Factorized Spread Sets and Translation Planes with Large Homology Groups

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Abstract

In this note we construct some series of translation planes of order q^2 which possess two commuting homology groups of order $q-1$ and $(q+1)(q-1,2)$ respectively. These translation planes can be considered as relatives of nearfield planes: for nearfield planes the nontrivial part of the associated spread set is a group whereas in our case this set is the product of two groups.

1 Introduction

In this note we construct series of nondesarguesian translation planes of order q^2 which have two commuting affine homology groups with an unfamiliar property: One homology group has order $q-1$ and the other has order $(q+1)(q-1,2)$. They generate a group which has an orbit of length q^2-1 on L_∞ and together with the kern homologies these groups generate the full linear translation complement or a subgroup of index 2. Note that for odd q the extra factor $(q-1,2)=2$ is needed as both homology groups contain -1. In the next section we describe a special construction of translation planes of order q^n, n arbitrary. The nature of this construction implies the existence of two homology groups which together produce an orbit of length q^n-1 on L_∞. From section 3 on we specialize to the case $n=2$. We present in section 3 various series of such planes (of order q^2). In section 4 we show that these planes are nondesarguesian, we compute their automorphism groups, and we determine isomorphisms. Our notation follows standard references like [4] or [6].

2 Notation, factorized spread sets and homologies

We start with a simple construction of spread sets. Rather surprisingly, it seems that this approach has never been considered before. Let L, R be subgroups of $\mathrm{GL}(n, q)$ such that

$$\det(s-1) \neq 0 \text{ for } 1 \neq s \in S_0 = LR \text{ and } |S_0| = q^n - 1. \qquad (*)$$

Clearly, $(*)$ implies that $S = S_0 \cup \{0\}$ is a spread set. Set $K = \mathrm{GF}(q), U = K^n$ and $V = U \times U$. We define $V(\infty) = 0 \times U$ and $V(s) = \{(u, us) \mid u \in U\}$ for $s \in S$. Then $\Sigma = \Sigma_S = \{V(s), V(\infty) \mid s \in S\}$ is a spread of V defining a translation plane $\mathbf{P} = \mathbf{P}(V, \Sigma_S)$ of order q^n. For $x \in L$, $y \in R$ define linear maps $\overline{x}, \overline{y}$ by $\overline{x} : (u, w) \mapsto (ux, w)$ and $\overline{y} : (u, w) \mapsto (u, wy)$. Then $\Sigma = \Sigma\overline{x} = \Sigma\overline{y}$ and $\overline{L} = \{\overline{x} \mid x \in L\}$ is a group of $(V(\infty), (0))$-homologies while $\overline{R} = \{\overline{y} \mid y \in R\}$ is a group of $(V(0), (\infty))$-homologies.
Finally, $\overline{L} \times \overline{R}$ is transitive on $L_\infty - \{(0), (\infty)\}$: Let $s = xy, s' = x'y' \in S_0$ with $x, x' \in L$ and $y, y' \in R$. Set $x_0 = x(x')^{-1}$ and $y_0 = y^{-1}y'$. Then $V(s') = V(s)\overline{x_0 y_0}$. This follows from

$$(u, us)\overline{x_0 y_0} = (ux_0, usy_0) = (w, wx_0^{-1}sy_0) = (w, ws')$$

where $w = ux_0$ for $u \in U$. Hence:

Proposition 2.1 *Assume hypothesis $(*)$. The groups $\overline{L}, \overline{R}$ are groups of affine homologies on $\mathbf{P}(V, \Sigma_S)$. The group $\overline{L} \times \overline{R}$ fixes (0) and (∞) and is transitive on $L_\infty - \{(0), (\infty)\}$.*

Remarks. (a) Assume that S_0 is a group. Then $\mathbf{P}$ is a nearfield plane (as the multiplicative loop of the associated quasifield is associative). In this case any factorization of S_0 by subgroups L, R provides an example. Our aim is to construct examples where S_0 is not a group.
(b) Other translation planes closely related to the nearfield planes include for instance Lüneburg's "merkwürdige Translationsebenen" [7] and translation planes with large groups of $(V(\infty), (0))$-homologies and $(V(0), (\infty))$-homologies of the same order (see [8], chapters 20, 71 for references).

3 Translation planes with factorized spread sets

The translation planes we are going to construct will have order q^2 where q is a prime power with an odd exponent, i.e. $q = p^{2m+1}, p$ a prime. We set $K = \mathrm{GF}(q)$, $F = \mathrm{GF}(q^2)$, and $V = K^4$. Next we describe the subgroups $L, R \leq \mathrm{GL}(2, K)$. For an integer k which will be specified later we set

$$L = \left\{ x_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^k \end{pmatrix} \mid 0 \neq \lambda \in K \right\}.$$

We are looking for a subgroup $R \leq \mathrm{GL}(2, K)$ of order $(q+1)(q-1, 2)$ such that the pair L, R satisfies $(*)$. In order to conveniently control $\det(xy - 1)$, $x \in L, y \in R$, we follow an approach of Oyama [9]: any matrix of the form

$$D = \begin{pmatrix} 1 & 1 \\ a & a^q \end{pmatrix} \in \mathrm{GL}(2, F).$$

transforms $\mathrm{GL}(2, K)$ onto the group $\mathrm{GL}(2, K)^D$ of invertible matrices of the form

$$\begin{pmatrix} u & v^q \\ v & u^q \end{pmatrix}$$

with $u, v \in F$ and in this group we start with the definition of R. Set

$$\tilde{R} = \left\{ z_\rho = \begin{pmatrix} \rho & 0 \\ 0 & \rho^q \end{pmatrix} \mid \rho \in E \right\}$$

where $E \leq F^*$ has order $(q + 1)(q - 1, 2)$. Then $R = \tilde{R}^{D^{-1}} \leq \mathrm{GL}(2, K)$. We will show that we can choose a and k and thus D, L, and R such that $(*)$ holds. Take $x = x_\lambda, z = z_\rho$ with $(\lambda, \rho) \neq \pm(1, 1)$ and set $y = z^{D^{-1}}$. Then $xy - 1 = (xDz - D)D^{-1}$ and in order to satisfy $(*)$ we must have

$$d(\lambda, \rho) = \det(x_\lambda D z_\rho - D) = \begin{vmatrix} \lambda\rho - 1 & \lambda\rho^q - 1 \\ (\lambda^k\rho - 1)a & (\lambda^k\rho^q - 1)a^q \end{vmatrix} \qquad (**)$$
$$= (a^q - a)\rho^{q+1}\lambda^{k+1} + (a\rho - a^q\rho^q)\lambda^k + (a\rho^q - a^q\rho)\lambda + (a^q - a) \neq 0.$$

Remarks. (a) Take $k = 1$. Then $L = Z(\mathrm{GL}(2, K))$ and any cyclic group R of order $(q+1)(q-1, 2)$ produces a cyclic group RL of order $q^2 - 1$ (Singer cycle). Therefore our plane would be a desarguesian plane. We will take $1 < k < q$.

