma) \sum_{i=1}^{N} u_{x_i} z_{x_i} + (2f_u + g^{'} g^{-1} f(t,u)) z \\ &+ \left( \frac{4ug^{'2}}{g^3} - \frac{2ug^{''}}{g^2} + \frac{2\gamma g^{'}}{g^2} \right) z^2 + 2z\Delta u - ug(u) \sum_{i,j=1}^{N} u_{x_i x_j}^2. \end{aligned} \tag{17}$$

By choosing $g(u) = u^{\alpha}$, and since

$$\sum_{i,j=1}^{N} u_{x_i x_j}^2 \geq \frac{1}{N} (\Delta u)^2, \tag{18}$$

replacing $g$ in (17),(18) we have

$$\begin{aligned} z_t \leq& u\Delta z - 2(\alpha + \gamma) \sum_{i=1}^{N} u_{x_i} z_{x_i} + (2f_u + \alpha u^{-1} f(t,u)) z \\ &+ 2\alpha(\alpha + 1 + \gamma) u^{-\alpha - 1} z^2 + 2z\Delta u - \frac{u^{\alpha+1}}{N} (\Delta u)^2. \end{aligned} \tag{19}$$

For $\gamma \geq \sqrt{2N} - 1$, if $\alpha$ satisfies

$$\alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0, \tag{20}$$

where $\alpha^2 + (\gamma + 1)\alpha \leq -\frac{N}{2}$, then,

$$2\alpha(\alpha + \gamma + 1) u^{-\alpha - 1} z^2 + 2z\Delta u - \frac{u^{\alpha+1}}{N} (\Delta u)^2 \leq 0. \tag{21}$$

Therefore from (19) and (21) we have

$$z_t \leq u\Delta z - 2(\alpha+\gamma)\sum_{i=1}^{N} u_{x_i} z_{x_i} + (2f_u + \alpha u^{-1} f(t,u))z. \tag{22}$$

By an application of the maximum principle in (22) we have

$$|z|_\infty \leq |z_0|_\infty.$$

Now, from (8), (11), with $g(u) = u^\alpha$, since the initial data (2) satisifes

$$|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M,$$

with $M$ a positive constant and $\alpha$ satisfies (20), we have

$$\begin{aligned}
|\nabla(u^{1+\frac{\alpha}{2}})|^2 &= \left|\sum_{i=1}^{N}(u^{1+\frac{\alpha}{2}})_{x_i} e_i\right|^2 \\
&= \sum_{i=1}^{N}\left[(u^{1+\frac{\alpha}{2}})_{x_i}\right]^2 \\
&= \sum_{i=1}^{N}\left[\left(1+\frac{\alpha}{2}\right)u^{\frac{\alpha}{2}} u_{x_i}\right]^2 \\
&= \left(1+\frac{\alpha}{2}\right)^2 u^\alpha \sum_{i=1}^{N} u_{x_i}^2 \\
&= 2\left(1+\frac{\alpha}{2}\right)^2 u^\alpha w \\
&= 2\left(1+\frac{\alpha}{2}\right)^2 z,
\end{aligned}$$

therefore

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M.$$

□

## 4. Hölder Continuity of $u(x,t)$

Now, using Theorem 3.1, we have the following corollary about the regularity of the viscosity solution $u(x,t)$ to the Cauchy problem (1),(2).

**Corollary 4.1.** *Let $f$ be a continuous fuctions such that*

$$|f(t,w)| \leq k|w|^m,$$

*where $w$ is a real value function and $m$, $k$ non-negative constants. Under conditions of the Theorem 3.1 the viscosity solution $u(x,t)$ of the Cauchy problem (1), (2) is Lipschitz continuous with respect to $x$ and locally Hölder continuous with exponent $\frac{1}{2}$ with respect to $t$ in $\overline{\Omega}$.*

*Proof.* From Theorem 3.1 there exists $\alpha \in \mathbb{R}$ with $\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \leq 0$, with $\alpha < 0$, or,

$$-\frac{\sqrt{(\gamma+1)^2 - 2N}}{2} - \frac{\gamma+1}{2} \leq \alpha \leq -\frac{\gamma+1}{2} + \frac{\sqrt{(\gamma+1)^2-2N}}{2} < 0.$$

Since $\alpha < 0$, taking $\alpha \neq -2$, we have the estimate,

$$\begin{aligned}
|\nabla(u^{1+\frac{\alpha}{2}})| &= \left|(1+\frac{\alpha}{2})u^{\frac{\alpha}{2}}\nabla u\right| \\
&= \left|1+\frac{\alpha}{2}\right| u^{\frac{\alpha}{2}} |\nabla u| \leq M.
\end{aligned}$$

