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L^1 singular limit for relaxation and viscosity approximations of extended traffic flow models *

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Abstract

This paper considers the Cauchy problem for an extended traffic flow model with L^1-bounded initial data. A solution of the corresponding equilibrium equation with L^1-bounded initial data is given by the limit of solutions of viscous approximations of the original system as the dissipation parameter ϵ tends to zero more slowly than the response time τ. The proof of convergence is obtained by applying the Young measure to solutions introduced by DiPerna and, based on the estimate

$$|\rho(t,x)| \le \sqrt{|\rho_0(x)|_1/(ct)}$$

derived from one of Lax's results and Diller's idea, the limit function $\rho(t,x)$ is shown to be a L^1-entropy week solution. A direct byproduct is that we can get the existence of L^1-entropy solutions for the Cauchy problem of the scalar conservation law with L^1-bounded initial data without any restriction on the growth exponent of the flux function provided that the flux function is strictly convex. Our result shows that, unlike the weak solutions of the incompressible fluid flow equations studied by DiPerna and Majda in [6], for convex scalar conservation laws with L^1-bounded initial data, the concentration phenomenon will never occur in its global entropy solutions.

1 Introduction

In this paper we are concerned with the Cauchy problem for the extended traffic flow model

$$\begin{gathered} \rho_t + (\rho u)_x = 0, \\ u_t + \big(\frac{u^2}{2} + f(\rho)\big)_x + \frac{h(\rho)(u - c\rho)}{\tau} = 0, \end{gathered} \tag{1.1}$$

with L^1-bounded initial data

$$(\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)) \in L^1(\mathbb{R}, \mathbb{R}^2), \tag{1.2}$$

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where c is a constant and τ denotes the response-time.

The existence of global classical solution of (1.1) was obtained by Schochet [15] for the case $f(\rho) = \frac{\mu}{\tau}\log\rho$ under the assumptions that τ is sufficiently small and $\tau \le \mu^{3+\alpha} (\alpha > 0)$. The zero relaxation limit in the L^∞ setting for related systems of (1.1) was considered in the recent paper [13].

In this paper we show that an L^1-solution of the equilibrium equation

$$\rho_t + (c\rho^2)_x = 0 \tag{1.3}$$

with L^1-bounded initial data $\rho_0(x)$ can be obtained by the limit of viscous solutions of the original system (1.1)

$$\begin{gathered}\rho_t + (\rho u)_x = \epsilon\rho_{xx},\\ u_t + \big(\frac{u^2}{2} + f(\rho)\big)_x + \frac{h(\rho)(u - c\rho)}{\tau} = \epsilon u_{xx},\end{gathered} \tag{1.4}$$

as the dissipation parameter ϵ and the response-time τ tend to zero, with $\tau = o(\epsilon)$.

For a large class of functions $f(\rho)$ the solutions of the parabolic systems (1.4) have no *a-priori* L^∞-estimates which are independent of the viscous parameter ϵ even if the initial data are bounded in L^∞ and sufficiently smooth. Fortunately under suitable restrictions on the nonlinear functions f and h, we can get the following estimates, in which $|\cdot|_p$ denotes the norm on L^p and $|\cdot|_{p,q}$ equals $|\cdot|_p + |\cdot|_q$,

$$|\rho^{\epsilon,\tau,m}(t,x)|_p \le M\,|\rho_0^m(x)|_{1,\infty}, \quad p > 1, \tag{1.5}$$

for solutions $(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))$ of the Cauchy problem (1.4) with initial data

$$(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))|_{t=0} = (\rho_0^m(x), u_0^m(x)), \tag{1.6}$$

where $\rho_0^m(x), u_0^m(x)$, which are smooth functions obtained by smoothing the initial data $(\rho_0(x), u_0(x))$ with a mollifier, satisfy

$$\begin{gathered}(\rho_0^m(x), u_0^m(x)) \in L^1 \cap L^\infty \cap C^\infty(\mathbb{R}, \mathbb{R}^2),\\ |\rho_0^m(x)|_1 \le |\rho_0(x)|_1, \quad |u_0^m(x)|_1 \le |u_0(x)|_1,\end{gathered} \tag{1.7}$$

for any fixed $m > 0$, and

$$\rho_0^m(x) \to \rho_0(x), \quad u_0^m(x) \to u_0(x) \quad \text{a.e. in } L^1 \text{ as } m \to 0. \tag{1.8}$$