(b) Ranging with the parameter a over $F - K$ means that we range over conjugates of the subgroup R in $\mathrm{GL}(2, K)$. In fact it is not hard to see that one obtains the full conjugacy class.

(c) Our specific choices of the parameters were suggested by computer experiments in small fields. They gave evidence for the following assumptions: (1) q has the form p^{2m+1}, p a prime, and if p is odd one has also $p \equiv 3 \pmod 4$. (2) The parameter a can be chosen in $\mathrm{GF}(p^2) - K$.

(d) Suppose q and k are given and $a \in F - K$ leads to spread set. It is easy to see that any element of the form $a^\gamma u$, $\gamma \in \mathrm{Gal}(F)$, $u \in K^*$, produces an equivalent spread set. However our experiments suggest that no other elements in $F - K$ will result in a spread set for the pair (q, k).

(e) Assume that the parameter $k = k_0$ leads to a spread set $S = LR \cup 0$. Clearly, k_0 and $q - 1$ must be coprime (otherwise a nontrivial element in L has eigenvalue 1). Then there exists a unique $k_1 \in \{1, \ldots, q - 2\}$ with $k_0 k_1 \equiv 1 \pmod{q - 1}$. It is not hard to see that by conjugating S with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ one obtains the spread set which is defined by the parameter $k = k_1$ (with the same D and R).

Before we start with the concrete constructions we prove a technical lemma. This lemma will show that in the generic case our parameter choices used in the constructions never result in a *group* of the form LR.

Lemma 3.1 *Let $q \geq 4$ be a prime power, $1 < k < q - 1$, $(k, q - 1) = 1$, and $L \leq \mathrm{GL}(2, K)$ be defined as above. Let $R \leq \mathrm{GL}(2, K)$ be a group of order $(q + 1)(q - 1, 2)$ which has a cyclic subgroup C of order $q + 1$. Then one of the following holds.*

(a) LR is not a subgroup of $\mathrm{GL}(2, K)$.

(b) LR is a subgroup of $\mathrm{GL}(2, K)$ and q is odd. Moreover $q \equiv 1 \pmod 4$ or $q = 7$, $k = 5$, R is nonabelian, and $LR/\langle -1 \rangle \simeq \mathrm{Sym}(4)$.

Proof. Suppose that LR is a group. By our assumptions $|LR| = q^2 - 1$. Set $M = LR \cap \mathrm{SL}(2, K) \trianglelefteq LR$. The cyclic group C is irreducible. By [5], Kap. II, 7.3 Satz, p. 187, or [3], Chap. 5, 6.3-4, p. 212 we see that $C_1 = C_{\mathrm{GL}(2,K)}(C)$ is a cyclic group of order $q^2 - 1$ and $C_1 \cap \mathrm{SL}(2, K) = C \leq M$. We inspect

Dickson's list of subgroups of $\mathrm{SL}(2, K)$ (see [5], Kap. II, 8.27 Satz). Then $|M| = q+1$ or $2(q+1)$ or $|Z(M)| = 2$ and $M/Z(M) \simeq \mathrm{Alt}(4), \mathrm{Sym}(4)$, or $\mathrm{Alt}(5)$. In the first two cases $C \trianglelefteq LR$ (note if M is nonabelian then $M/\langle -1 \rangle$ is a dihedral group and $C/\langle -1 \rangle$ is the unique cyclic subgroup of maximal order as $|M/\langle -1 \rangle| > 4$ by the assumptions) and hence $LR \leq N_{\mathrm{GL}(2,K)}(C)$. As $|N_{\mathrm{GL}(2,K)}(C) : C_1| = 2$ we see $|LR : (LR \cap C_1)| \leq 2$ and thus $|L : (L \cap C_1)| \leq 2$. This shows $\lambda = \lambda^k$ for $x_\lambda \in L \cap C_1 \leq Z(\mathrm{GL}(2, K))$. As $k \neq 1$ we have $k = \frac{q-1}{2} + 1 = \frac{q+1}{2}$ and q is odd. We assume now that $q \equiv 3 \pmod 4$. Then k is even and k and $q-1$ are not coprime, a contradiction.
We turn to the case that $M/Z(M)$ is alternating or symmetric. As M contains an element of order $q+1$ we have $q+1 \leq 6$ if $M/Z(M) \simeq \mathrm{Alt}(4)$, $q+1 \leq 8$ if $M/Z(M) \simeq \mathrm{Sym}(4)$, and $q+1 \leq 10$ if $M/Z(M) \simeq \mathrm{Alt}(5)$. So if $q \equiv 3 \pmod 4$ we are left with $q = 7$ as $q \geq 4$. Also $|\mathrm{SL}(2,7)|$ is not divisible by 5. Hence $M/Z(M) \simeq \mathrm{Sym}(4)$ and R is not cyclic. Moreover $k = 5$ as k is coprime to $q-1 = 6$. □

Constructions: the case q even

We start with the construction of two series in even characteristic. We set $q = 2^{2m+1}$, $m \geq 1$ and we show that for $k = 3$ and 5 we obtain factorized spread sets. Take $a = \delta$ with $\delta \in F^*$ of order 3 and choose $E \leq F^*$ to be the subgroup of order $q+1$. Then our main equation $(**)$ becomes

$$d(\lambda, \rho) = \lambda^{k+1} + T(\delta\rho)\lambda^k + T(\delta\rho^q)\lambda + 1$$

where $T : F \to K$ is the trace map. For $\rho \in E$ define $f^k_{\delta,\rho} \in F[X]$ by $f^k_{\delta,\rho}(X) = X^{k+1} + T(\delta\rho^q)X^k + T(\delta\rho)X + 1$.

Lemma 3.2 *Let $k = 3$ or 5. Then $f^k_{\delta,\rho}(x) = 0$ for $x \in K$ iff $\rho = x = 1$.*

Proof. We have $\delta^q = \delta^{-1}$, $\delta^2 + \delta + 1 = 0$ and $\rho^q = \rho^{-1}$. The case $\rho = 1$ is obvious. So assume $\rho \neq 1$.

