CRITICAL POINTS OF INNER FUNCTIONS, NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS, AND AN EXTENSION OF LIOUVILLE’S THEOREM

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Abstract
We establish an extension of Liouville’s classical representation theorem for solutions of the partial differential equation \( \Delta u = 4 e^{2u} \) and combine this result with methods from nonlinear elliptic PDE to construct holomorphic maps with prescribed critical points and specified boundary behaviour. For instance, we show that for every Blaschke sequence \( \{z_j\} \) in the unit disk there is always a Blaschke product with \( \{z_j\} \) as its set of critical points.

Our work is closely related to the Berger–Nirenberg problem in differential geometry.

1. Introduction
In this paper we discuss a method for constructing holomorphic maps with prescribed critical points based on a study of the Gaussian curvature equation

\[ \Delta u = 4 |h(z)|^2 e^{2u}, \tag{1.1} \]

where \( h \) is a holomorphic function on a domain \( \Omega \subset \mathbb{C} \). This technique has several applications to free boundary value problems for holomorphic maps of Riemann–Hilbert–Poincaré type. For instance, we prove the existence of infinite Blaschke products with preassigned branch points satisfying the Blaschke condition. The construction is in two steps. In a first step we find a solution of the curvature equation (1.1) in the unit disk with degenerate boundary data \( u = +\infty \) on the unit circle when \( h(z) \) is an infinite Blaschke product (see Theorem 3.1 below). As we shall see, this is a special case of the Berger–Nirenberg problem in differential geometry, i.e., the question which functions \( \kappa : \mathbb{R} \to \mathbb{R} \) on a Riemann surface \( \mathbb{R} \) arise as the Gaussian curvature of a conformal Riemannian metric \( \lambda(z) |dz| \) on \( \mathbb{R} \). The Berger–Nirenberg problem is well-understood, when the Riemann surface \( \mathbb{R} \) is compact and not the sphere (see for instance Chang [10], and also Moser [27] and Struwe [37] for the case of the sphere), but is still not completely understood for noncompact Riemann surfaces in which case the Berger–Nirenberg problem is concerned with complete conformal Riemannian metrics having prescribed curvature (see for instance Hulin & Troyanov [19] as one of the many references). In a second step, we establish an extension of Liouville’s classical representation formula (see Liouville [23]) for the solutions of the Liouville equation

\[ \Delta u = 4 e^{2u} \tag{1.2} \]

to the more general equation (1.1) (see Theorem 3.3). These two steps combined with some standard results about bounded analytic functions allow a quick construction of the desired Blaschke product (Theorem 2.1).

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Building holomorphic maps with the help of the Liouville equation (1.2) is an old idea and can be traced back at least to the work of Schwarz [35], Poincaré [33], Picard [30, 31, 32] and Bieberbach [4, 5]. In fact, many of the first attempts to prove the Uniformization Theorem for Riemann surfaces were based on Schwarz’ suggestion in [35] to use the partial differential equation $\Delta u = 4e^{2u}$ for this purpose. After Poincaré and Koebe proved the Uniformization Theorem by different means, the method seemed to have only occasionally been used in complex analysis. One notable important exception is M. Heins celebrated paper [16] in which the Schwarz–Picard problem (a special case of the Berger–Nirenberg problem) was solved. We also note that a complete proof of the full Uniformization Theorem via Liouville equation can be found in a recent paper [24] by Mazzeo and Taylor. The main new aspect of the present work is to show that the same method can also be applied in situations when branch points occur even though branching complicates the treatment considerably. Roughly speaking, this is accomplished by replacing Liouville’s equation (1.2) by the Gaussian curvature equation (1.1) with the critical points encoded as the zeros of the holomorphic function $h(z)$.

This paper is organized as follows. In Section 2 we start with a discussion of some free and fixed boundary value problems for analytic maps. Besides the construction of infinite Blaschke products with preassigned critical points mentioned above, we also give the solution to a problem raised by Fournier and Ruscheweyh [12, 13] about “hyperbolic” finite Blaschke products, i.e., bounded analytic maps $f$ defined on a bounded simply connected domain $\Omega \subset \mathbb{C}$ such that

$$\lim_{z \to \partial \Omega} \frac{|f'(z)|}{1 - |f(z)|^2} = 1$$

with finitely many prescribed branch points. These and the other results of Section 2 are mainly intended as examples illustrating the interplay between the Gaussian curvature equation (1.1) and holomorphic functions with prescribed branching and specified boundary behaviour – the main topic of Section 3. There we discuss the basic ingredients we need for the proofs of the results of Section 2: a solution of a special case of the Berger–Nirenberg problem (Theorem 3.1) and an extension of Liouville’s theorem to the solutions of the variable curvature equation (1.1) (see Theorem 3.3). These results are proved in a final Section 4, which also includes a discussion of the necessary tools from nonlinear elliptic partial differential equations and conformal geometry.

