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A well-balanced scheme to capture non-explicit steady states in the Euler equations with gravity

Vivien Desveaux[1]*, Markus Zenk[2], Christophe Berthon[1] and Christian Klingenberg[2]

[1] *Université de Nantes, Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France*
[2] *Universität Würzburg, Campus Hubland Nord, Emil-Fischer-Strasse 30, 97074 Würzburg, Germany*

SUMMARY

This paper describes a numerical discretization of the compressible Euler equations with a gravitational potential. A pertinent feature of the solutions to these inhomogeneous equations is the special case of stationary solutions with zero velocity, described by a nonlinear PDE, whose solutions are called hydrostatic equilibria. We present a well-balanced method, meaning that besides discretizing the complete equations the method is also able to maintain all hydrostatic equilibria. The method is a finite volume method, whose Riemann solver is approximated by a so-called relaxation Riemann solution that takes all hydrostatic equilibria into account. Relaxation ensure robustness, accuracy and stability of our method, since it satisfies discrete entropy inequalities. We will present numerical examples, illustrating that our method works as promised. Copyright © 2010 John Wiley & Sons, Ltd.

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1. INTRODUCTION

This paper concerns the derivation of numerical schemes to approximate the solutions of the Euler equations endowed with a gravity source term. The system under consideration writes as follows:

$$\partial_t \rho + \partial_x \rho u = 0, \tag{1a}$$
$$\partial_t \rho u + \partial_x(\rho u^2 + p) = -\rho \partial_x \Phi, \tag{1b}$$
$$\partial_t E + \partial_x (E+p)u = -\rho u \partial_x \Phi, \tag{1c}$$

where $\rho(x,t) > 0$ denotes the density, $u(x,t) \in \mathbb{R}$ the velocity, $E(x,t) > 0$ the total energy that can be written as

$$E = \rho e + \frac{1}{2}\rho u^2,$$

where $e > 0$ is the internal energy, and the function $\Phi : \mathbb{R} \to \mathbb{R}$ stands for a given smooth gravitational potential. Concerning the pressure, it is given by a general pressure law $p(\tau, e) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ where we have introduce $\tau = 1/\rho$ the specific volume.

We assume the pressure function obeys the second law of thermodynamics. As a consequence, there exists a specific entropy $\eta(\tau, e) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies, for some temperature

*Correspondence to: Université de Nantes, Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France. E-mail: vivien.desveaux@inria.fr

$T(\tau,e)>0$, the following relation:

$$-Td\eta = de + pd\tau. \tag{2}$$

It results the following equalities:

$$\partial_\tau \eta(\tau,e) = -\frac{p(\tau,e)}{T(\tau,e)} < 0 \quad \text{and} \quad \partial_e \eta(\tau,e) = -\frac{1}{T(\tau,e)} < 0. \tag{3}$$

In this work, we impose the application $(\tau,e)\mapsto \eta(\tau,e)$ to be strictly convex.

To shorten the notations, the system (1) can be rewritten in the following condensed form:

$$\partial_t w + \partial_x f(w) = s(w), \tag{4}$$

where we have set

$$w = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}, \quad s(w) = \begin{pmatrix} 0 \\ -\rho\partial_x\Phi \\ -\rho u \partial_x \Phi \end{pmatrix}. \tag{5}$$

The system (4) is associated with the following phase space:

$$\Omega = \left\{ w\in\mathbb{R}^3; \quad \rho>0, \quad e>0 \right\}. \tag{6}$$

In addition, we impose a positive acoustic impedance as follows:

$$\tau(p\partial_e p - \partial_\tau p) > 0,$$

in order to enforce the system (1) to be hyperbolic. As a consequence, the solutions may become discontinuous within finite time. In order to avoid some unphysical solutions, the system is equipped with additional conservation laws satisfied by the smooth solutions. These additional conservation laws, stated in the following result, yield to the expected entropy inequalities.

Lemma 1
The smooth solutions of (1) satisfy the additional conservation laws

$$\partial_t \rho\mathcal{F}(\eta) + \partial_x \rho\mathcal{F}(\eta)u = 0, \tag{7}$$

for all smooth function $\mathcal{F}$.
Moreover, assume

$$\mathcal{F}'(\eta) > 0 \quad \text{and} \quad \frac{1}{c_p}\mathcal{F}'(\eta) + \mathcal{F}''(\eta) > 0, \tag{8}$$

where c_p is the specific heat at constant pressure, defined by

$$c_p = -T\left(\frac{\partial\eta}{\partial T}\right)_p,$$

then the application $w\mapsto \rho\mathcal{F}(\eta)$ is strictly convex. As a consequence, the pair $(\rho\mathcal{F}(\eta), \rho\mathcal{F}(\eta)u)$ defines a Lax entropy pair for system (1). Hence, the weak solutions of (1) satisfy in addition:

$$\partial_t \rho\mathcal{F}(\eta) + \partial_x \rho\mathcal{F}(\eta)u \le 0. \tag{9}$$

Proof
First, let us consider smooth solutions of (1). From the continuity equation (1a), we get

$$\partial_t \tau + u\partial_x u - u\partial_x\tau = 0, \tag{10}$$

and from both momentum equation (1b) and energy equation (1c), we get

$$\partial_t e + u\partial_x e + p\tau\partial_x u = 0. \tag{11}$$

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Next, multiplying (10) by $-\frac{p}{T}$ and (11) by $-\frac{1}{T}$ and using the relations (3), we obtain

$$\partial_\tau \eta \partial_t \tau + u \partial_\tau \eta \partial_x \tau - \tau \partial_\tau \eta \partial_x u = 0, \tag{12}$$

$$\partial_e \eta \partial_t e + u \partial_e \eta \partial_x e + \tau \partial_\tau \eta \partial_x u = 0. \tag{13}$$

The sum of (12) and (13) easily gives

$$\partial_t \eta + u \partial_x \eta = 0.$$

The result is then achieved by multiplying this relation by $\rho \mathcal{F}'(\eta)$ and combining with the continuity equation (1a).

The establishment of the Lax entropy pair comes from a straightforward study of the Hessian matrix associated to the function $w \mapsto \rho \mathcal{F}(\eta)$ (for instance, see [1, 2, 3, 4] and references therein). □

Let us emphasize that the additional conservation laws (7) are satisfied by the smooth solutions of the system (1) with source term. Indeed, the gravity source term does not participate to the entropy laws associated with (1).

Nevertheless, the source term drastically modifies the steady states. In the present work, we focus our attention on the steady states at rest, *i.e.* with $u \equiv 0$. These steady solutions of particular interest are characterized as follows:

$$\begin{cases} u = 0, \\ \partial_x p = -\rho \partial_x \Phi. \end{cases} \tag{14}$$

This PDE system can be easily solved as soon as the pressure law only depends on the density. For instance the reader is referred to works devoted to the well-known shallow-water system [5, 6, 7, 8] and some related model [9, 10, 11, 12] (see also [13] for isentropic steady states associated with (1)). Unfortunately, since the pressure function p depends on both τ and e, we cannot exhibit algebraic relations satisfied by the solutions of (14), except under restrictive assumptions.

However, among the whole set of solutions of (14), a particular family is of prime importance, specially for astrophysics applications. It is known as the polytropic equilibrium, defined by

$$u(x) = 0, \qquad p(x) = K \rho(x)^\Gamma, \tag{15}$$

for $K > 0$ and $\Gamma \in (0, +\infty]$. Let us underline that here, Γ represents the polytropic coefficient, which is in general different from the adiabatic coefficient γ coming from an ideal gas law. Equation (14), together with the additional condition (15), can be solved explicitly, though the formulation differs according to the value of Γ:

- For $\Gamma = 1$, we obtain the isothermal equilibrium, which is defined by

$$\begin{cases} u(x) = 0, \\ \rho(x) = \exp\left(\dfrac{C - \Phi(x)}{K}\right), \\ p(x) = K \exp\left(\dfrac{C - \Phi(x)}{K}\right), \end{cases} \tag{16}$$

  for a given constant $C \in \mathbb{R}$.
- Finally, for $\Gamma \in (0,1) \cup (1, +\infty)$, we get the following expression:

$$\begin{cases} u(x) = 0, \\ \rho(x) = \left(\dfrac{\Gamma - 1}{\Gamma K}(C - \Phi(x))\right)^{\frac{1}{\Gamma - 1}}, \\ p(x) = K^{\frac{1}{1-\Gamma}} \left(\dfrac{\Gamma - 1}{\Gamma}(C - \Phi(x))\right)^{\frac{\Gamma}{\Gamma - 1}}, \end{cases} \tag{17}$$

  for a given constant $C \in \mathbb{R}$.

- The case of an incompressible equilibrium with a constant density:

$$\begin{cases} u(x) = 0, \\ \rho(x) = \text{constant}, \\ p(x) + \rho(x)\Phi(x) = \text{constant}. \end{cases} \tag{18}$$

Some authors also consider the specific case of the isentropic equilibrium. We underline that in the case of an ideal gas law, with adiabatic coefficient γ, the isentropic equilibrium coincides with the polytropic equilibrium (15), with a polytropic coefficient $\Gamma = \gamma$.

The objective of this paper is to derive a finite volume method to approximate the weak solutions of (1), which is able to capture exactly the polytropic equilibria (16), (17) and (18). In addition, we also require that the numerical scheme approximate accurately all the solutions of (14). Moreover, the derived scheme must be entropy preserving.

During the last two decades, numerous techniques were proposed in the literature to derive well-balanced schemes. Most of them concerned the shallow-water equations or related models. For a non-exhaustive list, the reader is referred, for instance, to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein.

However, adopting the Euler equations with gravity, the derivation of well-balanced schemes turns out to be more delicate since the steady states are not explicitly known and just given by the PDE system (14). Nevertheless, we can mention the pioneering work by Cargo and LeRoux [20]. By introducing cleverly an evolution equation for the hydrostatic potential, they are able to reformulate the system in a much simpler way. From this equivalent reformulation, they derive a simple and relevant well-balanced scheme. Unfortunately, this reformulation technique only works for a constant gravity field, namely $\Phi(x) = gx$ with $g > 0$ a given constant. Moreover, it is quite difficult to extend this scheme to two or three dimensions.

The Cargo-LeRoux's technique was recently revisited in the work by Chalons *et al* [16]. These authors proposed a suitable well-balanced relaxation scheme and they establish that the resulting numerical method is entropy preserving.