When ϵ and τ tend to zero related by $\tau = o(\epsilon)$, for any fixed $m > 0$, we can prove that the Young measure $\nu_{(t,x)}^m$ associated to the sequence $\{\rho^{\epsilon,\tau,m}(t,x)\}$ is an entropy measure valued solution of (1.3). Then by applying the results in [16], $\nu_{(t,x)}^m$ is a Dirac measure and the limit $\rho^m(t,x)$ of $\rho^{\epsilon,\tau,m}(t,x)$ is the unique L^∞-entropy solution of the Cauchy problem (1.3) with the initial data $\rho_0^m(x)$. Furthermore, according to the results obtained in [16], we have that such a solution $\rho^m(t,x)$ can be obtained as the strong limit of the solution sequence $\{\rho^{\beta,m}(t,x)\}$ to the following Cauchy problem

$$\begin{aligned}\rho_t + \left(c\rho^2\right)_x &= \beta\rho_{xx},\\ \rho(t,x)|_{t=0} &= \rho_0^m(x),\end{aligned}$$

as $\beta \to 0^+$. Since $\rho_0^m(x)$ satisfies (1.7), we have from the well-known result of Kruzkov [8] that

$$\int_{\mathbb{R}} |\rho^{m_1}(t,x) - \rho^{m_2}(t,x)|dx \leq \int_{\mathbb{R}} |\rho_0^{m_1}(x) - \rho_0^{m_2}(x)|dx, \tag{1.9}$$

which means that $\rho^m(t,x)$ is a Cauchy sequence in L^1.

Note that the flux function $c\rho^2$ in (1.3) is a strictly convex function, we have from the results obtained by Lax in [10] that $\rho^m(t,x)$ is the almost everywhere unique minimizer of the functional

$$\phi(t,x,v) = \int_x^{x+2cv} \rho_0^m(x)dx + cv^2t. \tag{1.10}$$

However, since for any fixed point (t,x), $\phi(t,x,0) = 0$, and

$$\phi(t,x,v) \geq -\int_{\mathbb{R}} |\rho_0^m(x)|dx + cv^2t > 0 \tag{1.11}$$

if

$$|v| \geq \sqrt{\frac{|\rho_0(x)|_1}{ct}}.$$

Consequently $\rho^m(t,x)$ must satisfy the estimate

$$|\rho^m(t,x)| \leq \sqrt{\frac{|\rho_0^m(x)|_1}{ct}} \leq \sqrt{\frac{|\rho_0(x)|_1}{ct}}. \tag{1.12}$$

Thus from (1.9) and (1.12), there exists a function $\rho(t,x) \in L^1_{\rm loc}(\mathbb{R}^+ \times \mathbb{R})$ such that $\rho^m(t,x) \to \rho(t,x)$ and the limit function $\rho(t,x)$ is a L^1-entropy solution, in the sense of Szepessy [16] and Diller [4], of the Cauchy problem (1.3) with L^1-initial data $\rho_0(x)$. Furthermore, following the arguments developed by Diller in [4], the L^1-entropy weak solution, which satisfies the estimate (1.12), to the Cauchy problem (1.3) with L^1-initial data $\rho_0(x)$ is unique and depends continuously in L^1-norm on the initial data and such a uniqueness result guarantees that the whole sequence of $\{(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))\}$ converges strongly to $(\rho(t,x), u(t,x))$.

In this paper we assume f, h and the initial data $(\rho_0(x), u_0(x))$ satisfy the following hypotheses: For $q < 4$, $p < 8$ and positive constants $c_1 \dots c_4$ we have

A1 $\frac{f'(\rho)}{\rho} \geq c_1 > c^2$, $|f'(\rho)| \leq M(1 + |\rho|^q)$;

A2 $c_2(1 + |\rho|^4) \leq h(\rho) \leq c_3(1 + |\rho|^p)$;

A3 $|\rho_0|_1 \leq c_4, |u_0|_1 \leq c_4$,

2 Viscous Solutions

In this section, we consider global existence results for the parabolic system (1.4) with initial data (1.6). Since for any fixed $m > 0$, $(\rho_0^m(x),\ u_0^m(x))$ are bounded in L^∞, by applying the standard contracting map principle to an integral representation of (1.4), the local existence of L^∞ solutions, for fixed ϵ, τ, to the Cauchy problem (1.4), (1.6) can be easily established.