In Section 3 we also indicate how methods from complex analysis can be used to some extent to obtain new information about the Berger–Nirenberg problem. Thus the interaction between the complex–analytic and the differential–geometric aspects of the curvature equation works in both ways. For instance, Theorem 3.2 shows that there is always a unique complete conformal Riemannian metric on the unit disk with curvature $\kappa(z) = -4|h(z)|^2$ when $h(z)$ is a Blaschke product.$^\dagger$ Uniqueness results of this kind are usually obtained by making use of Yau’s generalized maximum principle [43, 44], but require that the curvature is bounded above and below by negative constants near the boundary (see Bland & Kalka [7] and Troyanov [39]). Theorem 3.2 allows instead infinitely many zeros of the curvature function, which accumulate at the boundary. It hinges not only on Yau’s maximum principle, but also on a recent boundary version of Ahlfors’ lemma (see Kraus, Roth and Ruscheweyh [20]).

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$^\dagger$On the other hand, when $h$ is a singular inner function, then in general there is more than one such metric, see Example 2.
2. Free and fixed boundary value problems for holomorphic maps with preassigned critical points

A proper holomorphic self-map of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ of degree $n$ has always a representation in terms of its $n$ zeros $a_1, \ldots, a_n \in \mathbb{D}$ (with possible repetitions) as a finite Blaschke product of the form

$$f(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j}z}, \quad |\lambda| = 1. \quad (2.1)$$

While for many questions in classical complex analysis such a representation is perfectly well suited, it is not very helpful to study Blaschke products in their dependence on their critical points. Knowledge about the critical points of finite Blaschke products, however, is crucial in a number of applications e.g. when one studies the parameter space for complex polynomials (see Milnor [25]). A second motivation for considering critical points of Blaschke products comes from the new theory of discrete analytic functions, where discrete Blaschke products are most naturally defined in terms of their branch points (see Stephenson [36]).

In fact finite Blaschke products are uniquely determined up to normalization by their critical points and these critical points can always be prescribed arbitrarily:

**Theorem A.** Let $z_1, \ldots, z_n \in \mathbb{D}$ be given (not necessarily distinct) points. Then there is always a finite Blaschke product $f$ of degree $n + 1$ with critical points $z_j$ and no others. The Blaschke product $f$ is uniquely determined up to postcomposition with a conformal automorphism of the unit disk.

The uniqueness statement in Theorem A is fairly straightforward and follows easily for instance from Nehari's generalization of Schwarz' lemma (see [28, Corollary to Theorem 1]). To the best of our knowledge, the existence–part of Theorem A was first proved by M. Heins [16, Theorem 29.1]. Heins' argument is purely topological. He showed that the set of critical points of all finite Blaschke products of degree $n + 1$, which is clearly closed, is also open in the polydisk $\mathbb{D}^n$ by using Brouwer's fixed point theorem. Similar proofs were later given by Q. Wang & J. Peng [40], by T. Bousch in his thesis [8], and by S. Zakeri [45]. They consider the map $\Phi$ from $(n+1)$-tuples of zeros to $n$-tuples of critical points (one degree of freedom being used for normalization) and show that $\Phi$ is proper from the polydisk to the polydisk. Again, with invariance of domain, this implies $\Phi$ is onto; see [45] for the details. Bousch shows that $\Phi$ is even an analytic diffeomorphism [8]. In particular, all these proofs are nonconstructive.

An entirely novel and constructive approach to Theorem A based on Circle Packing has recently been devised by Stephenson (see [36, Theorem 21.1]), who builds discrete finite Blaschke products with prescribed branch set and shows that they converge locally uniformly in $\mathbb{D}$ to a finite classical Blaschke product with specified critical points.

The method we employ in the present paper for constructing Blaschke products with pre-scribed critical points differs considerably from the techniques described above and is also constructive in nature.\(^\dagger\) In addition, it has the advantage that it permits the construction of infinite Blaschke products with infinitely many prescribed critical points $z_1, z_2, \ldots$ provided

\(^\dagger\)However, although our method is constructive, it is nevertheless not really suitable to find a Blaschke product with prescribed critical points in an explicit form, even for finitely many critical points. Is there a finite algorithm, which allows one to compute a finite Blaschke product from its critical points?
\{z_j\} \subset \mathbb{D} is a Blaschke sequence, i.e., it satisfies the Blaschke condition

\[ \sum_{j=1}^{\infty} 1 - |z_j| < \infty. \tag{2.2} \]

Thus we have the following generalization of Theorem A.