Another technique, based on a local hydrostatic reconstruction, was also developed by Käppeli and Mishra [13]. Although this method extends very easily to second-order and to two space dimensions. It only preserves the *isentropic* steady states. In our work, we would like to preserve exactly a much wider family of steady states.

Following the ideas introduced in the companion paper [21] devoted to the Ripa model, we propose to design a Godunov-type scheme. To address such an issue, the key point stays in the derivation of an approximate Riemann solver which contains the source term in order to preserve the steady states defined by (14) (see [13, 14, 22]).

To obtain such a required approximate Riemann solver consistent with the gravity source term, we consider a Suliciu-type relaxation strategy. The resulting numerical scheme turns out to be positive preserving, entropy satisfying and well-balanced with the exact capture of the polytropic, isothermal and incompressible equilibria.

The paper is organized as follows. The next section is devoted to the derivation of the relaxation model (see [23, 24, 16, 25, 26, 27, 6, 28, 29, 30, 31, 32]) in order to get the required approximate Riemann solver. Here we extend the Suliciu type model introduced in [21] where a *transport property* is enforced to be satisfied by the source term. Such a model is governed by a system, which is *more linear* than the usual Suliciu model [24, 26, 6]. The main advantage of this new relaxation model stays for an easy algebra. However, the Riemann problem for this relaxation system is under-determined and a closure relation is missing. Actually, this failure turns out to be beneficial, since the missing relation can be relevantly defined to enforce the well-balanced property in a sense to be prescribed. As a consequence, we obtain an approximate Riemann solver which contains the source term. The end of the section is devoted to prove the properties of robustness and well-balancedness satisfied by the approximate solver.

Section 3 is devoted to prove technical results to establish that the derived approximate Riemann solver is consistent with the entropy inequalities (7) in the sense of Harten, Lax and van Leer [33].

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In Section 4, we present the Godunov-type scheme associated with the derived approximate solver. In addition, we prove the main properties satisfied by the obtained scheme. As expected, we establish that the scheme preserves the set of admissible states. Moreover, we show that the scheme is entropy preserving. Concerning the well-balanced property, we prove that the scheme is able to capture *exactly* the polytropic (17), isothermal (16) and incompressible (18) equilibria, as well as a suitable approximation of the solutions of (14)

Finally, Section 5 is devoted to illustrate the good behavior of the scheme and several numerical experiments are performed.

2. THE RELAXATION MODEL

In order to design a Godunov-type scheme, the present section is devoted to derive an approximate Riemann solver. To address such an issue, we here adopt a Suliciu-type relaxation strategy [26, 6]. More precisely, we approximate the weak solutions of the initial system (1) by the weak solutions of a first-order system endowed with a relaxation source term, namely the relaxation model. In a limit of a relaxation parameter, the relaxation model must restore, in a sense to be prescribed, the initial system.

In the previous work [21], a relaxation model was developed in the framework of the shallow-water model with horizontal temperature gradients, also known as the Ripa model. Its particularity is that the associated Riemann problem is under-determined and there is one relation missing. Actually, this turns out to be beneficial, since the missing relation can be relevantly defined to enforce the well-balanced property.

Here, we follow the same strategy for the Euler equation with gravity and the obtained relaxation model as well as the Riemann solution turn out to be in close relationship with the models derived in [21]. However, the similarities stop as soon as one look at the properties satisfied by the approximate Riemann solver. Indeed the required properties of robustness, stability and well-balancedness are quite different between the Ripa model and the Euler equations with gravity. As a consequence, we will skip the details in the presentation of the relaxation model and the computation of the Riemann solution, referring the reader to [21], whereas the properties satisfied by the approximate Riemann solver will be carefully examined.

According to the Suliciu relaxation approach (see [26, 6, 34, 35]), we suggest to approximate the pressure p by a new variable π governed by the following evolution law:

$$\partial_t \pi + u \partial_x \pi + \frac{a^2}{\rho} \partial_x u = \frac{1}{\varepsilon} \left(p(\tau, e) - \pi \right).$$

The relaxation parameter $a > 0$ will be fixed later in order to satisfy some robustness and stability conditions.

Now we decide to approximate the gravity by a new variable Z governed by a transport relaxation equation as follows:

$$\partial_t Z + u \partial_x Z = \frac{1}{\varepsilon} \left(\Phi - Z \right).$$

This leads to the following relaxation model:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \pi) = -\rho \partial_x Z, \\ \partial_t E + \partial_x (E + \pi) u = -\rho u \partial_x Z, \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \dfrac{\rho}{\varepsilon} (p(\tau, e) - \pi), \\ \partial_t \rho Z + \partial_x \rho Z u = \dfrac{\rho}{\varepsilon} (\Phi - Z). \end{cases} \tag{19}$$

Let us notice that when the parameter ε goes to zero, formally the new unknowns π and Z respectively converge to p and Φ. As a consequence, we recover the system (1) from the first three equations of the above system (19).

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In order to simplify the notations, we introduce

$$W = (\rho, \rho u, E, \rho\pi, \rho Z)^T$$

to designate the state vectors in the phase space

$$\mathcal{O} = \left\{W \in \mathbb{R}^5, \quad \rho > 0, \quad e > 0\right\}.$$

From each state $w \in \Omega$, defined by (5) and a given gravity function Φ, we define an equilibrium state for the relaxation model as follows:

$$W^{eq}(w) = (\rho, \rho u, E, \rho p(\tau, e), \rho\Phi)^T. \tag{20}$$

The following statement gives the algebra of the homogeneous system extracted from (19), which we denote with unambiguous notations by $(19)_{\varepsilon=\infty}$.

Lemma 2
Let $a > 0$ be given. The homogeneous system extracted from (19) is hyperbolic for all $W \in \mathcal{O}$. The eigenvalues of the system are $\lambda^\pm = u \pm \frac{a}{\rho}$ and $\lambda^u = u$, the last one being of multiplicity three. All the fields are linearly degenerate. The characteristic fields associated with the eigenvalues $\lambda^\pm$ admit the following Riemann invariants:

$$I_1^\pm = u \pm \frac{a}{\rho}, \qquad I_2^\pm = \pi \mp au, \qquad I_3^\pm = e - \frac{\pi^2}{2a^2}, \qquad I_4^\pm = Z, \tag{21}$$

while the characteristic field associated with the eigenvalue λ^u admits a unique (independent) Riemann invariant:

$$I_1^u = u. \tag{22}$$

We skip the proof of this result and the reader is referred to [21] (see also [26, 16, 1]).

We observe there is a missing Riemann invariant for the field associated with the eigenvalue λ^u, since one could expect two independent Riemann invariants. This has a direct implication when we want to compute the solution of the Riemann problem associated to $(19)_{\varepsilon=\infty}$. Indeed, let us consider an initial data made of two constant states separated by a discontinuity located at $x = 0$:

$$W_0(x) = \begin{cases} W_L & \text{if } x < 0, \\ W_R & \text{if } x > 0. \end{cases} \tag{23}$$

According to Lemma 2, if a solution exists, it is made of four constant states separated by three contact discontinuities as follows:

$$W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) = \begin{cases} W_L & \text{if } x/t < \lambda^-, \\ W_L^\star & \text{if } \lambda^- < x/t < \lambda^u, \\ W_R^\star & \text{if } \lambda^u < x/t < \lambda^+, \\ W_R & \text{if } \lambda^+ < x/t. \end{cases} \tag{24}$$

There are ten unknowns in this Riemann problem, five for both $W_L^\star$ and $W_R^\star$. However, the continuity of the Riemann invariants (21) and (22) only provides nine relations. Thus there is a missing relation in order to determine a unique Riemann solution. We are going to take advantage of this failure to enforce the well-balanced property.

Using the relaxation variables, equation (14), which defines the steady states at rest, writes

$$\partial_x \pi = -\rho \partial_x Z.$$

We chose as the closure equation, the following discretization of this relation:

$$\pi_R^\star - \pi_L^\star = -\overline{\rho}(W_L, W_R)(Z_R^\star - Z_L^\star), \tag{25}$$

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where the function $\overline{\rho} : \mathcal{O} \times \mathcal{O} \to \mathbb{R}^+$ denotes a ρ-average function which will be defined later. For now, we assume that this function satisfies the following consistency property:

$$\rho_L = \rho_R = \rho \quad \Rightarrow \quad \overline{\rho}(W_L, W_R) = \rho. \tag{26}$$

Moreover, it is assumed to be symmetric:

$$\overline{\rho}(W_R, W_L) = \overline{\rho}(W_L, W_R).$$

Equipped with the additional law (25), we solve the Riemann problem associated with (19)$_{\varepsilon=\infty}$ – (23).

Lemma 3
The Riemann problem of the system (19)$_{\varepsilon=\infty}$ –(23) completed by the relation (25) admits a unique solution, which is given by (24) where the intermediate states $W_L^\star$ and $W_R^\star$ are defined by

$$Z_L^\star = Z_L, \qquad Z_R^\star = Z_R, \tag{27a}$$

$$\begin{aligned} u^\star &= u_L^\star = u_R^\star \\ &= \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(\pi_R - \pi_L) - \frac{1}{2a}\overline{\rho}(W_L, W_R)(Z_R - Z_L), \end{aligned} \tag{27b}$$

$$\pi_L^\star = \pi_L + a(u_L - u^\star), \qquad \pi_R^\star = \pi_R + a(u^\star - u_R), \tag{27c}$$

$$\frac{1}{\rho_L^\star} = \frac{1}{\rho_L} + \frac{1}{a}(u^\star - u_L), \qquad \frac{1}{\rho_R^\star} = \frac{1}{\rho_R} + \frac{1}{a}(u_R - u^\star), \tag{27d}$$

$$e_L^\star = e_L + \frac{1}{2a^2}\left({\pi_L^\star}^2 - {\pi_L}^2\right), \qquad e_R^\star = e_R + \frac{1}{2a^2}\left({\pi_R^\star}^2 - {\pi_R}^2\right). \tag{27e}$$

Once again, we skip the proof of this standard result (see [26, 16, 21, 1]).

Using this Riemann solution for the relaxation system (19), we can define an approximate Riemann solver for the initial system (1) as follows:

$$w^{eq}\left(\frac{x}{t}; w_L, w_R\right) = W_{\mathcal{R}}^{(\rho,\rho u,E)}\left(\frac{x}{t}; W^{eq}(w_L), W^{eq}(w_R)\right), \tag{28}$$

where the exponent $(\rho, \rho u, E)$ denotes the projection on the first three components and $W^{eq}(w)$ is defined by (20).