To extend a local solution to a global solution, we use the the following *a-priori* L^∞-estimates.

Lemma 2.1 *If $h(\rho)$ and $f(\rho)$ satisfy the assumptions A1, A2, $(\rho_0^m(x), u_0^m(x))$ satisfies (1.7) and the smooth solutions $(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))$ of the Cauchy problem (1.4), (1.6) exist in $[0,T]\times\mathbb{R}$, then the following estimates hold*

$$|\rho^{\epsilon,\tau,m}(t,x)| \le C(T,\epsilon,\tau,m), \quad |u^{\epsilon,\tau,m}(t,x)| \le C(T,\epsilon,\tau,m), \tag{2.1}$$

where $C(T,\epsilon,\tau,m)$ is a positive constant depending on T,ϵ,τ and m. Furthermore if $M_1\tau \le \epsilon$ for a suitable large constant M_1, then

$$|\rho^{\epsilon,\tau,m}(t,x)|_3 \le M(m), \quad |\rho^{\epsilon,\tau,m}(t,x)|_1 \le |\rho_0(x)|_1, \tag{2.2}$$

where $M(m)$ is a positive constant independent of ϵ and τ, but depending on m.

Proof For simplicity, we omit superscripts in $(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))$. Multiplying the first equation in (1.4) by $\int_0^\rho \frac{f'(s)}{s}ds$ and the second equation by $u - c\rho$, then adding the results and integrating on $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\frac{1}{2}u^2 + \int_0^\rho\int_0^y \frac{f'(s)}{s}\,ds\,dy - c\rho u\Big)dx + \int_0^t\int_{\mathbb{R}} \frac{h(\rho)(u-c\rho)^2}{\tau}\,dx\,dt \\
&+ \epsilon\int_0^t\int_{\mathbb{R}}\Big(u_x^2 + 2c\rho_x u_x + \frac{f'(\rho)}{\rho}\rho_x^2\Big)\,dx\,dt \\
&\le \int_{\mathbb{R}}\Big(\frac{1}{2}\,(u_0^m)^2 + \int_0^{\rho_0^m}\int_0^y \frac{f'(s)}{s}dsdy - c\rho_0^m u_0^m\Big)dx.
\end{aligned} \tag{2.3}$$

Multiplying the first equation in (1.4) by $-\rho_{xx}$ and then integrating on $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{1}{2}\rho_x^2 dx + \epsilon\int_0^t\int_{\mathbb{R}}\rho_{xx}^2\,dx\,dt \\
&= \int_{\mathbb{R}}\frac{1}{2}(\rho_{0x}^m)^2 dx + \int_0^t\int_{\mathbb{R}}\rho_{xx}(\rho u)_x\,dx\,dt \\
&\le \int_{\mathbb{R}}\frac{1}{2}(\rho_{0x}^m)^2 dx + \frac{\epsilon}{2}\int_0^t\int_{\mathbb{R}}(\rho_{xx})^2\,dx\,dt + \frac{2}{\epsilon}\int_0^t\int_{\mathbb{R}}|(\rho u)_x|^2\,dx\,dt \\
&\le \int_{\mathbb{R}}\frac{1}{2}(\rho_{0x}^m)^2 dx + \frac{\epsilon}{2}\int_0^t\int_{\mathbb{R}}(\rho_{xx})^2\,dx\,dt + \frac{1}{\epsilon}|\rho|_\infty^2\int_0^t\int_{\mathbb{R}}(u_x)^2\,dx\,dt \\
&\quad + \frac{1}{\epsilon}|u|_\infty^2\int_0^t\int_{\mathbb{R}}(\rho_x)^2\,dx\,dt\,.
\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and A1,

$$\int_{\mathbb{R}} (\rho_x)^2 dx + \epsilon \int_0^t \int_{\mathbb{R}} (\rho_{xx})^2 \, dx \, dt \leq \int_{\mathbb{R}} (\rho_{0x}^m)^2 dx + C(\epsilon) \left(|\rho|_\infty^2 + |u|_\infty^2 \right). \tag{2.5}$$