**Theorem 2.1.** Let \{z_j\} \subseteq \mathbb{D} be a Blaschke sequence. Then there exists a Blaschke product with critical points \{z_j\} (counted with multiplicity) and no others.

Some remarks are in order. First note that unlike Theorem A there is no corresponding uniqueness statement in Theorem 2.1. In fact, an infinite Blaschke product is not necessarily determined by its critical points up to postcomposition with a conformal disk automorphism. Here is a very simple example, when there are no critical points at all.

**Example 1.** Let \( f \) be a Blaschke product which is also a universal covering map of the unit disk onto a punctured disk \( \mathbb{D} \setminus \{a\}, a \neq 0 \). For instance, one can take for \( f \) any Frostman shift \( \tau_\alpha \circ F \) of the standard universal covering \( D \to D \setminus \{0\} \), \( F(z) = \exp\left(-\frac{1+z}{1-z}\right) \) with a disk automorphism \( \tau_\alpha(z) := \frac{z + \alpha}{1 + \alpha z} \), provided \( \alpha \in \mathbb{D} \setminus \{0\} \). To check that \( \tau_\alpha \circ F \) is a Blaschke product for every \( \alpha \in \mathbb{D} \setminus \{0\} \) it suffices to note that none of its angular limits is 0, so it can have no singular inner factor (see for instance [14, Ch. II, Theorem 6.2]). Now, \( g(z) := f(-z) \) is also an infinite Blaschke product with the same critical points as \( f \), but clearly \( f \neq T \circ g \) for any disk automorphism \( T \).

Secondly, for infinitely many branch points one cannot completely distinguish between inner functions and Blaschke products in Theorem 2.1. In fact, for each inner function there are many Blaschke products with exactly the same critical points. This is an immediate consequence of Frostman’s theorem ([14, Ch. II, Theorem 6.4]) that for any inner function \( F \) every Frostman shift \( \tau_\alpha \circ F \) is a Blaschke product for all \( \alpha \in \mathbb{D} \) except for a set of capacity zero.

Thirdly, the condition in Theorem 2.1 that \( \{z_j\} \) is a Blaschke sequence might be compared with a result of M. Heins [16], who showed that for any bounded nonconstant holomorphic map \( f \) defined in \( \mathbb{D} \) the critical points which are contained in some fixed horocycle \( H(\omega, \lambda) := \left\{ z \in \mathbb{D} : |1 - z \omega|^2 < \lambda (1 - |z|^2) \right\} \), \( \omega \in \partial \mathbb{D}, \lambda > 0 \), satisfy the Blaschke condition.\(^{\dagger}\) Hence there is a considerable gap between the sufficient condition of Theorem 2.1 for the critical points of an infinite Blaschke product and the above necessary condition of Heins. One is inclined to ask whether for each nonconstant analytic map \( f : \mathbb{D} \to \mathbb{D} \) there is always an inner function or a Blaschke product with the same critical points as \( f \).

Fourthly, we wish to point out that the proof we give for Theorem 2.1 will show that in the special case of finitely many branch points \( z_1, \ldots, z_n \in \mathbb{D} \) there is always a finite Blaschke product (of degree \( N \) say) with exactly these critical points. The Riemann–Hurwitz formula then implies \( N = n + 1 \). Thus the existence–part of Theorem A might be considered as a special case of Theorem 2.1, see Remark 5 below.

\(^{\dagger}\)Even more is true: \( f' \) restricted to any horocycle is of bounded characteristic.
The method we use to establish Theorem 2.1 is not restricted to the construction of Blaschke products. It can also be used to prove the following extension of results due to Fournier and Ruscheweyh [12, 13] and Kühnau [21] (see also Agranovsky and Bandman [1]).

**Theorem 2.2.** Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected domain, $z_1, \ldots, z_n$ finitely many points in $\Omega$, and $\phi : \partial \Omega \to \mathbb{R}$ a continuous positive function. Then there exists a holomorphic function $f : \Omega \to \mathbb{D}$ with critical points at $z_j$ (counted with multiplicities) and no others such that

$$
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi), \quad \xi \in \partial \Omega.
$$

(2.3)

If $g : \Omega \to \mathbb{D}$ is another holomorphic function with these properties, then $g = T \circ f$ for some conformal disk automorphism $T : \mathbb{D} \to \mathbb{D}$.