Consider first $k = 3$. A computation shows

$$\begin{aligned} f^3_{\delta,\rho}(X) f^3_{\delta^{-1},\rho}(X) &= X^8 + bX^7 + (b^2+1)X^6 + bX^5 + b^2X^4 + bX^3 \\ &\quad +(b^2+1)X^2 + bX + 1 \\ &= (X^4 + \delta bX^3 + \delta^2 X^2 + bX + \delta) \\ &\quad \cdot (X^4 + \delta^2 bX^3 + \delta X^2 + bX + \delta^2) \end{aligned}$$

with $b = \rho + \rho^q \in K$. Assume that $x \in K$ is a root of $f^3_{\delta,\rho}$. Then $x^4 + \delta b x^3 + \delta^2 x^2 + bx + \delta = 0$ (the other case follows by symmetry) and hence

$$b = \frac{x^4 + \delta^2 x^2 + x}{x(\delta x^2 + 1)}.$$

Multiply nominator and denominator by $\delta^2 x^2 + 1$ and we obtain

$$b = \frac{x^6 + x^4 + x^2 + 1}{x^5 + x^3 + x}\delta + \frac{x^5 + x^3}{x^4 + x^2 + 1}.$$

As b lies in K we get $0 = x^6 + x^4 + x^2 + 1 = (x+1)^6$. But $x = 1$ implies $b = 0$ and hence $\rho = 1$, a contradiction.

Assume next $k = 5$. Now

$$\begin{aligned} f^5_{\delta,\rho}(X) f^5_{\delta^{-1},\rho}(X) &= X^{12} + bX^{11} + (b^2+1)X^{10} + bX^7 + b^2X^6 \\ &\quad + bX^5 + (b^2+1)X^4 + bX + 1 \\ &= (X^6 + \delta b X^5 + \delta^2 X^4 + \delta b X^3 + \delta^2 X^2 + \delta b X + 1) \\ &\quad \cdot (X^6 + \delta^2 b X^5 + \delta X^4 + \delta^2 b X^3 + \delta X^2 + \delta^2 b X + 1). \end{aligned}$$

Assume that $x \in K$ is a root of $f^5_{\delta,\rho}$. Then wlog. $x^6 + \delta b x^5 + \delta^2 x^4 + \delta b x^3 + \delta^2 x^2 + \delta b x + 1 = 0$ and hence

$$b = \frac{x^6 + \delta^2 x^4 + \delta^2 x^2 + 1}{(x^5 + x^3 + x)\delta} = \frac{x^6 + x^4 + x^2 + 1}{x^5 + x^3 + x}\delta + \frac{x^6 + 1}{x^5 + x^3 + x}.$$

We obtain the same contradiction as above. □

Result. For every $q = 2^{2m+1}$, $m \geq 1$ and $k = 3$ or 5 there exist translation planes $\mathbf{P}^k(q) = \mathbf{P}(V, \Sigma_{S^k})$ of order q^2 with a spread set S^k of the desired form $LR \cup \{0\}$. By remark (e) above there are two more parameters k which lead to equivalent spread sets.

The case q odd

In our first construction p is a prime with $p \equiv 3 \pmod 4$. We set $q = p^{2m+1}$, $m \geq 1$ and choose $E \leq F^*$ as the subgroup of order $2(q+1)$. Moreover for any $0 < \ell \leq 2m$ we set $k = p^\ell$ and choose $a \in F - K$ as an element of order 4. Replace in our main equation $(**)$ λ by X and we obtain the polynomials

$$f_\rho^{p,\ell}(X) = 2\rho^{q+1}X^{p^\ell+1} - (\rho + \rho^q)X^{p^\ell} - (\rho + \rho^q)X + 2,$$

$\rho \in E$. The following lemma shows $d(\lambda, \rho) = 0$ iff $(\lambda, \rho) = \pm(1, 1)$ so that we obtain factorized spread sets.

Lemma 3.3 *Let $1 < \ell \leq 2m$. The polynomial $f_\rho^{p,\ell}$ has no roots in K except the in case $\rho = \pm 1$ when the roots are ± 1.*

Proof. Consider first the case $\rho^{q+1} = 1$ and assume that $x \in K$ is a root of $f = f_\rho^{p,\ell}$. Since $\rho^q = \rho^{-1}$ we obtain

$$2x^{p^\ell+1} - (\rho + \rho^{-1})x^{p^\ell} - (\rho + \rho^{-1})x + 2 = 0.$$

Multiplying this equation by ρ we obtain a quadratic equation in ρ whose discriminant is $(2(x^{p^\ell+1} + 1))^2 - 4(x^{p^\ell} + x)^2 = 4(x^2 - 1)^{p^\ell+1}$ which is a square in K. Hence ρ lies in K, i.e. $\rho = \pm 1$. But for $\rho = \pm 1$ we obtain $f = 2(X \pm 1)^{p^\ell+1}$.

Assume now $\rho^{q+1} = -1$ and again assume that $x \in K$ is a root of f. This time we get for ρ the quadratic equation

$$-\rho^2(x^{p^\ell} + x) + \rho(2 - 2x^{p^\ell+1}) + (x^{p^\ell} + x) = 0$$

whose discriminant is $4(x^2 + 1)^{p^\ell+1}$. Thus $\rho \in K$, a contradiction. □

Result. Let p be a prime with $p \equiv 3 \pmod 4$ and set $q = 2^{2m+1}$, $m \geq 1$. For $k = p^\ell$, $0 < \ell \leq 2m$, there exist translation planes $\mathbf{P}^{p^\ell}(q) = \mathbf{P}(V, \Sigma_{S^{p^\ell}})$ of order q^2 with a spread set S^{p^ℓ} of the form $LR \cup \{0\}$.

In our second construction p is again an odd prime with $p \equiv 3 \pmod 4$. In addition we assume $p \not\equiv \pm 1 \pmod 5$. We set $q = p^{2m+1}$, $m \geq 0$ and choose E as before. Also we take $k = 5$. By our assumption and the quadratic reciprocity law -5 is a square in GF(p). Choose $b \in \mathrm{GF}(p)$ with $b^2 = -5$. Then the polynomials $g_\pm = X^2 \pm bX - 1$ have discriminant -1 and are therefore irreducible over K. We pick $a \in F$ to be a root of g_+. Replace in $d(\lambda, \rho)$ the λ by X so that we obtain the polynomials

$$f_\rho^{(5)}(X) = (a^q - a)\rho^{q+1}X^6 + (a\rho - a^q\rho^q)X^5 + (a\rho^q - a^q\rho)X + (a^q - a)$$

$\rho \in E$. The following lemma shows $d(\lambda, \rho) = 0$ iff $(\lambda, \rho) = \pm(1, 1)$ so that we obtain factorized spread sets.