In [21], in the framework of the Ripa model, we were able to reformulate the closure relation as relaxation equations. In other terms, we were able to exhibit a full relaxation model which admits a unique solution and which leads to the same approximate Riemann solver. Moreover, a third interpretation of the approximate Riemann solver was given by a different relaxation approach based on the work of Cargo and LeRoux [20]. We underline that such reformulations are also possible for the Euler equations with gravity. However, once again, the details are very similar and the reader is referred to [21].

We now study the approximate Riemann solver w^{eq}. First, we establish that w^{eq} preserves the set Ω.

Lemma 4
Let be given w_L and w_R in Ω, defined by (6). If the relaxation parameter a is large enough to ensure the following inequalities:

$$u_L - \frac{a}{\rho_L} < u^\star < u_R + \frac{a}{\rho_R}, \tag{29}$$

$$e_L + \frac{{\pi_L^\star}^2 - {\pi_L}^2}{2a^2} > 0, \qquad e_R + \frac{{\pi_R^\star}^2 - {\pi_R}^2}{2a^2} > 0, \tag{30}$$

where $u^\star$ and $\pi_{L,R}^\star$ are respectively defined by (27b) and (27c), then $w^{eq}(x/t; w_L, w_R)$ belongs to Ω.

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Proof
First, the continuity of the Riemann invariants $I_1^{\pm}$ writes

$$u_L - \frac{a}{\rho_L} = u^\star - \frac{a}{\rho_L^\star} \qquad \text{and} \qquad u_R + \frac{a}{\rho_R} = u^\star + \frac{a}{\rho_R^\star},$$

so that (29) leads to $-\rho_L^\star < 0 < \rho_R^\star$. The positiveness of the density is thus established. In the same way, thanks to the continuity of the Riemann invariants $I_3^{\pm}$, we have

$$e_L - \frac{\pi_L^2}{2a^2} = e_L^\star - \frac{{\pi_L^\star}^2}{2a^2} \qquad \text{and} \qquad e_R - \frac{\pi_R^2}{2a^2} = e_R^\star - \frac{{\pi_R^\star}^2}{2a^2},$$

so that (30) implies $e_L^\star > 0$ and $e_R^\star > 0$. Hence the intermediate internal energies are positive and the proof is completed. □

The next Lemma concerns the well-balanced property of the approximate Riemann solver w^{eq}.

Lemma 5
Let w_L and w_R be given in Ω such that

$$u_L = u_R = 0, \tag{31}$$
$$p_R - p_L + \overline{\rho}(W_L, W_R)(\Phi_R - \Phi_L) = 0. \tag{32}$$

Then we get w^{eq} at rest:

$$w^{eq}(x/t; w_L, w_R) = \begin{cases} w_L & \text{if } x/t < 0, \\ w_R & \text{if } x/t > 0. \end{cases} \tag{33}$$

Proof
According to (27), and since $u_L = u_R = 0$, it suffices to prove that $u^\star = 0$ to ensures (33) is satisfied. The speed $u^\star$ writes

$$u^\star = -\frac{p_R - p_L}{2a} - \frac{1}{2a}\overline{\rho}(\rho_L, \rho_R)(\Phi_R - \Phi_L),$$

which vanishes thanks to (31)-(32). □

In general, starting from a continuous steady state, its piecewise constant projection does not necessarily satisfy (32). As a consequence, Lemma 5 states that a well-chosen approximation of any continuous steady state is preserved, so it is only an approximate result. Now, we will see, that w^{eq} is able to preserve exactly the polytropic steady states (16), (17) and (18).

Lemma 6
The approximate Riemann solver w^{eq} satisfies the following properties:

(i) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_{L,R} = \exp\left(\dfrac{C - \Phi_{L,R}}{K}\right), \\ p_{L,R} = K \exp\left(\dfrac{C - \Phi_{L,R}}{K}\right), \end{cases} \tag{34}$$

with $K > 0$ and $C \in \mathbb{R}$. Assume the ρ–average $\overline{\rho}$ is defined by

$$\overline{\rho}(W_L, W_R) = \begin{cases} \dfrac{\rho_R - \rho_L}{\ln(\rho_R) - \ln(\rho_L)} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases} \tag{35}$$

Then we get w^{eq} at rest given by (33).

(ii) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_{L,R} = \left(\dfrac{\Gamma-1}{\Gamma K}(C-\Phi_{L,R})\right)^{\frac{1}{\Gamma-1}}, \\ p_{L,R} = K^{\frac{1}{1-\Gamma}}\left(\dfrac{\Gamma-1}{\Gamma}(C-\Phi_{L,R})\right)^{\frac{\Gamma}{\Gamma-1}}, \end{cases} \tag{36}$$

with $\Gamma \in (0,1)\cup(1,+\infty)$, $K>0$ and $C\in\mathbb{R}$. Assume the ρ–average $\overline{\rho}$ is defined by

$$\overline{\rho}(W_L, W_R) = \begin{cases} \dfrac{\Gamma-1}{\Gamma}\dfrac{\rho_R^{\Gamma}-\rho_L^{\Gamma}}{\rho_R^{\Gamma-1}-\rho_L^{\Gamma-1}} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases} \tag{37}$$

Then we get w^{eq} at rest given by (33).

(iii) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_L = \rho_R, \\ p_L + \rho_L\Phi_L = p_R + \rho_R\Phi_R. \end{cases} \tag{38}$$

Then we get w^{eq} at rest given by (33) independently from the definition of $\overline{\rho}$.

Proof
As in Lemma 5, we just need to prove that $u^\star = 0$. Since in all cases we have $u_L = u_R = 0$, the intermediate speed $u^\star$ writes

$$u^\star = -\frac{1}{2a}\left(p_R - p_L + \overline{\rho}(W_L, W_R)(\Phi_R - \Phi_L)\right).$$

In case (i), the hypothesis (34) implies

$$\Phi_R - \Phi_L = K(\ln(\rho_R) - \ln(\rho_L)).$$

In addition, we have

$$p_R - p_L = K(\rho_R - \rho_L).$$

Combining the last two relations together with the definition (35) of $\overline{\rho}(W_L, W_R)$, we get $u^\star = 0$.
In case (ii), the hypothesis (36) leads to

$$\Phi_R - \Phi_L = K\frac{\Gamma}{\Gamma-1}\left(\rho_R^{\Gamma-1} - \rho_L^{\Gamma-1}\right)$$

and

$$p_R - p_L = K\left(\rho_R^{\Gamma} - \rho_L^{\Gamma}\right).$$

We conclude from the two last equations and from the definition (37) that $u^\star = 0$. The proof is thus achieved
Finally, case (iii) is immediate thanks to the consistency property (26) satisfied by $\overline{\rho}$. □

The last property we study concerns the entropy consistency of w^{eq}. However, it is much more complex to establish than the other properties and it is the object of next section.

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3. CONSISTENCY WITH THE ENTROPY INEQUALITIES

This section is dedicated to prove that the approximate Riemann solver w^{eq}, given by (28), is consistent with the entropy inequalities (7) in the sense of Harten, Lax and van Leer [33]. Indeed, such an entropy consistency turns out to be the main ingredient establishing the required discrete entropy inequalities satisfied by the numerical scheme associated to the approximate Riemann solver w^{eq}.

We first state the main result of this section, where w^{eq}, given by (28), is shown to be entropy consistent in the sense of Harten, Lax and van Leer. To address such an issue, an additional property must be imposed to be satisfied by the pressure function.

First, we introduce a set defined by

$$\mathcal{A} = \Big\{(I,J)\in\mathbb{R}^2;\quad \exists\tau>0,\quad \exists e>0 \text{ such that:}$$

$$I = p(\tau,e) + a^2\tau, \tag{39}$$

$$J = e - \frac{p(\tau,e)^2}{2a^2}, \tag{40}$$

$$a^2 > p(\tau,e)\,\partial_e p(\tau,e) - \partial_\tau p(\tau,e)\Big\}. \tag{41}$$

We underline that the inequality (41), known as the sub-characteristic Whitham condition [36], imposes that the sound speed $a\tau$ of the system $(19)_{\varepsilon=\infty}$ has to be greater than the sound speed $c = \tau\sqrt{p\partial_e p - \partial_\tau p}$ of the original model (1).

Next, for all pair (I,J) in $\mathcal{A}$, we introduce the function $f_{I,J} : \mathbb{R}^+ \to \mathbb{R}$ defined as follows:

$$f_{I,J}(\tau) = \tau p\left(\tau, J + \frac{(I-a^2\tau)^2}{2a^2}\right) + a^2\tau^2 - I\tau. \tag{42}$$

We impose the following additional assumption to be satisfied by the pressure law:

Assumption 7
We assume the pressure law is such that the function $\tau \mapsto f_{I,J}(\tau)$, defined by (42), is strictly convex for all pair (I,J) fixed in $\mathcal{A}$.

Nevertheless, such a pressure restriction is satisfied for most usual pressure laws. For instance, in the case of a perfect gas law, given by

$$p(\tau,e) = (\gamma-1)\frac{e}{\tau},$$

the function $f_{I,J}$ writes

$$f_{I,J}(\tau) = \frac{\gamma+1}{2}a^2\tau^2 - \gamma I\tau + (\gamma-1)J + \frac{\gamma-1}{2a^2}I^2.$$

We immediately see that $f_{I,J}$ is a second-order polynomial with a positive highest degree coefficient, and thus it is strictly convex.

Equipped with Assumption 7, we now give our main result.

Theorem 8
Let w_L and w_R be two states of Ω. We consider a smooth function $\mathcal{F}$ such that the hypotheses (8) are satisfied. Let $a>0$ be a parameter such that the following sub-characteristic Whitham conditions hold:

$$a^2 > p(\tau_L,e_L)\,\partial_e p(\tau_L,e_L) - \partial_\tau p(\tau_L,e_L), \tag{43a}$$

$$a^2 > p(\tau_L^*,e_L^*)\,\partial_e p(\tau_L^*,e_L^*) - \partial_\tau p(\tau_L*,e_L^*), \tag{43b}$$

$$a^2 > p(\tau_R^*,e_R^*)\,\partial_e p(\tau_R^*,e_R^*) - \partial_\tau p(\tau_R^*,e_R^*), \tag{43c}$$

$$a^2 > p(\tau_R,e_R)\,\partial_e p(\tau_R,e_R) - \partial_\tau p(\tau_R,e_R). \tag{43d}$$

Fix $\Delta t > 0$ and $\Delta x > 0$ two constants such that the following (CFL) restriction is satisfied:

$$\frac{\Delta t}{\Delta x} \max \left\{ \left| u_L - \frac{a}{\rho_L} \right|, \left| u_R + \frac{a}{\rho_R} \right| \right\} \leq \frac{1}{2}. \tag{44}$$

Moreover, we assume that the pressure law satisfies Assumption 7.