Since

$$|\rho|^2 = 2 \int_{-\infty}^{x} \rho \rho_x dx \leq |\rho|_2 |\rho_x|_2 \leq C(\epsilon)(1 + |\rho|_\infty + |u|_\infty), \tag{2.6}$$

we have

$$|\rho|_\infty \leq C(\epsilon)\big(1 + |u|_\infty^{1/2}\big). \tag{2.7}$$

Multiplying the second equation in (1.4) by $-u_{xx}$ and then integrating, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2} u_x^2 dx + \epsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt \\ &= \int_{\mathbb{R}} \frac{1}{2} (u_{0x}^m)^2 dx + \int_0^t \int_{\mathbb{R}} \frac{h(\rho)(u - c\rho) u_{xx}}{\tau} \, dx \, dt \\ &\quad + \int_0^t \int_{\mathbb{R}} u u_x u_{xx} \, dx \, dt + \int_0^t \int_{\mathbb{R}} f'(\rho) \rho_x u_{xx} \, dx \, dt. \end{aligned} \tag{2.8}$$

Due to (2.3) and A2,

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \frac{h(\rho)(u - c\rho) u_{xx}}{\tau} \, dx \, dt \\ &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + \frac{1}{\epsilon \tau^2} \int_0^t \int_{\mathbb{R}} |h(\rho)|^2 (u - c\rho)^2 \, dx \, dt \\ &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + \frac{C}{\epsilon \tau} |h(\rho)|_\infty \\ &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + C(\epsilon, \tau) \left(1 + |\rho|_\infty^p\right) \\ &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + C(\epsilon, \tau) \left(1 + |u|_\infty^{\frac{p}{2}}\right), \end{aligned} \tag{2.9}$$

$$\Big| \int_0^t \int_{\mathbb{R}} u u_x u_{xx} \, dx \, dt \Big| \leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + C(\epsilon, \tau) \left(1 + |u|_\infty^2\right) \tag{2.10}$$

and

$$\begin{aligned} \Big| \int_0^t \int_{\mathbb{R}} f'(\rho) \rho_x u_{xx} \, dx \, dt \Big| &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + C(\epsilon) |f'(\rho)|_\infty^2 \\ &\leq \frac{\epsilon}{4} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt + C(\epsilon) \left(1 + |u|_\infty^q\right), \end{aligned} \tag{2.11}$$

we have

$$\int_{\mathbb{R}} u_x^2 dx + \epsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, dt \leq \int_{\mathbb{R}} (u_{0x}^m)^2 dx + C(\epsilon, \tau) \left(1 + |u|_\infty^2 + |u|_\infty^{\frac{p}{2}} + |u|_\infty^q\right). \tag{2.12}$$

Since

$$u^2 \le |u|_2|u_x|_2 \le C(\epsilon,\tau)\Big(1+|u|_\infty^2+|u|_\infty^{\frac{p}{2}}+|u|_\infty^{q}\Big)^{1/2} \tag{2.13}$$

and $p<8$, $q<4$, we have by (2.3) and $A1$,

$$|u|_\infty \le C_1(\epsilon,\tau) < \infty.$$

Combining the above result with (2.7), we conclude $|\rho|_\infty \le C_1(\epsilon,\tau)$. Therefore, (2.1) is proved.

Now we prove (2.2). Multiplying the first equation in (1.4) by $|\rho|\rho$ and then integrating, we have

$$\begin{aligned}
&\frac{1}{3}\int_{\mathbb{R}} |\rho|^3 dx + \epsilon\int_0^t\int_{\mathbb{R}} |\rho|\rho_x^2\,dx\,dt\\
&= \frac{1}{3}\int_{\mathbb{R}} |\rho_0^m|^3 dx + \int_0^t\int_{\mathbb{R}} \rho u(|\rho|\rho)_x dxdt\\
&= \frac{1}{3}\int_{\mathbb{R}} |\rho_0^m|^3 dx + \int_0^t\int_{\mathbb{R}} \rho(u-c\rho)(|\rho|\rho)_x\,dx\,dt + c\int_0^t\int_{\mathbb{R}} (\rho)^2(|\rho|\rho)_x dxdt\\
&= \frac{1}{3}\int_{\mathbb{R}} |\rho_0^m|^3 dx + \int_0^t\int_{\mathbb{R}} 2|\rho|\rho(u-c\rho)\rho_x dxdt\\
&\le \frac{1}{3}\int_{\mathbb{R}} |\rho_0^m|^3 dx + \tau\int_0^t\int_{\mathbb{R}} \rho_x^2 dxdt + \int_0^t\int_{\mathbb{R}} \frac{\rho^4(u-c\rho)^2}{\tau}\,dx\,dt\\
&\le M\left(|\rho_0^m|_{1,\infty}\right)\left(1+\frac{\tau}{\epsilon}\right).
\end{aligned} \tag{2.14}$$