Now, as $u \geq 0$, we have that

$$|\nabla u| \leq \left|1+\frac{\alpha}{2}\right|^{-1} u^{-\frac{\alpha}{2}} M \leq M_1 \text{ in } \overline{\Omega}, \tag{23}$$

since $u$ is bounded.
Using the value mean theorem we have

$$u(x_1,t)-u(x_2,t) = \nabla u(x_1+\theta(x_2-x_1),t)\cdot(x_1-x_2), \tag{24}$$

for any $\theta \in (0,1)$. From (23), (24) we have,

$$\begin{aligned} |u(x_1,t)-u(x_2,t)| &\leq |\nabla u(x_1+\theta(x_2-x_1),t)||x_1-x_2| \\ &\leq M_1|x_1-x_2|, \qquad \forall (x_1,t),(x_2,t)\in\Omega. \end{aligned}$$

Therefore $u(x,t)$ is a Lipschitz continuous with respect to the spatial variable.

For Hölder continuity of $u(x,t)$ with respect to the temporary variable, we are going to use the ideas developed in [5]. Let $u_\epsilon(x,t) \in C^{2.1}(\Omega)\bigcap C(\overline{\Omega})\bigcap L^\infty(\Omega)$ the classical solution to the Cauchy problem problem (1), (2), namely,

$$\begin{cases} u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u) & \text{in } \Omega \\ u(x,0) = u_0(x)+\epsilon & \text{on } \mathbb{R}^N, \end{cases}$$

We have that

$$\begin{aligned} \left|\nabla(u_0+\epsilon)^{1+\frac{\alpha}{2}}\right| &= \left|\left(1+\frac{\alpha}{2}\right)(u_0+\epsilon)^{\frac{\alpha}{2}}\nabla u_0\right| \\ &\leq \left|1+\frac{\alpha}{2}\right|(u_0)^{\frac{\alpha}{2}}|\nabla u_0| \\ &= \left|\nabla\left(u_0^{1+\frac{\alpha}{2}}\right)\right|. \\ &\leq M, \end{aligned}$$

Then, the conditions of Theorem 3.1 holds. Thereby

$$\left|\nabla(u_0+\epsilon)^{1+\frac{\alpha}{2}}\right| \leq M.$$

Since $u_\epsilon$ is a classical solution, $u$ is also a weak solution of the Cauchy problem (6), (7). Hence, using the same arguments in the proof of Theorem 3.1, we have that $u_\epsilon$ is a Lipschitz continuous with respect to the spatial variable, with constant $M$, namely

$$|u_\epsilon(x_1,t)-u_\epsilon(x_2,t)| \leq M|x_1-x_2| \;\; \forall\; (x_1,t),(x_2,t)\in\Omega. \tag{25}$$

Now, let $z = u_\epsilon$ be, then we have,

$$z_t = u_{\epsilon_t} = u_\epsilon\Delta u_\epsilon - \gamma|\nabla u_\epsilon|^2 + f(t,u_\epsilon)$$

or,

$$u_\epsilon\Delta z - z_t = \gamma|\nabla u_\epsilon|^2 - f(t,u_\epsilon) \text{ in } \Omega. \tag{26}$$

Using (26) we have that for all $T>0$,$R>0$, $z$ satisfies the equation

$$u_\epsilon\Delta z - z_t = \gamma|\nabla u_\epsilon|^2 - f(t,u_\epsilon) \text{ in } B_{2R}(0)\times(0,T], \tag{27}$$

where $B_{2R}(0)$ is the open ball centered in 0, with radius $2R$ in $\mathbb{R}^N$. Noticing that $u_\epsilon \in C^{2.1}\Big(B_{2R}(0)) \times (0,T]\Big)$.

Now, since $u_\epsilon$ and $\nabla u_\epsilon$ are bounded in $\overline{B_{2R}(0)} \times (0,T]$, there exists a constant $\mu > 0$, such that

$$\sum_{i=1}^{N} u_\epsilon(x,t) = N u_\epsilon(x,t) \le \mu,$$

$$\gamma|\nabla u_\epsilon(x,t)| \le \mu, \qquad \forall (x,t) \in B_{2R}(0) \times (0,T],$$

and

$$f(t,u_\epsilon) \le \mu.$$

From (25), we have also

$$|z(x_1,t) - z(x_2,t)| \le M|x_1 - x_2| \qquad \forall (x,t) \in B_{2R}(0)) \times (0,T].$$

In acording with [5], there exists a positive constant $\delta$ (which depends only of $\mu$ and $R$) and a positive constant $K$, which depends only of $\mu$, $R$ and $M$, such that

$$|z(x,t) - z(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$.

That is,

$$|u_\epsilon(x,t) - u_\epsilon(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$.

Whenever $K$ is independent of $\epsilon$, taken $\epsilon \searrow 0$, we obtain

$$|u(x,t) - u(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t),(x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$. □

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