Then the approximate Riemann solver w^{eq}, defined by (28), satisfies the inequalities

$$\frac{1}{\Delta x} \int_0^{\Delta x/2} (\rho \mathcal{F}(\eta)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{\rho_R \mathcal{F}(\eta_R)}{2} - \frac{\Delta t}{\Delta x} \left(\rho_R \mathcal{F}(\eta_R) u_R - \{\rho \mathcal{F}(\eta) u\}_{L,R} \right), \tag{45}$$

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^0 (\rho \mathcal{F}(\eta)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{\rho_L \mathcal{F}(\eta_L)}{2} - \frac{\Delta t}{\Delta x} \left(\{\rho \mathcal{F}(\eta) u\}_{L,R} - \rho_L \mathcal{F}(\eta_L) u_L \right), \tag{46}$$

where we have set

$$\{\rho \mathcal{F}(\eta) u\}_{L,R} = \begin{cases} \rho_L \mathcal{F}(\eta_L) u_L & \text{if } 0 < u_L - \frac{a}{\rho_L}, \\ \rho_L^\star \mathcal{F}(\eta_L) u^\star & \text{if } u_L - \frac{a}{\rho_L} < 0 < u^\star, \\ \rho_R^\star \mathcal{F}(\eta_R) u^\star & \text{if } u^\star < 0 < u_R + \frac{a}{\rho_R}, \\ \rho_R \mathcal{F}(\eta_R) u_R & \text{if } u_R + \frac{a}{\rho_R} < 0. \end{cases} \tag{47}$$

Let us notice that the usual Harten, Lax and van Leer entropy consistency [33] reads:

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} (\rho \mathcal{F}(\eta)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{1}{2} \left(\rho_L \mathcal{F}(\eta_L) + \rho_R \mathcal{F}(\eta_R) \right) - \frac{\Delta t}{\Delta x} \left(\rho_R \mathcal{F}(\eta_R) u_R - \rho_L \mathcal{F}(\eta_L) u_L \right).$$

Of course this last inequality directly comes from the sum of (45) and (46). In fact, the formulation (45) and (46) will be more convenient to derive the expected discrete entropy inequalities.

In order to establish Theorem 8, we now give four successive technical lemmas devoted to establish and analyze a suitable relaxation entropy function (see [26, 37] to similar arguments).

First, we set

$$I(W) := I(\pi, \tau) = \pi + a^2 \tau, \tag{48}$$

$$J(W) := J(\pi, e) = e - \frac{\pi^2}{2a^2}. \tag{49}$$

In fact, these quantities are strong Riemann invariants of the first-order homogeneous system $(19)_{\varepsilon=\infty}$ in the following sense:

Lemma 9

The weak solutions of the relaxation model $(19)_{\varepsilon=\infty}$ satisfy

$$\partial_t \rho \Psi(I, J) + \partial_x \rho \Psi(I, J) u = 0, \tag{50}$$

for all smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$.

Proof

First, let us consider a smooth solution W of the system $(19)_{\varepsilon=\infty}$. An easy computation gives the

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following evolution law satisfied by the internal energy $e = E/\rho - u^2/2$:

$$\partial_t e + \pi\tau\partial_x u + u\partial_x e = 0.$$

From the density equation, we deduce the evolution equation for the specific volume given by

$$\partial_t \tau + u\partial_x \tau - \tau\partial_x u = 0.$$

Finally, from the evolution equation of π, we get

$$\partial_t \frac{\pi^2}{2} + a^2\pi\tau\partial_x u + u\partial_x \frac{\pi^2}{2} = 0.$$

From the three above identities, we immediately obtain

$$\partial_t I + u\partial_x I = 0 \qquad \text{and} \qquad \partial_t J + u\partial_x J = 0.$$

As a consequence, for all smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$, the following transport equation is satisfied:

$$\partial_t \Psi(I,J) + u\partial_x \Psi(I,J) = 0,$$

to directly deduce the conservation equation (50).

To conclude the proof, we just have to mention that the system $(19)_{\varepsilon=\infty}$ is only made of linearly degenerate fields. Then the relation (50) is also satisfied by the weak solutions of $(19)_{\varepsilon=\infty}$. □

The following result connects Assumption 7 with the uniqueness of the couple (τ, e) which appears in the definition of $\mathcal{A}$.

Lemma 10
For all $(I,J) \in \mathcal{A}$, there exists a unique couple $(\tau, e) \in (0, +\infty)^2$ satisfying (39), (40) and (41). This couple will be denoted by $(\overline{\tau}(I,J), \overline{e}(I,J))$.
Moreover, for all $\tau > 0$ and $e > 0$, which satisfy the Whitham condition (41), the couple $(I(p(\tau,e),\tau), J(p(\tau,e),e))$ belongs to $\mathcal{A}$ and we have

$$\overline{\tau}(I(p(\tau,e),\tau), J(p(\tau,e),e)) = \tau, \tag{51}$$

$$\overline{e}(I(p(\tau,e),\tau), J(p(\tau,e),e)) = e. \tag{52}$$

Proof
Let (I,J) be a couple in $\mathcal{A}$. The existence of a couple $(\tau, e) \in (0, +\infty)^2$ satisfying (39), (40) and (41) comes directly from the definition of $\mathcal{A}$. To prove the uniqueness, let (τ, e) be such a couple. Combining relations (39) and (40), we get

$$p\left(\tau, J + \frac{(I - a^2\tau)^2}{2a^2}\right) + a^2\tau - I = 0.$$

In other terms, it means that τ is a root of function $f_{I,J}$. Moreover, the derivative of $f_{I,J}$ at point τ writes

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f_{I,J}(\tau) = \tau(a^2 + \partial_\tau p(\tau,e) - p(\tau,e)\partial_e p(\tau,e)),$$

which is positive according to (41). We have thus proven that τ is a root of $f_{I,J}$ where the derivative is positive. Since $f_{I,J}$ is strictly convex from Assumption 7, τ is unique. The quantity e is then uniquely defined by

$$e = J + \frac{(I - a^2\tau)^2}{2a^2}.$$

Finally, for $\tau > 0$ and $e > 0$ satisfying (41), the definitions (48) and (49) of I and J imply that the pair $(I(p(\tau,e),\tau), J(p(\tau,e),e))$ belongs to $\mathcal{A}$ and (51) and (52) come directly from the uniqueness of $(\overline{\tau}, \overline{e})$. The proof is thus achieved. □

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Now, we introduce the set

$$\mathcal{E} = \left\{ W \in \mathcal{O}; \quad (I(W), J(W)) \in \mathcal{A}, \quad a^2 > p(\tau, e)\, \partial_e p(\tau, e) - \partial_\tau p(\tau, e) \right\}. \tag{53}$$

Let us notice that, according to Lemma 10, both quantities $\overline{\tau}(I(W), J(W))$ and $\overline{e}(I(W), J(W))$ are well-defined as soon as W belongs to the set $\mathcal{E}$. As a consequence, for all state $W \in \mathcal{E}$, we can define

$$\overline{\eta}(W) = \eta(\overline{\tau}(I(W), J(W)), \overline{e}(I(W), J(W))), \tag{54}$$

where η is the specific entropy defined according to (2). From Lemma 10, we can immediately remark that the functions η and $\overline{\eta}$ coincide as soon as relaxation equilibrium states are assumed; namely $I := I(p(\tau, e), \tau)$ and $J := J(p(\tau, e), e)$. Indeed, under the sub-characteristic Whitham condition (41) imposed by definition of $\mathcal{E}$, from (51) and (52) we get

$$\overline{\tau}\left(W_{|\pi = p(\tau,e)}\right) = \tau \qquad \text{and} \qquad \overline{e}\left(W_{|\pi = p(\tau,e)}\right) = e.$$

As a consequence, we have

$$\overline{\eta}\left(W_{|\pi = p(\tau,e)}\right) = \eta(\tau, e).$$

For the sake of simplicity in the forthcoming developments, we shorten the notations as follows:

$$\overline{\tau} = \overline{\tau}(I(W), J(W)), \quad \overline{e} = \overline{e}(I(W), J(W)), \quad \overline{p} = p(\overline{\tau}, \overline{e}).$$

We now prove that the function $\overline{\eta}$ reaches its minimum when the relaxation equilibrium holds.

Lemma 11
For all $W \in \mathcal{E}$, we have

$$\overline{\eta}(W) \geq \eta(\tau, e). \tag{55}$$

Proof
To establish this result, we first evaluate the successive derivative of the function $\pi \mapsto \overline{\eta}(W)$. To access such an issue, we detail the following sequence of derivation.