Lemma 3.4 *The polynomial* $f_\rho^{(5)}$ *has no roots in* $K = \mathrm{GF}(q)$ *except in the case* $\rho = \pm 1$ *when the roots are* ± 1.

Proof. Note that $a^q = -a^{-1}$ as $aa^q = aa^p = -1$. Consider first the case $\rho^{q+1} = 1$ and assume that $x \in K$ is a root of $f = f_\rho^{(5)}$. Then

$$\begin{aligned} 0 &= (a^q\rho^{q+1} - a\rho^{q+1})x^6 + (a\rho - a^q\rho^q)x^5 + (a\rho^q - a^q\rho)x + (a^q - a) \\ &= -(a + \frac{1}{a})x^6 + (a\rho + \frac{1}{a\rho})x^5 + (\frac{a}{\rho} + \frac{\rho}{a})x - (a + \frac{1}{a}). \end{aligned}$$

Multiply by ρ and we obtain a quadratic equation in ρ

$$0 = -(a + \frac{1}{a})(x^6 + 1)\rho + (ax^5 + \frac{x}{a})\rho^2 + (\frac{x^5}{a} + ax).$$

As g_+ has roots a and $-a^{-1}$ one has $b = a - a^{-1}$. Squaring implies $a^2 + a^{-2} = -3$. The determinant of D is $-a - a^{-1}$. Hence $\det D^{-1} = a + a^{-1}$ and multiplying our equation with $\det D^{-1}$ the coefficients lie in K. The discriminant of this quadratic polynomial in ρ is

$$\begin{aligned} (x^6+1)^2 - 4(a + \frac{1}{a})^2(ax^5 + \frac{x}{a})(\frac{x^5}{a} + ax) &= x^{12} + 4x^{10} - 10x^6 + 4x^2 + 1 \\ &= (x+1)^2(x-1)^2(x^4 + 3x^2 + 1)^2. \end{aligned}$$

As the discriminant is a square we see that $\rho = \rho^{-1}$ lies in K, i.e, $\rho = \pm 1$. Clearing the leading coefficient in $f = f_{\pm 1}^{(5)}$ we get

$$\begin{aligned} X^6 - X^5 - X + 1 &= (X-1)^2(X^4 + X^3 + X^2 + X + 1), \\ X^6 + X^5 + X + 1 &= (X+1)^2(X^4 - X^3 + X^2 - X + 1). \end{aligned}$$

Then $f_1^{(5)} f_{-1}^{(5)} = (X-1)(X+1)(X^{10} - 1)$. By our assumptions K^* does not contain elements of order divisible by 5. But this forces $\rho = x = \pm 1$.

Assume now $\rho^{q+1} = -1$ and that $x \in K$ is a root of f. A similar computation as in the first case yields

$$\begin{aligned} 0 &= (a^q\rho^{q+1} - a\rho^{q+1})x^6 + (a\rho - a^q\rho^q)x^5 + (a\rho^q - a^q\rho)x + (a^q - a) \\ &= (a + \frac{1}{a})x^6 + (a\rho - \frac{1}{a\rho})x^5 + (\frac{\rho}{a} - \frac{a}{\rho})x - (a + \frac{1}{a}) \end{aligned}$$

Again multiply by $\det D^{-1}$ to obtain coefficients in K. This time the discriminant is

$$(x^6-1)^2 - 4(a+\frac{1}{a})^2(ax^5+\frac{x}{a})(-\frac{x^5}{a}-ax) = x^{12}-4x^{10}+10x^6-4x^2+1$$
$$= (x^2+1)^2(x^2+x-1)^2(x^2-x-1)^2.$$

So $\rho \in K$, a contradiction. □

Result. Let p be a prime with $p \equiv 3 \pmod 4$ and $p \not\equiv \pm 1 \pmod 5$. Set $q = p^{2m+1}$, $m \geq 0$ and $k = 5$. There exist exist translation planes $\mathbf{P}^5(q) = \mathbf{P}(V, \Sigma_{S^5})$ of order q^2 with a spread set S^5 of the form $LR \cup \{0\}$. By remark (e) above there is one more parameter k which produces an equivalent spread set.

Nonabelian variations

Again we assume that q is odd and we make precisely the same choices of q, p, m, k as in the two constructions before. We show that we can replace R by a nonabelian group of the same order. Let $E \leq F^*$ be again the group of order $2(q+1)$ and denote by E_0 the subgroup of index 2. For $\rho \in E_0$ the element z_ρ is defined as above. For $\rho \in E - E_0$ however we set

$$z_\rho = \begin{pmatrix} 0 & \rho^q \\ \rho & 0 \end{pmatrix}.$$

Set further $\tilde{R} = \{z_\rho \mid \rho \in E\}$ and $R = \tilde{R}^{D^{-1}}$ where D is chosen (depending on k, p) in the same fashion as before. Then $R, \tilde{R}$ are nonabelian groups of order $2(q+1)$. For $\rho \in E_0$ we have $d_0(\lambda, \rho) = \det(x_\lambda D z_\rho - D) = d(\lambda, \rho)$ where d is the function d defined by $(**)$. Hence $d_0(\lambda, \rho) = 0$ iff $(\lambda, \rho) = \pm(1, 1)$. For $\rho \in E - E_0$ we obtain $(\rho^{q+1} = -1)$

$$\begin{aligned} d_0(\lambda, \rho) &= \begin{vmatrix} \lambda\rho - 1 & \lambda\rho^q - 1 \\ \lambda^k \rho a^q - a & \lambda^k \rho^q a - a^q \end{vmatrix} \\ &= (a^q - a)\lambda^{k+1} + (a^q\rho - a\rho^q)\lambda^k + (a\rho^q - a^q\rho)\lambda + (a^q - a). \end{aligned}$$

The following lemma shows that $LR \cup 0$ is a spread set.

Lemma 3.5 $d_0(\lambda, \rho) \neq 0$ *for* $(\lambda, \rho) \in K^* \times (E - E_0)$.

Proof. Consider first the case $q = p^{2m+1}$, $m \geq 1$, $k = p^\ell$, and $|a| = 4$ (as in lemma 3.3). Assume that the polynomial (use $a^q = -a$)

$$f_\rho^{p,\ell}(X) = -2X^{p^\ell+1} - (\rho + \rho^q)X^{p^\ell} + (\rho + \rho^q)X - 2,$$

has the root $x \in K$. Arguing similarly as in the proof of lemma 3.3 we obtain the quadratic equation

$$(x - x^{p^\ell})\rho^2 - 2(x^{p^\ell+1} + 1)\rho + (x^{p^\ell} - x) = 0$$

in ρ. It's discriminant is $4(x^2 + 1)^{p^\ell+1}$, i.e. a square in K. But then $\rho \in K$, a contradiction.