So the first estimate in (2.2) is proved. Similarly we can prove the second estimate which completes the proof of Lemma 2.1. ♢

From the *a-priori* L^∞ estimates (2.1) we can extend the local solution step by step and get the following global existence theorem.

Theorem 2.2 *If $h(\rho), f(\rho)$ and the initial data satisfy the conditions A1, A2, and A3, then for any fixed ϵ,τ,m satisfying $\tau=o(\epsilon)$, the Cauchy problem (1.4), (1.6) admits a unique, global smooth solution $(\rho^{\epsilon,\tau,m}(t,x),\ u^{\epsilon,\tau,m}(t,x))$ which satisfies the estimates (2.1), (2.2).*

3 Zero Relaxation and Dissipation Limit

In this section, we consider the convergence of solutions $(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))$ to the Cauchy problem (1.4), (1.6) as the dissipation parameter ϵ and the response time τ tend to zero. We show that a L^1-solution of (1.3) with L^1-bounded initial data $\rho_0(x)$ can be given by the limit of $\rho^{\epsilon,\tau,m}(t,x)$ as $\epsilon+\tau+m\to 0+$. The technique to prove the strong convergence is to employ the concept of entropy measure valued solution to (1.3) with initial data $\rho_0^m(x)$ introduced by DiPerna [5].

We show that the Young measure $\nu^m_{(t,x)}$ associated with $\{\rho^{\epsilon,\tau,m}(t,x)\}$ is an entropy measure valued solution of (1.3) with initial data $\rho^m_0(x)$. Then by applying the results given in [16], we get that $\nu^m_{(t,x)}$ is a Dirac measure and the limit function $\rho^m(t,x)$ of $\{\rho^{\epsilon,\tau,m}(t,x)\}$ as ϵ,τ tend to zero related by $\tau = o(\epsilon)$ is the unique L^3-entropy solution of (1.3) with the initial data $\rho^m_0(x)$. To prove that the limit function $\rho(t,x)$ of $\rho^m(t,x)$ as m tends to zero is a L^1-solution of (1.3) with the initial data $\rho_0(x)$, we need the following results of Lax

Lemma 3.1 ([10]) *Let $u(t,x)$ be the entropy solution of the Cauchy problem for the scalar conservation law*

$$\begin{gathered} u_t + (cu^2)_x = 0, \\ u(t,x)|_{t=0} = u_0(x) \in L^1 \cap L^\infty. \end{gathered} \tag{3.1}$$

obtained by Kruzkov in [8]. Then $u(t,x)$ is the almost everywhere unique minimizer of the functional

$$\phi(t,x,v) = \int_x^{x+2cv} u_0(x)dx + cv^2 t. \tag{3.2}$$

If $\rho^m(t,x)$ is the $L^1 \cap L^\infty$-entropy solution for the equation (1.3) with $L^1 \cap L^\infty$ initial data $\rho^m_0(x)$, then from Lemma 3.1 and Diller's result in [4], we have

$$|\rho^m(t,x)| \le \sqrt{\frac{|\rho^m_0(x)|_1}{ct}} \le \sqrt{\frac{|\rho_0(x)|_1}{ct}}\,. \tag{3.3}$$

In fact, we have that for any fixed point (t,x), $\phi(t,x,0) = 0$, and

$$\phi(t,x,v) \ge -\int_{\mathbb{R}} |\rho^m_0(x)|dx + cv^2 t > 0$$

if $|v| \ge \sqrt{|\rho^m_0(x)|_1/(ct)}$, and as an immediate consequence, we have that (3.3) holds almost everywhere.