First, we derive relation

$$I = p(\overline{\tau}(I, J), \overline{e}(I, J)) + a^2 \overline{\tau}(I, J)$$

with respect to π to get

$$\partial_\pi I = \partial_\pi \overline{\tau} \partial_\tau \overline{p} + \partial_\pi \overline{e} \partial_e \overline{p} + a^2 \partial_\pi \overline{\tau}.$$

But from (48), we have $\partial_\pi I = 1$ and thus we deduce

$$\partial_\pi \overline{e} = \frac{1}{\partial_e \overline{p}} \left(1 - (\partial_\tau \overline{p} + a^2) \partial_\pi \overline{\tau}\right). \tag{56}$$

Next, we derive relation

$$J = \overline{e}(I, J) - \frac{p(\overline{\tau}(I, J), \overline{e}(I, J))^2}{2a^2}$$

with respect to π to get

$$-\frac{\pi}{a^2} = \left(1 - \frac{\overline{p}\partial_e \overline{p}}{a^2}\right) \partial_\pi \overline{e} - \frac{\overline{p}}{a^2} \partial_\tau \overline{p} \partial_\pi \overline{\tau}.$$

Plugging (56) into this relation gives

$$\partial_\pi \overline{\tau} = \frac{(\overline{p} - \pi)\partial_e \overline{p} - a^2}{a^2 \left(\overline{p}\partial_e \overline{p} - \partial_\tau \overline{p} - a^2\right)}. \tag{57}$$

We now consider the function $\overline{\eta}$ defined by (54). Deriving $\overline{\eta}$ with respect to π, we obtain

$$\partial_\pi \overline{\eta} = \partial_\tau \overline{\eta} \partial_\pi \overline{\tau} + \partial_e \overline{\eta} \partial_\pi \overline{e},$$

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where we substitute $\partial_\pi \overline{e}$ by the expression (56) to get

$$\partial_\pi \overline{\eta} = \left(\partial_\tau \overline{\eta} - \frac{\partial_e \overline{\eta}}{\partial_e \overline{p}} (\partial_\tau \overline{p} + a^2) \right) \partial_\pi \overline{\tau} + \frac{\partial_e \overline{\eta}}{\partial_e \overline{p}}.$$

Next, we plug the definition of $\partial_\pi \overline{\tau}$, given by (57), into this relation to obtain, after a straightforward calculation:

$$\partial_\pi \overline{\eta} = \frac{(\overline{p} - \pi)(\partial_\tau \overline{\eta} \partial_e \overline{p} - \partial_\tau \overline{p} \partial_e \overline{\eta} - a^2 \partial_e \overline{\eta}) + a^2 (\overline{p} \partial_e \overline{\eta} - \partial_\tau \overline{\eta})}{a^2 (\overline{p} \partial_e \overline{p} - \partial_\tau \overline{p} - a^2)}.$$

Since by definition of the specific entropy, given by (3), we have

$$\partial_\tau \eta = p \partial_e \eta, \tag{58}$$

we immediately deduce the first-order derivative of $\overline{\eta}$ with respect to π as follows:

$$\partial_\pi \overline{\eta} = \frac{\overline{p} - \pi}{a^2} \partial_e \overline{\eta}. \tag{59}$$

Finally, deriving again this expression with respect to π, we obtain

$$\partial_{\pi\pi} \overline{\eta} = \frac{\partial_e \overline{\eta}}{a^2} \left(\partial_e \overline{\eta} \partial_\tau \overline{p} + (\overline{p} - \pi) \partial_{\tau e} \overline{\eta} \right) \partial_\pi \overline{\tau} + \frac{1}{a^2} \left(\partial_e \overline{\eta} \partial_e \overline{p} + (\overline{p} - \pi) \partial_{ee} \overline{\eta} \right) \partial_\pi \overline{e} - \frac{\partial_e \overline{\eta}}{a^2}.$$

Once again we substitute the expressions (56) of $\partial_\pi \overline{e}$ and (57) of $\partial_\pi \overline{\tau}$ to get

$$\begin{aligned} \partial_{\pi\pi} \overline{\eta} = \frac{1}{a^4 (\overline{p} \partial_e \overline{p} - \partial_\tau \overline{p} - a^2)} \Big((\overline{p} - \pi)^2 \left(\partial_e \overline{p} \partial_{\tau e} \overline{\eta} - \partial_{ee} \overline{\eta} \partial_\tau \overline{p} - a^2 \partial_{ee} \overline{\eta} \right) \\ + a^2 (\overline{p} - \pi) \left(\overline{p} \partial_{ee} \overline{\eta} - \partial_{\tau e} \overline{\eta} - \partial_e \overline{p} \partial_e \overline{\eta} \right) + a^4 \partial_e \overline{\eta} \Big). \end{aligned} \tag{60}$$

Let us notice that deriving (58) with respect to e gives

$$\partial_{\tau e} \eta = p \partial_{ee} \eta + \partial_e p \partial_e \eta,$$

to write (60) as follows:

$$\partial_{\pi\pi} \overline{\eta} = \frac{(\overline{p} - \pi)^2}{a^4} \partial_{ee} \overline{\eta} + \frac{\partial_e \overline{\eta}}{a^4 (\overline{p} \partial_e \overline{p} - \partial_\tau \overline{p} - a^2)} \left((\overline{p} - \pi) \partial_e \overline{p} - a^2 \right)^2. \tag{61}$$

Since $(\tau, e) \mapsto \eta(\tau, e)$ is a strictly convex function, we have $\partial_{ee} \overline{\eta} \geq 0$. On the other hand, from the inequalities (3), we get $\partial_e \overline{\eta} < 0$. Finally, under the sub-characteristic Whitham condition (41), we deduce that $\partial_{\pi\pi} \overline{\eta} \geq 0$. As a consequence, the function $\pi \mapsto \overline{\eta}$ is convex. Therefore, the proof will be concluded as soon as we establish $\partial_\pi \overline{\eta}|_{\pi = p(\tau, e)} = 0$.

Let us notice that we have

$$\begin{aligned} \overline{p}_{|\pi = p(\tau, e)} &= p \left(\overline{\tau} \left(p(\tau, e) + a^2 \tau, e - \frac{p(\tau, e)^2}{2a^2} \right), \overline{e} \left(p(\tau, e) + a^2 \tau, e - \frac{p(\tau, e)^2}{2a^2} \right) \right), \\ &= p(\tau, e). \end{aligned}$$

After (59), we see that $\partial_\pi \overline{\eta}$ vanishes at the point $\pi = p(\tau, e)$. So we have

$$\overline{\eta}(W) \geq \overline{\eta} \left(W_{\pi = p(\tau, e)} \right).$$

Since the state $W_{|\pi = p(\tau, e)}$ coincides to the relaxation equilibrium, necessarily we have $\overline{\eta} \left(W_{|\pi = p(\tau, e)} \right) = \eta (\tau, e)$, which concludes the proof. □

Next, we prove that the solution of the Riemann problem (23) satisfies a property of decreasing entropy.

Lemma 12
Let W_L and W_R be two states in $\mathcal{O}$ at the relaxation equilibrium: $\pi_L = p(\tau_L, e_L)$ and $\pi_R = p(\tau_R, e_R)$. Assume that the sub-characteristic Whitham conditions (43) are satisfied. Let $W_{\mathcal{R}}$, defined by (24) and (27), be the solution of the Riemann problem (23)–(25). Then $\overline{\eta}\left(W_{\mathcal{R}}\left(x/t; W_L, W_R\right)\right)$ is defined for all (x,t) in $\mathbb{R}\times\mathbb{R}^+$ and satisfies

$$\overline{\eta}\left(W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right)\right) \geq \eta\left((\tau_{\mathcal{R}}, e_{\mathcal{R}})\left(\frac{x}{t}; W_L, W_R\right)_{|\pi=p(\tau,e)}\right). \tag{62}$$

Proof
Since the function $W_{\mathcal{R}}$ is made of four constant states W_L, $W_L^\star$, $W_R^\star$ and W_R, according to (24), the proof will be achieved as soon as the following inequalities will be established:

$$\overline{\eta}(W_{L,R}) \geq \eta\left(\tau_{L,R}, e_{L,R}\right) \qquad \text{and} \qquad \overline{\eta}(W_{L,R}^\star) \geq \eta\left(\tau^\star_{L,R|\pi=p^\star_{L,R}}, e^\star_{L,R|\pi=p^\star_{L,R}}\right).$$

After Lemma 11, such inequalities are satisfied whenever the involved states $W_{L,R}$ and $W_{L,R}^\star$ stay in $\mathcal{E}$.

With W_L and W_R given at the relaxation equilibrium and the Whitham conditions (43a) and (43d), the states W_L and W_R immediately belong to the set $\mathcal{E}$. Next, we turn establishing that $W_L^\star$ belongs to $\mathcal{E}$. According to (53), we have to prove that $(I(W_L^\star), J(W_L^\star))$ is in $\mathcal{A}$.

According to Lemma 9, I and J are strong Riemann invariants, so we have

$$I(W_L^\star) = I(W_L) \qquad \text{and} \qquad J(W_L^\star) = J(W_L).$$

Next, since $\pi_L = p(\tau_L, e_L)$, we get the two following relations:

$$I(W_L^\star) = p(\tau_L, e_L) + a^2\tau_L \qquad \text{and} \qquad J(W_L^\star) = e_L - \frac{p(\tau_L, e_L)^2}{2a^2}.$$

Finally, the Whitham condition (43a) enforces $(I(W_L^\star), J(W_L^\star))$ to belong to $\mathcal{A}$. To conclude, the condition (43b) then ensures that $W_L^\star$ belongs to $\mathcal{E}$. A similar reasoning leads to $W_R^\star \in \mathcal{E}$. Arguing Lemma 11, the proof is completed. □

Equipped with the above entropy minimum principle, given Lemma 12, we are able to establish Theorem 8.

Proof of Theorem 8
First, let us consider the weak solutions of the relaxation model (19), with an initial condition given by

$$W(x,0) = \begin{cases} W^{eq}(w_L) & \text{if } x < 0, \\ W^{eq}(w_R) & \text{if } x > 0. \end{cases}$$

The function $W \mapsto \overline{\eta}(W)$, defined by (54), only depends of I and J, so Lemma 9 ensures that the weak solutions of (19) satisfy the additional following conservation law:

$$\partial_t \rho\mathcal{F}(\overline{\eta}) + \partial_x \rho\mathcal{F}(\overline{\eta})u = 0.$$

We integrate this equation over $[0, \Delta x/2) \times [0, \Delta t)$ to get

$$\begin{aligned}\int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{\eta}))\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) dx = \\ \int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{\eta}))(W(x,0))dx - \Delta t(\rho\mathcal{F}(\overline{\eta})u)\left(W_{\mathcal{R}}\left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) \\ + \Delta t(\rho\mathcal{F}(\overline{\eta})u)\left(W_{\mathcal{R}}\left(0; W^{eq}(w_L), W^{eq}(w_R)\right)\right).\end{aligned} \tag{63}$$

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Since the state $W^{eq}(w_R)$ is at the relaxation equilibrium, the following sequence of equalities holds for $x \in [0, \Delta x/2)$:

$$(\rho\mathcal{F}(\overline{\eta}))(W(x,0)) = (\rho\mathcal{F}(\overline{\eta}))(W^{eq}(w_R)) = (\rho\mathcal{F}(\eta))(W^{eq}(w_R)) = \rho_R\mathcal{F}(\eta_R). \tag{64}$$

In the other hand, the CFL restriction (44) implies for all $x \in [0, \Delta x/2)$:

$$W_{\mathcal{R}}\left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right) = W^{eq}(w_R).$$

As a consequence, the first flux term in (63) writes

$$(\rho\mathcal{F}(\overline{\eta})u)\left(W_{\mathcal{R}}\left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) = \rho_R\mathcal{F}(\eta_R)u_R. \tag{65}$$

Now, since $\overline{\eta}$ only depends of I and J and these two variables are continuous through the waves of speed $u_L - a/\rho_L$ and $u_R + a/\rho_R$, we have

$$\overline{\eta}\left(W_{\mathcal{R}}\left(0; W^{eq}(w_L), W^{eq}(w_R)\right)\right) = \begin{cases} \overline{\eta}(W^{eq}(w_L)) = \eta(w_L) & \text{if } u^\star > 0, \\ \overline{\eta}(W^{eq}(w_R)) = \eta(w_R) & \text{if } u^\star < 0. \end{cases}$$

We immediately deduce that the second flux term in (63) writes

$$(\rho\mathcal{F}(\overline{\eta})u)\left(W_{\mathcal{R}}\left(0; W^{eq}(w_L), W^{eq}(w_R)\right)\right) = \{\rho\mathcal{F}(\eta)u\}_{L,R}, \tag{66}$$

where $\{\rho\mathcal{F}(\eta)u\}_{L,R}$ is defined by (47).