Consider now the case $q = p^{2m+1}$, $m \geq 0$, $k = 5$, and a is a root of $g_+ = X^2 + bX - 1$ (as in lemma 3.4). Assume that the polynomial

$$f_\rho^{(5)}(X) = (a^q - a)X^6 + (a^q\rho - a\rho^q)X^5 + (a\rho^q - a^q\rho)X + (a^q - a)$$

has the root $x \in K$. This time we obtain the quadratic equation

$$(\frac{x}{a} - \frac{x^5}{a})(a + \frac{1}{a})\rho^2 + (x^6 + 1)\rho + (ax^5 - ax)(a + \frac{1}{a}) = 0$$

in ρ whose discriminant is $x^{12} - 4x^{10} + 10x^6 - 4x^2 + 1$. We have observed in the proof of lemma 3.4 that this element is a square in K. Again we get $\rho \in K$, a contradiction. Thus the claim is proved in all cases. □

Results. (a) Let p be a prime with $p \equiv 3 \pmod 4$ and set $q = p^{2m+1}$, $m \geq 1$. For $k = p^\ell$, $0 < \ell \leq 2m$, there exist translation planes $\mathbf{P}_0^{p^\ell}(q) = \mathbf{P}(V, \Sigma_{S^{p^\ell,0}})$ of order q^2 with a spread set $S^{p^\ell,0}$ of the form $LR \cup \{0\}$ with a nonabelian group R of order $2(q + 1)$.

(b) Let p be a prime with $p \equiv 3 \pmod 4$ and $p \not\equiv \pm 1 \pmod 5$. Set $q = p^{2m+1}$, $m \geq 0$ and set $k = 5$. There exist translation planes $\mathbf{P}_0^5(q) = \mathbf{P}(V, \Sigma_{S^{5,0}})$ of order q^2 with a spread set $S^{5,0}$ of the form $LR \cup \{0\}$ with a nonabelian group R of order $2(q + 1)$.

Remark. Let $q > 3$. By lemma 3.1 all our planes are not nearfield planes with the only possible exception of $\mathbf{P}_0^5(7)$. We inspect the translation planes of order 49 [2] and denote by L the group of $(V(\infty), (0))$-homologies and by

R the group of $(V(0),(\infty))$-homologies and assume that $\langle L, R\rangle$ has an orbit of length 48. If we exclude the nearfield case ($|L| = |R| = 48$) we are left with three cases and for only one of them $|LR| = 48$ holds and the plane in this case is $\mathbf{P}^5(7)$. This shows that $\mathbf{P}_0^5(7)$ is a nearfield plane.

Summary. We have constructed the following series of translation planes with factorized spread sets $S = LR \cup 0$.

(1) $\mathbf{P}^3(2^{2m+1})$, $\mathbf{P}^5(2^{2m+1})$, $m \geq 1$.

(2) $\mathbf{P}^{p^\ell}(p^{2m+1})$, $\mathbf{P}_0^{p^\ell}(p^{2m+1})$, $m \geq 1$, $0 < \ell \leq 2m$. Here p is a prime with $p \equiv 3 \,(\mathrm{mod}\, 4)$.

(3) $\mathbf{P}^5(p^{2m+1})$, $\mathbf{P}_0^5(p^{2m+1})$, $m \geq 0$. Here p is a prime with $p \equiv 3 \,(\mathrm{mod}\, 4)$ and $p \not\equiv \pm 1 \,(\mathrm{mod}\, 5)$ but $\mathbf{P}^5(3)$, $\mathbf{P}_0^5(3)$, $\mathbf{P}_0^5(7)$ are excluded.

The three excluded planes in (3) are nearfield planes. By lemma 3.1 for none of the remaining planes LR forms a group. In the sequel we call a plane of type (*) if it lies in series (*). The problem of isomorphisms between these planes is addressed in corollary 4.8 in the next section.

4 Automorphism groups

In this section we compute the automorphism groups of the translation planes constructed in section 3. First we present some "obvious" automorphisms and show subsequently that we have obtained all automorphisms. The first result concerns the kernel.

Lemma 4.1 *Consider a translation plane constructed in section 3. Let $\mathcal{K}$ be the group of kern homologies with center $(0,0)$. Then $\mathcal{K} = \{\lambda 1_V \mid \lambda \in K^*\}$. In particular each of the constructed translation planes has kernel* $\mathrm{GF}(q)$.

Proof. Clearly, $\mathcal{K}$ contains $\{\lambda 1_V \mid \lambda \in K^*\}$. Hence the kernel has at least size q. By lemma 3.1 and the summary in section 3 the planes are not desarguesian. The assertion follows. □

4.2 Some automorphisms. We keep the notation of section 3. In particular $V = K^4$, $K = \mathrm{GF}(q)$, and $q = p^{2m+1}$. We denote by H the translation complement fixing $0 \in V$ and by $H_0 = H \cap \mathrm{GL}(V)$ the linear translation complement. We have $M = \overline{LR}\mathcal{K} \le H_0$. If $x \in \overline{L}$, $y \in \overline{R}$ and $xy \in \mathcal{K}$ we must have $|x| = |y|$. Thus $xy \in \mathcal{K}$ iff $x = y = \pm 1$. Hence $|\overline{LR} \cap \mathcal{K}| = (q-1, 2)$ and $|M| = (q^2-1)(q-1)$ follows. Turn now to the planes of type (2). Set

$$\tilde{u} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We compute $wD\tilde{u} = -D$ and observe that $\tilde{u}$ normalizes $\tilde{R}$ and w centralizes L. Set $u = \tilde{u}^{D^{-1}}$. Then:

$$wLRu = LwD\tilde{R}\tilde{u}D^{-1} = LwD\tilde{u}\tilde{R}D^{-1} = LR$$

This shows that $\sigma \in \mathrm{GL}(V)$ defined by $(x, y)\sigma = (xw, yu)$ lies in H_0. The involution σ normalizes M.

Finally we come to semilinear automorphisms. Consider the automorphism of $\mathrm{GL}(2, F)$ which acts on the entries of the matrices as the field automorphism $x \mapsto x^{p^2}$. Clearly, L and $\tilde{R}$ are normalized and D is centralized by this automorphism. This shows that the map $\tau : V \to V$ defined by $(v_1, \ldots, v_4) \mapsto (v_1^{p^2}, \ldots, v_4^{p^2})$ induces an automorphism of order $2m+1$. We will show that we have constructed all automorphisms.