Theorem 3.2 *The Young measure $\nu^m_{(t,x)}$ associated to the sequence $\{\rho^{\epsilon,\tau,m}(t,x)\}$ is an entropic measure valued solution of the Cauchy problem (1.3) with the initial data $\rho^m_0(x)$.*

Proof It is sufficient to prove the following two estimates [16]:

$$\frac{\partial}{\partial t}\langle \nu^m_{(t,x)}(\lambda), |\lambda - k|\rangle + \frac{\partial}{\partial x}\langle \nu^m_{(t,x)}(\lambda), \operatorname{sign}(\lambda - k)(c\lambda^2 - ck^2)\rangle \le 0 \tag{3.4}$$

for all $k \in \mathbb{R}^1$ in the sense of distributions, and

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I \langle \nu^m_{(t,x)}(\lambda), |\lambda - v_0(x)|\rangle\, dx\, dt = 0 \tag{3.5}$$

for any compact interval $I \in \mathbb{R}$. Since the function $|\rho - k|$ can be approximated by smooth bounded convex functions $\eta(\rho)$ whose first and second derivatives are bounded in $\mathbb{R}$, the following inequality with the Young measure representing weak limit theorem [5, 16] will give the proof of (3.4):

$$\eta(\rho^{\epsilon,\tau,m})_t + q(\rho^{\epsilon,\tau,m})_x \le 0 \tag{3.6}$$

in the sense of distributions, where $q(\rho)$ is a entropy flux of (1.3) corresponding to $\eta(\rho)$. For brevity, we will omit the superscripts ϵ, τ and m in the following.

To prove (3.6), multiplying the first equation in (1.4) by $\eta'(\rho)$, we have

$$\begin{aligned} &\eta(\rho)_t + q(\rho)_x \\ &= -\eta'(\rho)(\rho(u - c\rho))_x + \epsilon\eta'(\rho)\rho_{xx} \\ &= -(\eta'(\rho)\rho(u - c\rho))_x + \eta''(\rho)\rho(u - c\rho)\rho_x - \epsilon\eta''(\rho)\rho_x^2 + \epsilon\eta_{xx}(\rho). \end{aligned} \tag{3.7}$$

From estimates in (2.4) and $\tau = o(\epsilon)$, we have that

$$\begin{aligned} &\int\int_\Omega |\rho\eta''(\rho)(u - c\rho)\rho_x|\,dx\,dt \\ &\le M\Big(\int\int_\Omega \frac{h(\rho)(u - c\rho)^2}{\tau}\,dx\,dt\Big)^{1/2}\Big(\int\int_\Omega \frac{\tau\rho^2\rho_x^2}{h(\rho)}\,dx\,dt\Big)^{1/2} \to 0, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\Big|\int\int_\Omega (\eta'(\rho)\rho(u - c\rho))_x\phi\,dx\,dt\Big| \\ &= \Big|\int\int_\Omega \eta'(\rho)\rho(u - c\rho)\phi_x\,dx\,dt\Big| \\ &\le M\Big(\int\int_\Omega \frac{\tau\rho^2\phi_x^2}{h(\rho)}\,dx\,dt\Big)^{1/2}\Big(\int\int_\Omega \frac{h(\rho)(u - c\rho)^2}{\tau}\,dx\,dt\Big)^{1/2} \to 0 \end{aligned} \tag{3.9}$$

as $\tau = o(\epsilon)$ and ϵ tends to zero for any compact set Ω in $\mathbb{R}\times\mathbb{R}^+$. Moreover since $\eta''(\rho) \ge 0$, and $\epsilon\eta(\rho)_{xx} \to 0$ in the sense of distributions, then (3.6) is proved by letting $\epsilon \to 0$ in (3.7).

The proof of (3.5) can be obtained as in [16]; thus we omit the details. This completes the proof of Theorem 3.2. ♢

Now we give the main result in this section.