**Remark 1.**

(a) Choosing $\phi \equiv 1$, we see in particular that there is always a holomorphic solution $f : \Omega \to \mathbb{D}$ of the nonlinear boundary value problem of Riemann–Hilbert–Poincaré type

$$
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = 1, \quad \xi \in \partial \Omega,
$$

(2.4)

with prescribed finitely many critical points in $\Omega$. As indicated above, this solves a problem which arises from the work in [12, 13, 21]. There, Theorem 2.2 was proved for the special case $\phi \equiv 1$ and $\Omega = \mathbb{D}$ by using a completely different method, which can be traced back to Beurling’s celebrated extension of the Riemann mapping theorem [3]. First, the problem is transferred to an integral equation which is then solved by (i) iteration for a single critical point in [21] and by (ii) applying Schauder’s fixed point theorem in the general case in [13]. In fact, an inspection of this method shows that it requires at least some amount of regularity of $\Omega$ and doesn’t seem to be capable of yielding the full result of Theorem 2.2 for general bounded simply connected domains.

(b) In general Theorem 2.2 does not hold when $\Omega$ is not simply connected. See Example 3 below.

(c) Since (2.3) and (2.4) are free boundary value problems for the analytic map $f$ it is at first glance a little surprising that one needs no assumptions on the boundary regularity of $\Omega$ at all in Theorem 2.2. The point is that one can view (2.3) and (2.4) as a fixed boundary value problem for the conformal pseudo–metric

$$
\lambda(z) |dz| := \frac{|f'(z)|}{1 - |f(z)|^2} |dz|,
$$

i.e., $u(z) := \log \lambda(z)$ is a solution to Liouville’s equation $\Delta u = 4e^{2u}$ in $\Omega$ (except for the critical points) with fixed boundary values. In order to solve such a fixed boundary value problem it suffices that the domain $\Omega$ is regular for the Laplace operator $\Delta$ in the sense of potential theory.

The boundary conditions (2.3) and (2.4) involve unrestricted approach to $\xi \in \partial \Omega$ from inside. If $\Omega$ is a smooth domain, then we can relax this condition to nontangential limits and allow infinitely many critical points for $f$. This is the content of the following theorem, which we formulate for simplicity only for the case $\Omega = \mathbb{D}$. Here and in the sequel we use the notation $\angle \lim$ to indicate nontangential (angular) limits.

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†For instance $\Omega$ of Smirnov–type (see [34]) would suffice.
Theorem 2.3. Let \( \{z_j\} \) be a Blaschke sequence in \( \mathbb{D} \) and let \( \phi : \partial \mathbb{D} \to (0, \infty) \) be a function such that \( \log \phi \in L^\infty(\partial \mathbb{D}) \). Then there exists a holomorphic function \( f : \mathbb{D} \to \mathbb{D} \) with critical points \( z_j \) (counted with multiplicities) such that
\[
\sup_{z \in \mathbb{D}} \frac{|f'(z)|}{1 - |f(z)|^2} < \infty, \tag{2.5}
\]
and
\[
\angle \lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi) \quad \text{for a.e. } \xi \in \partial \mathbb{D}.
\]
If \( g : \mathbb{D} \to \mathbb{D} \) is another holomorphic function with these properties, then \( g = T \circ f \) for some conformal disk automorphism \( T : \mathbb{D} \to \mathbb{D} \).

Remark 2. Theorem 2.3 has the following obvious partial converse. If \( f : \mathbb{D} \to \mathbb{D} \) is a non–constant holomorphic function such that (2.5) holds, then the non–tangential limit
\[
\angle \lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} =: \phi(\xi)
\]
exists for a.e. \( \xi \in \partial \mathbb{D} \) and \( \log \phi \in L^1(\partial \mathbb{D}) \). Moreover, the critical points of \( f \) (counted with multiplicity) satisfy the Blaschke condition. To check this it suffices to say that (2.5) forces \( f' \) to be bounded on \( \mathbb{D} \), so \( f \) has a continuous extension to \( \overline{\mathbb{D}} \).

Theorems 2.1, 2.2 and 2.3 are all of a similar flavour and will be proved in a unified way in Section 4 below. However, there are also a number of differences. For instance, unlike Theorem 2.1 we also have a uniqueness statement in Theorem 2.3. On the other hand, Theorems 2.1 and 2.3 deal with analytic maps defined on the unit disk, whereas Theorem 2.2 is valid for any simply connected domain \( \Omega \) regardless of the complexity of its boundary.