First we determine the structure of affine homology groups and we introduce some notation for this purpose. Let L_1 be the group of $(V(\infty), (0))$-homologies and R_1 the group of $(V(0), (\infty))$-homologies. Clearly, $\overline{L} \le L_1$, $\overline{R} \le R_1$, and L_1 (respectively R_1) can be considered as a fixed-point-free subgroup of $\mathrm{GL}(V(0))$ (respectively $\mathrm{GL}(V(\infty))$). Therefore the orders of both homology groups are coprime to p and a Sylow r-subgroup of such a group is cyclic if r is odd and a generalized quaternion group or a cyclic group if $r = 2$ (see [5], Kap. V, (8.7) Hauptsatz, p. 499). We also consider the subgroup $\overline{L}^2$ of squares in $\overline{L}$. Note that $\widehat{L} = \overline{L}^2 \cap Z(\mathrm{GL}(V(0)))$ has order 1 if we have type (1) or (3) and if $q \equiv 1 \pmod 2$ and $k = p^\ell$ (type (2)) then $\widehat{L}$ is cyclic of order $\frac{p^r-1}{2}$ where $r = (2m+1, \ell)$.

Lemma 4.3 *The group of $(V(\infty), (0))$-homologies is $\overline{L}$ $(= L_1)$.*

Proof. Using [2] we can assume $q \ge 23$ for type (3). Assume first that L_1 is not solvable. By Dickson's theorem $N = L_1 \cap \mathrm{SL}(V(0)) \simeq \mathrm{SL}(2, 5)$, q

is odd. Also we have type (2) since 5 divides q^2-1. The group $\overline{L}^2/C_{\overline{L}^2}(N)$ is isomorphic to a cyclic group of automorphisms of N of odd order, i.e. $|\overline{L}^2 : C_{\overline{L}^2}(N)| = 1, 3$, or 5 (for $\mathrm{Aut}(N)$ see [1] for instance). On the other hand $C_{\mathrm{GL}(V(0))}(N) = Z(\mathrm{GL}(V(0)))$ by Schur's lemma. This shows $\widehat{L} = C_{\overline{L}^2}(N)$. Hence $\frac{p^{2m+1}-1}{2} = |\overline{L}^2| \leq 5\frac{p^r-1}{2}$ (r as above). This is impossible.
Therefore L_1 is solvable. Let $\mathcal{F} = F(L_1)$ be the Fitting subgroup. As $\mathcal{F}$ is nilpotent and fixed-point-free $\mathcal{F} = Z \times Q$ with a cyclic group Z of odd order and Q is a cyclic 2-group or a generalized quaternion group. If $\mathcal{F}$ is an irreducible cyclic group one has $|N_{\mathrm{GL}(V(0))}(\mathcal{F}) : C_{\mathrm{GL}(V(0))}(\mathcal{F})| = 2$ ([5], Kap. II, 7.3 Satz). If Q is a generalized quaternion group one has $|N_{\mathrm{GL}(V(0))}(Q) : C_{\mathrm{GL}(V(0))}(Q)|_{2'} \leq 3$ (it is well known that the odd part of the order of the automorphism group of a generalized quaternion group Q is 3 if $|Q| = 8$ and 1 for $|Q| > 8$). Also as Q is irreducible $Z \leq Z(\mathrm{GL}(V(0)))$ by Schur's lemma. So in both cases $|\overline{L}^2 : C_{\overline{L}^2}(\mathcal{F})| \leq 3$ and $C_{\overline{L}^2}(\mathcal{F}) = \widehat{L}$, which leads to the same contradiction as before.
Hence $\mathcal{F}$ is cyclic and reducible. Then $V(0)$ is the direct sum of two 1-dimensional $\mathcal{F}$-modules. As $\mathcal{F}$ is fixed-point-free it acts faithfully on each submodule, in particular $|\mathcal{F}|$ divides $q-1$ and $|\mathcal{F}|_2 = 2$. If both submodules are isomorphic one has $\mathcal{F} \leq Z(\mathrm{GL}(V(0)))$. Hence $L_1 = C_{L_1}(\mathcal{F}) \leq \mathcal{F}$ ([5], Kap. III, 4.2.b Satz, p. 277) and $\overline{L} = \mathcal{F}$, a contradiction. Hence the two $\mathcal{F}$-modules are nonisomorphic and $\overline{L}^2$ fixes both of them. Again $\overline{L}^2 \leq C_{L_1}(\mathcal{F}) \leq \mathcal{F}$ which implies $\mathcal{F} = \overline{L}$. As a generator of $\overline{L}$ has eigenvalues λ and λ^k and as $\lambda^k \neq \lambda^{-1}$ we must have

$$L_1 \leq N_{\mathrm{GL}(V(0))}(\overline{L}) = C_{\mathrm{GL}(V(0))}(\overline{L}) \simeq \mathrm{C}_{q-1} \times \mathrm{C}_{q-1}.$$

This shows $\overline{L} = L_1$ as L_1 is fixed-point-free. □

Lemma 4.4 *The group of $(V(0), (\infty))$-homologies is $R_1 = \overline{R} \times Z$ with a cyclic group Z isomorphic to $\widehat{L}$. Moreover $R_1 \leq M$.*

Proof. Set $R_0 = R_1 \cap \mathrm{SL}(V(\infty))$. We recall that $\overline{R} \cap R_0$ contains a cyclic group C of order $q+1$. We inspect the Dickson list.
Assume first that q is odd and $R_0/Z(R_0) \simeq \mathrm{Alt}(4), \mathrm{Sym}(4)$, or $\mathrm{Alt}(5)$. Then a cyclic group in R_0 has order at most 10. This shows $q = 7$ and therefore we have the case $\mathbf{P}^5(7)$. By [2] we see that $\overline{R} = R_1$ is a cyclic group of order 16, a contradiction.
So we now assume that $R_0 = C$ is cyclic or $\overline{R} = R_0$ is noncyclic and $|\overline{R} :$

$C| = 2$. Note that $\overline{R}$ can not be a quaternion group of order 8 as otherwise $q = 3$. We claim that C contains an irreducible subgroup C_1 which is normal in R_1:
Take $C_1 = C$ if $R_0 = C$. If $|C|$ is not a 2-power take C_1 as the subgroup of order $|C|_{2'}$. If however R_0 is a noncyclic 2-group, that is a generalized quaternion group of order > 8 we take again $C_1 = C$ which is characteristic in R_0. We have

$$R_1 \leq N = N_{\mathrm{GL}(V(\infty))}(C_1) \simeq \mathrm{C}_{q^2-1} \cdot \mathrm{C}_2$$