We plug (64), (65) and (66) into (63), to obtain

$$\frac{1}{\Delta x}\int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{\eta}))\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) dx = \frac{\rho_R\mathcal{F}(\eta_R)}{2} - \frac{\Delta t}{\Delta x}\left(\rho_R\mathcal{F}(\eta_R)u_R - \{\rho\mathcal{F}(\eta)u\}_{L,R}\right). \tag{67}$$

Finally, Lemma 12 ensures that

$$\overline{\eta}\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) \geq \eta\left((\tau^{eq}, e^{eq})\left(\frac{x}{\Delta t}; w_L, w_R\right)\right).$$

Moreover, from (8), the function $\mathcal{F}$ is increasing, and thus we get

$$\mathcal{F}(\overline{\eta})\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) \geq \mathcal{F}(\eta)\left(w^{eq}\left(\frac{x}{\Delta t}; w_L, w_R\right)\right),$$

to obtain the expected inequality (45).

The inequality (46) is proven in a very similar way by adopting an integration over $(-\Delta x/2, 0] \times [0, \Delta t)$. $\square$

4. THE RELAXATION SCHEME

Equipped with the approximate Riemann solver $w^{eq}(x/t; w_L, w_R)$, defined by (28), we now derive a finite volume scheme to discretize the Euler equation with gravity (1). First, we introduce some usual mesh notations. Concerning the space discretization, we adopt a uniform mesh with cells $(x_i - \Delta x/2, x_i + \Delta x/2)$ where $\Delta x > 0$ denotes the constant size of the mesh. The time discretization is given by $t^{n+1} = t^n + \Delta t$ where $\Delta t > 0$ is the time step restricted according the following CFL like condition:

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(\left|u_i^n - \frac{a_{i+1/2}}{\rho_i^n}\right|, \left|u_{i+1}^n + \frac{a_{i+1/2}}{\rho_{i+1}^n}\right|\right) \leq \frac{1}{2}, \tag{68}$$

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where $a_{i+1/2}$ will be detailed later on.

From an approximation of the solution at time t^n, given by

$$w^n(x,t^n) = w_i^n, \qquad x \in (x_{i-1/2}, x_{i+1/2}),$$

we define the updated state at time t^{n+1} as follows:

$$w_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w^n(x,t^n+\Delta t)dx. \tag{69}$$

Here, the function $w^n(x,t^n+t)$ is nothing but the sequence of the approximate Riemann solver (28) stated at each interface $x_{i+1/2}$:

$$w^n(x,t^n+t) = w^{eq}\left(\frac{x-x_{i+1/2}}{t}; w_i^n, w_{i+1}^n\right), \qquad x \in (x_i, x_{i+1}) \quad t \in (0,\Delta t). \tag{70}$$

In fact, for all t in $(0,\Delta t)$, the successive approximate Riemann solvers, involved to define $w^n(x,t^n+t)$, do not interact as long as the parameter $a_{i+1/2}$, introduced in (68), coincides with a local (defined interface per interface) relaxation parameter. As a consequence, $a_{i+1/2}$ is asked to satisfy at each interface both Ω-preserving conditions (29)-(30) and sub-characteristic Whitham conditions (43).

After a straightforward computation (for instance, see [38, 33, 39, 40]), the updated state w_i^{n+1} reads in the following more convenient form:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right) + \frac{\Delta t}{2}\left(S_{i-1/2} + S_{i+1/2}\right), \tag{71}$$

with an approximated source term defined by

$$S_{i+1/2} = \left(0, -\overline{\rho}(\rho_i^n,\rho_{i+1}^n)\frac{\Phi_{i+1}-\Phi_i}{\Delta x}, -\overline{\rho}(\rho_i^n,\rho_{i+1}^n)u^\star_{i+1/2}\frac{\Phi_{i+1}-\Phi_i}{\Delta x}\right)^T. \tag{72}$$

The detailed form of the numerical flux function $f_{i+1/2} := f_\Delta(w_i^n, \Phi_i, w_{i+1}^n, \Phi_{i+1})$ is given by

$$f_\Delta(w_L,\Phi_L,w_R,\Phi_R) = \begin{cases} \left(\rho_L u_L, \rho_L u_L^2 + p_L + s_{LR}, (E_L+p_L)u_L + u^\star s_{LR}\right)^T & \text{if } u_L - \dfrac{a}{\rho_L} > 0, \\ \left(\rho_L^\star u^\star, \rho_L^\star (u^\star)^2 + \pi_L^\star + s_{LR}, (E_L^\star + \pi_L^\star)u^\star + u^\star s_{LR}\right)^T & \text{if } u_L - \dfrac{a}{\rho_L} < 0 < u^\star, \\ \left(\rho_R^\star u^\star, \rho_R^\star (u^\star)^2 + \pi_R^\star - s_{LR}, (E_R^\star + \pi_R^\star)u^\star - u^\star s_{LR}\right)^T & \text{if } u^\star < 0 < u_R + \dfrac{a}{\rho_R}, \\ \left(\rho_R u_R, \rho_R u_R^2 + p_R - s_{LR}, (E_R+p_R)u_R - u^\star s_{LR}\right)^T & \text{if } u_R + \dfrac{a}{\rho_R} < 0, \end{cases} \tag{73}$$

where we have introduced

$$s_{LR} = -\frac{1}{2}\overline{\rho}(\rho_L,\rho_R)(\Phi_R - \Phi_L) \tag{74}$$

and the intermediate states are given by (27).

To conclude this presentation of the relaxation scheme (71)-(72)-(73)-(74), we now give its main properties. Indeed, in the following result we summarize the robustness, the stability and the well-balancedness of the derived numerical method.

Theorem 13

For all i in $\mathbb{Z}$, assume that the local relaxation parameter $a_{i+1/2}$ satisfies, at each interface $x_{i+1/2}$, the order conditions (29)-(30) and the sub-characteristic Whitham condition (43). Assume the pressure law satisfies Assumption 7. Assume w_i^n belongs to Ω for all $i \in \mathbb{Z}$. Then, under the CFL condition (68), the updated state w_i^{n+1}, defined by the relaxation scheme (71), satisfies the following properties:

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1. Robustness: For all i in $\mathbb{Z}$, w_i^{n+1} belongs to Ω.
2. Entropy preserving: For all smooth function $\mathcal{F}$ such that (8) is verified, w_i^{n+1} satisfies the following discrete entropy inequality:

$$\rho_i^{n+1}\mathcal{F}(\eta_i^{n+1}) \leq \rho_i^n\mathcal{F}(\eta_i^n) - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(\eta)u\}_{i+1/2}^n - \{\rho\mathcal{F}(\eta)u\}_{i+1/2}^n\right), \tag{75}$$

where the entropy numerical flux are defined by

$$\{\rho\mathcal{F}(\eta)u\}_{i+1/2}^n = f_{i+1/2}^\rho \times \begin{cases} \mathcal{F}(\eta_i^n) & \text{if } F_{i+1/2}^\rho > 0, \\ \mathcal{F}(\eta_{i+1}^n) & \text{if } F_{i+1/2}^\rho < 0. \end{cases} \tag{76}$$

3. General steady state preserving: Let us consider an initial data w_i^0 given by

$$\frac{1}{\Delta x}(p_{i+1}^0 - p_i^0) + \overline{\rho}(\rho_i^0, \rho_{i+1}^0)\frac{\Phi_{i+1} - \Phi_i}{\Delta x} = 0. \tag{77}$$

Then the updated state w_i^{n+1} stays at rest, and thus satisfies $w_i^{n+1} = w_i^n$ for all $i \in \mathbb{Z}$

Proof

First, let us establish the robustness of the scheme. Since we have imposed (29) and (30), Lemma 4 can be applied locally to each function $w^{eq}(x/t; w_i^n, w_{i+1}^n)$, which thus stays in Ω. The CFL restriction (68) ensures that the Riemann problems do not interact, so the function $x \mapsto w^n(x, t^n + \Delta t)$ is valued in Ω. Then the update formula (69) and the convexity of Ω imply that w_i^{n+1} belongs to Ω.