Theorem 3.3 *If $h(\rho), f(\rho)$ and the initial data satisfy the conditions A1-A3, then the whole solution sequence of $(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))$ to the Cauchy problem (1.4), (1.6) converges pointwise almost everywhere*

$$(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x)) \to (\rho(t,x), u(t,x))$$

as m, ϵ and τ tend to zero whose relation are given by $\tau = o(\epsilon)$. Here the limit functions $(\rho(t,x), u(t,x))$ satisfy

1. *$u(t,x) = c\rho(t,x)$ for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and*
2. *$\rho(t,x)$ is the unique L^1-entropy solution of the Cauchy problem (1.3) with L^1-bounded initial data $\rho_0(x)$, which satisfies the estimate (3.3).*

Proof From Theorem 3.2 and the results obtained in [16], we conclude that

$$\nu^m_{(t,x)} = \delta_{\rho^m(t,x)}, \quad \text{a. e.},$$

From Lemma 2.1, there exists a subsequence $\{\rho^{\epsilon_k,\tau_k,m}(t,x)\}$ of $\{\rho^{\epsilon,\tau,m}(t,x)\}$ such that

$$\rho^{\epsilon_k,\tau_k,m}(t,x) \to \rho^m(t,x) \quad \text{in } L^1(\mathbb{R}^+ \times \mathbb{R}) \quad \text{as } \epsilon_k + \tau_k \to 0^+ \tag{3.10}$$

provided $\tau_k = o(\epsilon_k)$. One can easily verify that $\rho^m(t,x)$ is a L^∞-entropy week solution, in the sense of [16], to (1.3) with initial data $\rho_0^m(x)$.

On the other hand, (2.3) and A2 imply

$$\lim_{\epsilon_k+\tau_k\to 0+} \int_0^t \int_{\mathbb{R}} |u^{\epsilon_k,\tau_k,m}(t,x) - c\rho^{\epsilon_k,\tau_k,m}(t,x)| \, dx\, dt = 0 \tag{3.11}$$

which means that there exists a function $u^m(t,x) = c\rho^m(t,x)$ such that

$$u^{\epsilon_k,\tau_k,m}(t,x) \to u^m(t,x) \quad \text{in } L^1(\mathbb{R}^+ \times \mathbb{R}) \quad \text{as } \epsilon_k + \tau_k \to 0^+. \tag{3.12}$$

Furthermore, by employing the results obtained in [16] again, we have that the solution $\rho^m(t,x)$ obtained above can be obtained as the strong limit of the solution sequence $\{\rho^{\beta,m}(t,x)\}$ to the following Cauchy problem

$$\begin{gathered} \rho_t + \left(c\rho^2\right)_x = \beta\rho_{xx}, \\ \rho(t,x)|_{t=0} = \rho_0^m(x), \end{gathered}$$

as $\beta \to 0^+$. Since $\rho_0^m(x) \in L^1 \cap L^\infty$, we have from the well-known result of Kruzkov [8] that

$$\int_{\mathbb{R}} |\rho^{m_2}(t,x) - \rho^{m_1}(t,x)| dx \le \int_{\mathbb{R}} |\rho_0^{m_2}(x) - \rho_0^{m_1}(x)| dx, \tag{3.13}$$

and from the discussions after Lemma 3.1, $\rho^m(t,x)$ must satisfy the estimate (3.3). Consequently

$$\rho^m(t,x) \to \rho(t,x) \quad \text{in } L^1(\mathbb{R}^+ \times \mathbb{R}), \tag{3.14}$$

and $\rho(t,x)$ satisfies

$$|\rho(t,x)| \le \sqrt{\frac{|\rho_0(x)|_1}{ct}}. \tag{3.15}$$

Diller's results in [4] show that $\rho(t,x)$ is a L^1-entropy weak solution of (1.3) with L^1-bounded initial data $\rho_0(x)$ in the sense of Kruzkov [8]. So the first assertion of Theorem 3.3 is easy to be verified by (3.11), (3.12) and (1.8).

To conclude that the whole sequence of $\{(\rho^{\epsilon,\tau,m}(t,x), u^{\epsilon,\tau,m}(t,x))\}$ converges almost everywhere to $(\rho(t,x), u(t,x))$, we need only to prove the uniqueness of the L^1-entropy weak solution $\rho(t,x)$, which satisfies (3.15), to the Cauchy problem (1.3) with L^1-bounded initial data $\rho_0(x)$. Due to the estimate (3.15), such a result follows from the same arguments developed by Diller in [4]. This completes the proof of Theorem 3.3.

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