(again [5], Kap. III, 4.2.b). Also R_1 can not contain a full Sylow 2-subgroup of N as otherwise it would contain involutions which have the eigenvalue 1. Hence $|R_1|_2 = |\overline{R}|_2$. On the other hand $|N|/2 = |Z(\mathrm{GL}(V(\infty)))\overline{R}|$ which implies $R_1 \leq Z(\mathrm{GL}(V(\infty)))\overline{R}$ and therefore $R_1 = \overline{R} \times Z$ with $Z \leq Z(\mathrm{GL}(V(\infty)))$ of odd order. Let z be a generator of Z. Now $\mathcal{K}$ contains an element z_0 which induces the same transformation as z on $V(\infty)$. This shows $zz_0^{-1} = z_1 \in L_1 = \overline{L}$. So we have even $z_1 \in \widehat{L}$ and of course $|z_1| = |z|$. On the other hand one can modify a generator of $\widehat{L}$ by an element of $\mathcal{K}$ to obtain an element in R_1 and thus in Z. Hence $Z \simeq \widehat{L}$. □

Lemma 4.5 *We have $N_{H_0}(M) = M$ for types (1) and (3) and $N_{H_0}(M) = M\langle\sigma\rangle$ holds for type (2). Moreover $N_{H_0}(M)$ is the stabilizer of the set $\{(0), (\infty)\}$ and fixes both points (0) and (∞).*

Proof. Let $\tilde{H}$ be the stabilizer of the set $\{(0), (\infty)\}$ in H_0. Of course $\tilde{H}$ normalizes M and as $N_{H_0}(M)$ must fix the set of fixed-points of M on L_∞ we obtain $\tilde{H} = N_{H_0}(M)$.

(1) $\tilde{H}$ fixes (0) and (∞): $\tilde{H}$ can not interchange the two points as the group of $(V(\infty), (0))$-homologies is not isomorphic to the group of $(V(0), (\infty))$-homologies.

Assume that z is an element in $\tilde{H}$ but not in M. We denote the restriction of z to $V(0)$ $(V(\infty))$ by z_0 (z_∞). We have already observed in the proofs of 4.3 and 4.4 that $z_0 \in N_{\mathrm{GL}(V(0))}(\overline{L}) \simeq \mathrm{C}_{q-1} \times \mathrm{C}_{q-1}$ and $z_\infty \in N_{\mathrm{GL}(V(\infty))}(C_1) \simeq \mathrm{C}_{q^2-1} \cdot \mathrm{C}_2$ where C_1 is a cyclic, irreducible subgroup of R_1. Restricting if necessary to the prime power parts of z we can assume that $|z|$ is the power of a prime r.

(2) $r = 2$: Assume that r is odd. Note that $(\mathcal{K}\overline{R})_{V(\infty)}$ already contains the full odd part of the group C_{q^2-1}. Hence there is a $s \in \mathcal{K}\overline{R}$ such that $(zs)_{V(\infty)} = 1_{V(\infty)}$, i.e. $zs \in L_1 = \overline{L} \leq M$. Hence $z \in M$, a contradiction.

(3) $q \equiv 1 \,(\mathrm{mod}\, 2)$: If q would be even then $q-1$ is odd and $z_0 = 1_{V(0)}$ and $z \in R_1 \leq M$ would follow, a contradiction.

(4) $z_\infty \in u(\mathcal{K}\overline{R})_{V(\infty)}$ with u as in 4.2: As $(\mathcal{K}\overline{R})_{V(\infty)}$ is a fixed-point-free subgroup of $\mathrm{GL}(V(\infty))$ ($\overline{R}$ and $\mathcal{K}$ are fixed-point-free subgroups and $(\mathcal{K}\cap\overline{R})_{V(\infty)} = \langle -1_{V(\infty)}\rangle$) this group does not contain u. Also $|N_{\mathrm{GL}(V(\infty))}(C_1) : (\mathcal{K}\overline{R})_{V(\infty)}| = 2$. So if the assertion is false we have $z_\infty \in (\mathcal{K}\overline{R})_{V(\infty)}$. But then we get the same contradiction as in (2).

Using (4) we adjust z by an element in $\mathcal{K}\overline{R}$ so that wlog. $z_\infty = u$. Hence $\dim C_{V(\infty)}(z) = 1$. Also a Sylow 2-subgroup of $\mathrm{C}_{q-1} \times \mathrm{C}_{q-1}$ is elementary abelian. This shows $|z_0| = 2$ and $\dim C_{V(0)}(z) = 1$. Adjusting if necessary z by the involution in $\overline{L}$ we also can assume $z_0 = w$ (w as in 4.2), i.e. $z = \sigma$. This implies $LR = wLRu = LwD\tilde{u}\tilde{R}D^{-1}$. Then the element

$$wD\tilde{u} = \begin{pmatrix} -1 & -1 \\ a^q & a \end{pmatrix}$$

lies in $LD\tilde{R}$. The equation $x_\lambda D z_\rho = wD\tilde{u}$ forces $\lambda\rho = -1$, i.e. $x_\lambda = \pm 1$, $z_\rho = \mp 1$, and $a^q = -a$. On the other hand $aa^q = \pm 1$. As a is not in K we get $aa^q = 1$, $|a| = 4$, and we have type (2). All assertions are proved. □

Lemma 4.6 *H_0 fixes (0) and (∞).*

Proof. Assume that H_0 does not fix both points.
Consider first the case that H_0 is transitive on L_∞. Pick $x \in H_0$ with $(0)^x = (\infty)$. Then $\overline{L}^x$ is a group of homologies with center (∞) and axis $V(\infty)^x$. As R_1 is the full group of $(V(0), (\infty))$-homologies we have $V(\infty)^x \neq V(0)$, in particular (0) is not fixed by $\overline{L}^x$. Hence H_0 is even 2-transitive. This shows $|(H_0)_{\{(0),(\infty)\}} : (H_0)_{(0),(\infty)}| = 2$, which contradicts 4.5.
So H_0 is not transitive on L_∞. This implies that H_0 has on L_∞ the orbits $L_\infty - \{(t)\}$ and $\{(t)\}$ for $t = 0$ or ∞. Also q^2 divides $|H_0|$. Both $V(t)$ and $V/V(t)$ are H_0-modules and we denote by W the module which is isomorphic to $V(\infty)$ as a R_1-module. Since $|\mathrm{GL}(W)|_p = q$ we see that H_0 contains a