Next, we prove the stability property satisfied by the scheme. Since the CFL restriction (68) and the Whitham conditions (43) are satisfied, we can apply Theorem 8. Inequality (45), where we have set $w_L = w_{i-1}^n$ and $w_R = w_i^n$, reads

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i} (\rho\mathcal{F}(\eta))\left(w^{eq}\left(\frac{x - x_{i-1/2}}{\Delta t}; w_{i-1}^n, w_i^n\right)\right) dx \\ \leq \frac{\rho_i^n\mathcal{F}(\eta_i^n)}{2} - \frac{\Delta t}{\Delta x}\left(\rho_i^n\mathcal{F}(\eta_i^n)u_i^n - \{\rho\mathcal{F}(\eta)u\}_{i-1/2}^n\right) dx, \tag{78}$$

whereas inequality (46), where we have set $w_L = w_i^n$ and $w_R = w_{i+1}^n$, reads

$$\frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}} (\rho\mathcal{F}(\eta))\left(w^{eq}\left(\frac{x - x_{i+1/2}}{\Delta t}; w_i^n, w_{i+1}^n\right)\right) \\ \leq \frac{\rho_i^n\mathcal{F}(\eta_i^n)}{2} - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(\eta)u\}_{i+1/2}^n - \rho_i^n\mathcal{F}(\eta_i^n)u_i^n\right). \tag{79}$$

Summing (78) and (79), we obtain

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} (\rho\mathcal{F}(\eta))\left(w^n(x, t^n + \Delta t)\right) dx \leq \rho_i^n\mathcal{F}(\eta_i^n) \\ - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(\eta)u\}_{i+1/2}^n - \{\rho\mathcal{F}(\eta)u\}_{i-1/2}^n\right). \tag{80}$$

Now, according to assumption (8), the function $w \mapsto \rho\mathcal{F}(\eta)$ is strictly convex, so we can apply the Jensen inequality to get

$$\rho\mathcal{F}(\eta)\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w^n(x, t^n + \Delta t)dx\right) \leq \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} (\rho\mathcal{F}(\eta))\left(w^n(x, t^n + \Delta t)\right) dx. \tag{81}$$

Using the definition (69) of the numerical scheme together with inequalities (80) and (81), we obtain the required discrete entropy inequality (75).

About the establishment of the well-balanced property, the proof directly comes from Lemma 5. Indeed, at each interface, the initial data satisfies (31)-(32) and Lemma 5 can be applied. As a consequence, at each interface the approximate Riemann solver stays at rest. Since the updated state w_i^1, at time $t = \Delta t$, is defined by (69), we immediately deduce $w_i^1 = w_i^0$ for all i in $\mathbb{Z}$. Arguing an induction procedure, the proof of the steady state preserving property is then completed. □

Concerning the steady state preserving property, we notice that no additional definition is imposed to the ρ-average. We just impose an initial data given by a specific approximation of the partial differential equation (14), given by (77). Then, this initial data is exactly preserved. Concerning the capture of the polytropic steady states, the ρ-average must be specified as presented next.

Theorem 14
Assume one of the following conditions occurs:

1. Isothermal equilibrium: The average function $\overline{\rho}$ is defined by (35) and the initial data w_i^0 satisfies

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = \exp\left(\dfrac{C-\Phi_i}{K}\right), \\ p_i^0 = K\exp\left(\dfrac{C-\Phi_i}{K}\right), \end{cases} \qquad \text{with } K>0 \quad \text{and} \quad C\in\mathbb{R};$$

2. General polytropic equilibrium: The average function $\overline{\rho}$ is defined by (37) and the initial data w_i^0 satisfies

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = \left(\dfrac{\Gamma-1}{\Gamma K}(C-\Phi_i)\right)^{\frac{1}{\Gamma-1}}, \\ p_i^0 = K^{\frac{1}{1-\Gamma}}\left(\dfrac{\Gamma-1}{\Gamma}(C-\Phi_i)\right)^{\frac{\Gamma}{\Gamma-1}}, \end{cases} \qquad \begin{array}{l}\text{with } \Gamma\in(0,1)\cup(1,+\infty),\\ K>0 \quad \text{and} \quad C\in\mathbb{R}.\end{array}$$

3. Incompressible equilibrium: The initial data w_i^0 satisfies

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = M, \\ p_i^0 + \rho_i^0\Phi_i = C, \end{cases} \qquad \text{with } M>0 \quad \text{and} \quad C\in\mathbb{R};$$

Then the updated state w_i^{n+1} stays at rest, and thus satisfies $w_i^{n+1} = w_i^n$ for all $i \in \mathbb{Z}$.

Proof
Since the initial data w_i^0 satisfies one of the conditions 1, 3 or 2, we can apply Lemma 6, so each approximate Riemann solver stays at rest. As a consequence, each approximate Riemann solver is given by (33). With the updated state w_i^1 defined by (69), we get immediately $w_i^1 = w_i^0$.

The proof is then easily achieved by an induction procedure. □

5. NUMERICAL RESULTS

The scheme (71)–(72)–(73) is now illustrated performing several numerical experiments. In all the applications, the pressure will be given by an ideal gas law

$$p = (\gamma-1)\rho e,$$

where the adiabatic coefficient is set to $\gamma = 1.4$.

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N	Density	Velocity
100	1.01E-16	1.41E-16
200	2.24E-16	1.42E-16
400	3.61E-16	1.38E-16
800	5.39E-16	1.71E-16
1600	9.28E-16	2.49E-16
3200	1.60E-15	2.50E-16

Table I. L^1 error in density and velocity for the isothermal atmosphere

5.1. *Isothermal atmosphere*

The aim of this experiment is to illustrate the exact preservation of the isothermal equilibrium. To underline that this property does not depend on the gravitational potential, we consider a gravity source term given by

$$\Phi(x) = x^2$$

on the computational domain $[0, 1]$. The initial condition is fixed to the isothermal equilibrium

$$(\rho_0, u_0, p_0)(x) = \left(e^{-x^2}, \quad 0, \quad e^{-x^2}\right).$$

As an isothermal equilibrium is considered, we choose the definition (35) for $\overline{\rho}$. At final time $T = 0.25$, we compute the L^1 error between the approximated solution and the exact solution. The results are given on Table I in density and velocity for an increasing number of cells N.

The results show that the isothermal atmosphere is preserved up to machine precision. This is coherent with Theorem 14. Of course a similar result would be achieved concerning an incompressible or polytropic atmosphere.

5.2. *General steady state*

Next, we investigate the behavior of the scheme on a general steady state, i.e. a steady state which does not belong to the polytropic family described by (15). We consider the computational domain $[0, 1]$ with periodic boundary conditions. The gravitational potential is here defined by

$$\Phi(x) = -\sin(2\pi x).$$

One can easily check that the initial condition

$$(\rho_0, u_0, p_0)(x) = (3 + 2\sin(2\pi x), \quad 0, \quad 3 + 3\sin(2\pi x) - 0.5\cos(4\pi x)) \tag{82}$$

defined a steady state solution.

Here, we choose the definition (35) for the ρ–average $\overline{\rho}$. We compute the L^1 error at time $T = 1$ for an increasing number of cells N. The results, as well as the convergence orders, are displayed in Table II for the density and the velocity. We observe that a second-order is achieved, although the scheme is only first-order. This can be formally explained by the fact that relation (77) is satisfied up to second-order. Indeed, a straightforward computation leads to

$$p_0(x + \Delta x) - p_0(x) + \overline{\rho}(\rho_0(x), \rho_0(x + \Delta x))(\Phi(x + \Delta x) - \Phi(x)) = O(\Delta x^2).$$

However, we notice that for this specific case, defining $\overline{\rho}$ by

$$\overline{\rho}(W_L, W_R) = \frac{\rho_L + \rho_R}{2}$$

leads to the exact satisfaction of relation (77). Thus with this choice, the steady state (82) would be exactly preserved.

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N	Density		Velocity	
100	4.46E-05		2.03E-05	
200	7.11E-06	2.65	5.29E-06	1.94
400	1.23E-06	2.53	1.34E-06	1.98
800	2.35E-07	2.39	3.37E-07	1.99
1600	5.02E-08	2.23	8.44E-08	2.00
3200	1.15E-08	2.13	2.11E-08	2.00

Table II. L^1 error in density and velocity for general steady state

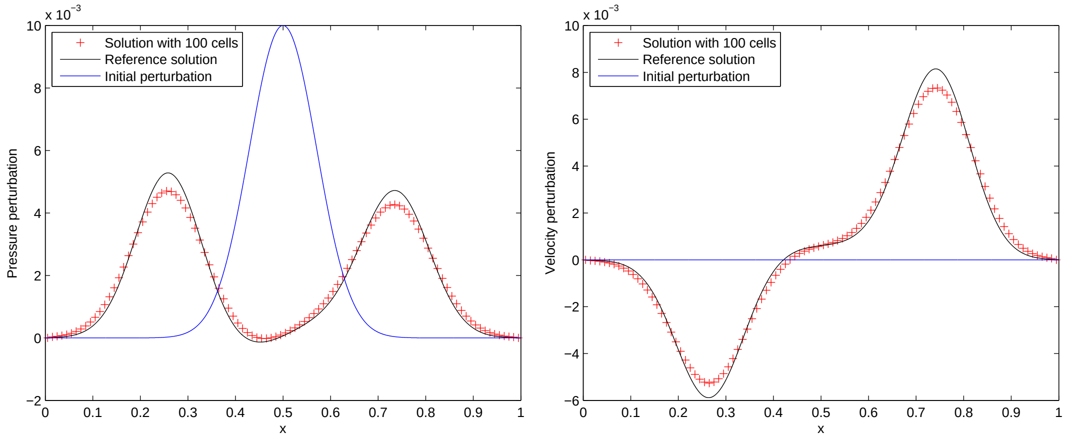

Figure 1. Perturbation in pressure (left) and in velocity (right)

5.3. *Perturbation of an isothermal atmosphere*

The following test case has been introduced in [41, 22]. Here we consider a classical constant gravitational field described by the potential $\Phi(x) = x$.The computational domain $[0, 1]$ is initially filled with a gas staying in an isothermal equilibrium

$$\rho_0(x) = p_0(x) = e^{-x}, \quad u_0(x) = 0.$$

At time $t = 0$, the pressure is perturbed by

$$p(x, t = 0) = p_0(x) + 0.01e^{-100(x-0.5)^2}.$$

For the average $\overline{\rho}$, we choose here the definition (35).

We compute the solution at time $T = 0.2$ using 100 cells. The results are compared with a reference solution, which is obtained using 30.000 cells. On Figure 1, we display the results in pressure perturbation (i.e. $p(x) - p_0(x)$) and in velocity perturbation.