p-group P, $|P| \geq q$, which acts trivially on W. Choose $1 \neq x \in P$ and $1 \neq y \in R_1$. Then $xy \neq yx$ as otherwise x would fix the axis $V(0)$ and the coaxis $V(\infty)$ of y and p would divide the order of $(H_0)_{(0),(\infty)}$, a contradiction to 4.5. Then

$$1 \neq e = [x, y] = x^{-1}x^y = (y^{-1})^x y$$

acts trivially on $V(t)$ and $V/V(t)$ as x acts trivially on W and y trivially on the other module. Hence e is an elation with axis $V(t)$ and center (t). Let E be the elation group with this axis and center. The homology groups $\overline{L}$ and R_1 act fixed-point-freely on E by conjugation. This shows $|E| = q^2$. Thus our plane is a semifield plane. This implies that $LR \cup 0$ or $(LR)^{-1} \cup 0 = RL \cup 0$ is an additive group. However the cyclic group of order $q+1$ in R generates as an additive group a field isomorphic to $\mathrm{GF}(q^2)$. Hence LR is a multiplicative group, a contradiction. □

We summarize:

Theorem 4.7 *Let* $\mathbf{P}$ *be a translation plane of order* q^2 *with factorized spread set which is described in section 3.*

(a) $\mathbf{P}$ *is a nondesarguesian translation plane with a kernel isomorphic to* $K = \mathrm{GF}(q)$*. In particular* $\mathrm{Aut}(\mathbf{P}) = H \cdot V$*.*

(b) $M = \overline{LR}\mathcal{K}$ *is a group of order* $(q^2-1)(q-1)$ *and* $M/\mathcal{K}$ *acts regularly on* $L_\infty - \{(0), (\infty)\}$*. The linear translation complement is* $H_0 = M\langle\sigma\rangle$ *(i.e.* $|H : M| = 2$*) if* $\mathbf{P}$ *has type (2) and* $H_0 = M$ *for the other types.*

(c) $H = H_0\langle\tau\rangle$ *and therefore* $H/H_0 \simeq \mathrm{C}_{2m+1}$*.*

(d) H *fixes the points* $(0), (\infty)$*.*

Proof. (a) follows from 4.1 and (b) from 4.3-6. As $x \mapsto x^{p^2}$ generates the Galois group of K assertion (c) follows too. (d) is a consequence of the other assertions. □

The translation planes of section 3 are described by the symbols $\mathbf{P}^k(q)$ and $\mathbf{P}^k_0(q)$. We call the numbers q, k, and 0 the parameters of the plane. The next result settles the equivalence problem for our planes.

Corollary 4.8 *Let* $\mathbf{P}$ *and* $\mathbf{P}'$ *be isomorphic translation planes with factorized spread sets as described in section 3. Then both planes have the same parameters or:*

(a) $\mathbf{P} = \mathbf{P}^{p^\ell}(q)$, $\mathbf{P}' = \mathbf{P}^{p^{2m+1-\ell}}(q)$ *or* $\mathbf{P} = \mathbf{P}_0^{p^\ell}(q)$, $\mathbf{P}' = \mathbf{P}_0^{p^{2m+1-\ell}}(q)$ *respectively for all admissible* $q = p^{2m+1}$ *and* $0 < \ell \leq 2m$.

(b) $\mathbf{P} = \mathbf{P}^3(8)$ *and* $\mathbf{P}' = \mathbf{P}^5(8)$.

Proof. Let $S = S_0 \cup 0$, $S_0 = LR$, be the spread set associated with $\mathbf{P}$ and $S' = S_0' \cup 0$, $S_0' = L'R'$, be the spread set associated with $\mathbf{P}'$. An isomorphism from $\mathbf{P}$ to $\mathbf{P}'$ is induced by a transformation ϕ in $\Gamma\mathrm{L}(V)$ which maps Σ_S onto $\Sigma_{S'}$ (see [6], theorem 1.10). As $\mathrm{Aut}(\mathbf{P}')$ covers $\Gamma\mathrm{L}(V)/\mathrm{GL}(V)$ we can adjust ϕ by an element from $\mathrm{Aut}(\mathbf{P}')$ and assume wlog. that $\phi \in \mathrm{GL}(V)$. By 4.3, 4.4 and 4.7 we have $V(0)\phi = V(0)$, $V(\infty)\phi = V(\infty)$ and $\overline{L}^\phi = \overline{L'}$, $R_1^\phi = R_1'$ (of course the symbol $*'$ denotes an object from $\mathbf{P}'$ which corresponds to the object $*$ from $\mathbf{P}$). Hence either both R_1, R_1' are abelian or both are nonabelian, i.e. either both planes have type $\mathbf{P}^*(q)$ or both planes have type $\mathbf{P}_0^*(q)$. By symmetry it is enough to treat the first case.

So assume $\mathbf{P} = \mathbf{P}^k(q)$, $\mathbf{P}' = \mathbf{P}^{k'}(q)$. Let $\overline{x}_\lambda$ be a generator of $\overline{L}$. Then $\overline{x}_\lambda^\phi = \overline{x}'_\mu$ is a generator of $\overline{L'}$. The eigenvalues λ, λ^k of $\overline{x}_\lambda$ have to coincide with the eigenvalues $\mu, \mu^{k'}$ of $\overline{x}'_\mu$. Then either $\lambda = \mu$ and $\lambda^k = \mu^{k'}$ or $\lambda = \mu^{k'}$ and $\lambda^k = \mu$. In the first case $\mathbf{P}$ and $\mathbf{P}'$ have the same parameters.

So assume that we are in the second case. Then $\lambda = \lambda^{kk'}$ and k' is the unique number between 1 and $q-1$ with $kk' \equiv 1 \,(\mathrm{mod}\, q-1)$. Assume q is even and $k = 3$, $k' = 5$. Then $15 \equiv 1 \,(\mathrm{mod}\, q-1)$ which implies $q = 8$ and assertion (b) follows. Assume q odd and $k = p^\ell$. Then $k' = p^{2m+1-\ell}$ (in particular $k' \neq 5$). Now assertion (a) follows.

On the other hand by remark (e) in the introduction of section 3 we have indeed $\mathbf{P}^{p^\ell}(q) \simeq \mathbf{P}^{p^{2m+1-\ell}}(q)$, $\mathbf{P}_0^{p^\ell}(q) \simeq \mathbf{P}_0^{p^{2m+1-\ell}}(q)$, and $\mathbf{P}^3(8) \simeq \mathbf{P}^5(8)$. □

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