The initial perturbation splits into two waves moving in opposite directions. We can see that the scheme does not create spurious oscillations near the bumps. In additions, the size of the bumps decrease with time. This shows the stability of the numerical scheme with respect to this steady state. Finally, comparison with the reference solution shows that the general shape is well captured by the numerical scheme, although it is only a first order scheme

5.4. *Perturbation of a polytropic equilibrium*

The last test is designed to investigate the ability of the scheme to propagate small disturbances in a polytropic atmosphere. We consider once again a constant gravitational field given by $\Phi(x) = x$. We

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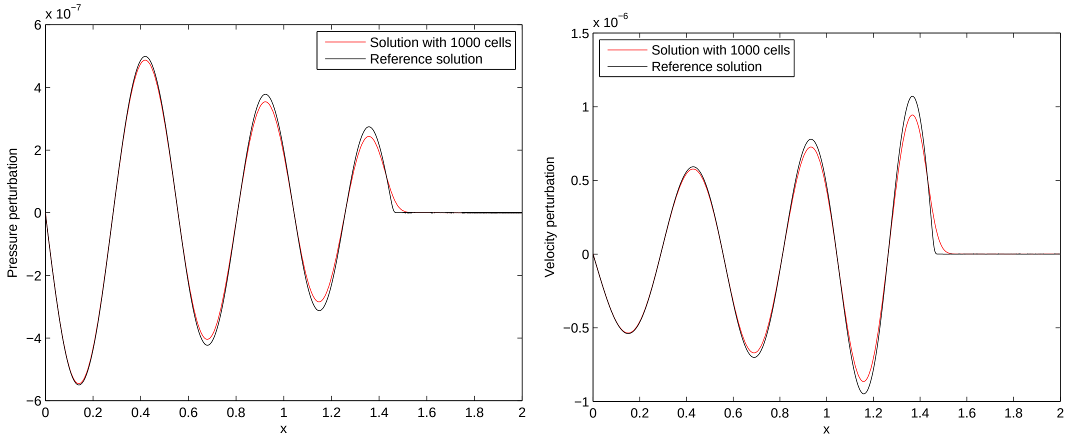

Figure 2. Perturbation in pressure (left) and in velocity (right)

consider the computational domain $[0, 2]$ initially filled with a gas staying in a polytropic equilibrium

$$\rho_0(x) = \left(1 - \frac{2}{5}\Phi(x)\right)^{3/2}, \quad u_0(x) = 9, \quad p_0(x) = \left(1 - \frac{2}{5}\Phi(x)\right)^{5/2},$$

which corresponds to a polytropic coefficient $\Gamma = 5/3$.

This time, the perturbation is applied as a boundary condition:

$$u(0, t) = A\sin(4\pi t),$$

where the amplitude is set to $A = 10^{-6}$. We stop the computation at $t = 1.5$, shortly before the perturbation reaches the right boundary. As the excited wave move through the domain, its amplitude is softened by the stratification due to gravity.

The final perturbation in pressure and velocity are displayed in Figure 2. Here we compare the results obtained with 1.000 cells with a reference solution computed with 30.000 cells.

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REFERENCES

1. Godlewski E, Raviart PA. *Numerical approximation of hyperbolic systems of conservation laws*, *Applied Mathematical Sciences*, vol. 118. Springer-Verlag: New York, 1996.
2. LeFloch PG. *Hyperbolic systems of conservation laws*. Lectures in Mathematics ETH Zürich, Birkhäuser Verlag: Basel, 2002, doi:10.1007/978-3-0348-8150-0. The theory of classical and nonclassical shock waves.
3. Dafermos CM. *Hyperbolic conservation laws in continuum physics*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 325. Third edn., Springer-Verlag: Berlin, 2010, doi:10.1007/978-3-642-04048-1.
4. Harten A, Lax PD, Levermore CD, Morokoff WJ. Convex entropies and hyperbolicity for general Euler equations. *SIAM J. Numer. Anal.* 1998; **35**(6):2117–2127, doi:10.1137/S0036142997316700.
5. Bermudez A, Vazquez ME. Upwind methods for hyperbolic conservation laws with source terms. *Computers & Fluids* 1994; **23**(8):1049–1071.
6. Bouchut F. *Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources*. Frontiers in Mathematics, Birkhäuser Verlag: Basel, 2004, doi:10.1007/b93802.
7. Greenberg JM, Leroux AY. A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. *SIAM Journal on Numerical Analysis* 1996; **33**(1):1–16.
8. Liang Q, Marche F. Numerical resolution of well-balanced shallow water equations with complex source terms. *Advances in water resources* 2009; **32**(6):873–884.
9. Castro M, Frings JT, Noelle S, Parés C, Puppo G. On the hyperbolicity of two- and three-layer shallow water equations. *Hyperbolic problems, theory, numerics and applications. Volume 1, Ser. Contemp. Appl. Math. CAM*, vol. 17. World Sci. Publishing: Singapore, 2012; 337–345.

DOI: 10.1002/fld

10. Chertock A, Dudzinski M, Kurganov A, Lukáčová-Medvidová M. Well-balanced schemes for the shallow water equations with coriolis forces xxxxxx; .
11. Chertock A, Kurganov A, Liu Y. Central-upwind schemes for the system of shallow water equations with horizontal temperature gradients. *Numerische Mathematik* 2013; :1–45doi:10.1007/s00211-013-0597-6.
12. Natalini R, Ribot M, Twarogowska M. A well-balanced numerical scheme for a one dimensional quasilinear hyperbolic model of chemotaxis. *Commun. Math. Sci.* 2014; **12**(1):13–39, doi:10.4310/CMS.2014.v12.n1.a2.
13. Käppeli R, Mishra S. Well-balanced schemes for the Euler equations with gravitation. *Journal of Computational Physics* 2014; **259**(0):199 – 219, doi:http://dx.doi.org/10.1016/j.jcp.2013.11.028.
14. Luo J, Xu K, Liu N. A well-balanced symplecticity-preserving gas-kinetic scheme for hydrodynamic equations under gravitational field. *SIAM Journal on Scientific Computing* 2011; **33**(5):2356–2381.
15. Xu K. A well-balanced gas-kinetic scheme for the shallow-water equations with source terms. *Journal of Computational Physics* 2002; **178**(2):533–562.
16. Chalons C, Coquel F, Godlewski E, Raviart PA, Seguin N. Godunov-type schemes for hyperbolic systems with parameter-dependent source. The case of Euler system with friction. *Math. Models Methods Appl. Sci.* 2010; **20**(11):2109–2166, doi:10.1142/S021820251000488X.
17. Berthon C, Foucher F. Efficient well-balanced hydrostatic upwind schemes for shallow-water equations. *Journal of Computational Physics* 2012; **231**(15):4993–5015.
18. Berthon C, Foucher F. Hydrostatic upwind schemes for shallow–water equations. *Finite Volumes for Complex Applications VI Problems & Perspectives*. Springer, 2011; 97–105.
19. Noelle S, Xing Y, Shu CW. High-order well-balanced finite volume WENO schemes for shallow water equation with moving water. *J. Comput. Phys.* 2007; **226**(1):29–58, doi:10.1016/j.jcp.2007.03.031.
20. Cargo P, Le Roux AY. Un schéma équilibre adapté au modèle d'atmosphère avec termes de gravité. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* 1994; **318**(1):73–76.
21. Desveaux V, Zenk M, Berthon C, Klingenberg C. Well-balanced schemes to capture non-explicit steady states: Ripa model. *Submitted* ; .
22. Xu K, Luo J, Chen S. A well-balanced kinetic scheme for gas dynamic equations under gravitational field. *Advances in Applied Mathematics and Mechanics* 2010; **2**:200–210.
23. Jin S, Xin Z. The relaxation scheme for systems of conservation laws in arbitrary space dimension. *Comm. Pure Appl. Math.* 1995; **45**:235–276.
24. Coquel F, Perthame B. Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics. *SIAM Journal on Numerical Analysis* 1998; **35**(6):2223–2249.
25. Baudin M, Berthon C, Coquel F, Masson R, Tran QH. A relaxation method for two-phase flow models with hydrodynamic closure law. *Numerische Mathematik* 2005; **99**(3):411–440.
26. Berthon C. Numerical approximations of the 10-moment gaussian closure. *Mathematics of Computation* 2006; **75**(256):1809–1832.
27. Berthon C, Breuß M, Titeux MO. A relaxation scheme for the approximation of the pressureless Euler equations. *Numerical Methods for Partial Differential Equations* 2006; **22**(2):484–505.
28. Bouchut F, Morales T. A subsonic-well-balanced reconstruction scheme for shallow water flows. *SIAM Journal on Numerical Analysis* 2010; **48**(5):1733–1758.
29. Bouchut F, Klingenberg C, Waagan K. A multiwave approximate Riemann solver for ideal MHD based on relaxation II: numerical implementation with 3 and 5 waves. *Numer. Math.* 2010; **115**(4):647–679, doi:10.1007/s00211-010-0289-4.
30. Bouchut F, Klingenberg C, Waagan K. A multiwave approximate Riemann solver for ideal MHD based on relaxation. I: theoretical framework. *Numerische Mathematik* 2007; **108**(1):7–42.
31. LeVeque RJ, Pelanti M. A class of approximate Riemann solvers and their relation to relaxation schemes. *Journal of Computational Physics* 2001; **172**(2):572–591.
32. Coquel F, Hérard JM, Saleh K, Seguin N. A robust entropy-satisfying finite volume scheme for the isentropic Baer-Nunziato model. *ESAIM Math. Model. Numer. Anal.* 2014; **48**(1):165–206, doi:10.1051/m2an/2013101.
33. Harten A, Lax P, Van Leer B. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM review* 1983; **25**:35–61.
34. Ambroso A, Chalons C, Coquel F, Galié T. Relaxation and numerical approximation of a two-fluid two-pressure diphasic model. *ESAIM: Mathematical Modelling and Numerical Analysis* 2009; **43**(06):1063–1097.
35. Chalons C, Coquel F, Marmignon C. Well-balanced time implicit formulation of relaxation schemes for the Euler equations. *SIAM Journal on Scientific Computing* 2008; **30**(1):394–415.
36. Whitham GB. *Linear and nonlinear waves*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.
37. Berthon C, Dubroca B, Sangam A. A local entropy minimum principle for deriving entropy preserving schemes. *SIAM Journal on Numerical Analysis* 2012; **50**(2):468–491.
38. LeVeque RJ. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics, Cambridge University Press: Cambridge, 2002, doi:10.1017/CBO9780511791253.
39. Toro EF. *Riemann solvers and numerical methods for fluid dynamics*. Third edn., Springer-Verlag: Berlin, 2009, doi:10.1007/b79761. A practical introduction.
40. Kröner D. *Numerical schemes for conservation laws*. Wiley-Teubner Series Advances in Numerical Mathematics, John Wiley & Sons Ltd.: Chichester, 1997.
41. LeVeque RJ, Bale DS. Wave propagation methods for conservation laws with source terms. *International series of numerical mathematics* 1999; :609–618.