3. The Gaussian curvature equation, critical points of holomorphic maps, and the Berger–Nirenberg problem

The idea of the proof of Theorem 2.1 is based on the following simple observation. If \( f : \mathbb{D} \to \mathbb{D} \) is a bounded holomorphic map and the zeros \( \{z_j\} \) of \( f' \) form a Blaschke sequence, then the Blaschke product
\[
B(z) := \prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z}_j z}
\]
(3.1)
is a holomorphic self–map of \( \mathbb{D} \). In particular,
\[
\lambda(z) := \frac{|f'(z)|}{1 - |f(z)|^2} \frac{1}{|B(z)|}
\]
defines the density of a conformal Riemannian metric \( \lambda(z)|dz| \) on \( \mathbb{D} \). A quick computation shows that the Gaussian curvature
\[
\kappa_\lambda(z) := -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}
\]
of this metric is
\[
\kappa_\lambda(z) = -4 |B(z)|^2.
\]
Hence the metric \( \lambda(z)|dz| \) has nonpositive curvature and the curvature function \( \kappa_\lambda \) is the negative square–modulus of a bounded holomorphic function, which vanishes exactly at the critical points of \( f \). In other words, the function
\[
u(z) := \log \lambda(z)
\]
is a smooth solution to the Gaussian curvature equation \( \Delta u = 4 |B(z)|^2 e^{2u} \). Note that the critical points \( z_j \) are encoded as the zeros of the function \( B(z) \). Thus this PDE (or, what is the same, the curvature of the metric \( \lambda(z) \) \( |dz| \)) is uniquely determined by the critical points \( \{z_j\} \) of \( f \). If we assume momentarily that \( f \) is a finite Blaschke product, then \( B \) is also a finite Blaschke product, and the metric \( \lambda(z) \) \( |dz| \) clearly blows up at the unit circle:

\[
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} \frac{1}{|B(z)|} = +\infty, \quad \xi \in \partial \mathbb{D}.
\]

Thus the function \( u(z) = \log \lambda(z) \) is a smooth solution to the boundary value problem

\[
\Delta u = 4 |B(z)|^2 e^{2u} \quad \text{in } \mathbb{D},
\]

\[
\lim_{z \to \xi} u(z) = +\infty \quad \text{for every } \xi \in \partial \mathbb{D}.
\]

(3.3)

The key idea of the proof of Theorem 2.1 is now to reverse these considerations. Given a Blaschke sequence \( \{z_j\} \) we form the Blaschke product \( B \) via (3.1). In a first step we then show that the boundary value problem (3.3) always has a \( C^2 \)-solution \( u : \mathbb{D} \to \mathbb{R} \) (see Theorem 3.1 below). In a second step, we reconstruct from the corresponding conformal Riemannian metric \( \lambda(z) \) \( |dz| := e^{u(z)} |dz| \) a holomorphic function by solving the equation (3.2) for \( f \) (see Theorem 3.3). In order to complete the proof of Theorem 2.1 it finally remains to exploit the boundary condition in (3.3) to show that \( f \) may be taken to be a Blaschke product.

In fact, the two main steps of the proof of Theorem 2.1 as outlined above might both be stated in more general form which we shall need later. For instance, the boundary value problem (3.3) might be solved for any bounded regular domain, that is, for any bounded domain possessing a Green’s function which vanishes continuously on the boundary.

**Theorem 3.1.** Let \( \Omega \subseteq \mathbb{C} \) be a bounded regular domain and \( h : \Omega \to \mathbb{C} \), \( h \neq 0 \), a bounded holomorphic function. Then there exists a \( C^2 \)-solution \( u : \Omega \to \mathbb{R} \) to

\[
\Delta u = 4 |h(z)|^2 e^{2u} \quad \text{in } \Omega
\]

(3.4)

such that

\[
\lim_{z \to \xi} u(z) = +\infty \quad \text{for every } \xi \in \partial \Omega.
\]

(3.5)

**Remark 3.** Thus if \( u : \Omega \to \mathbb{R} \) is a \( C^2 \)-solution to the boundary value problem (3.4)–(3.5), then the conformal Riemannian metric \( \lambda(z) \) \( |dz| := e^{u(z)} |dz| \) has Gaussian curvature \( -4 \) \( |h(z)|^2 \).

If \( \Omega \) is in addition a smooth domain (e.g. \( \partial \Omega \) is of class \( C^2 \)), then the boundary condition (3.5) implies that this metric is even a complete conformal Riemannian metric for \( \Omega \). This follows e.g. from the boundary version of Ahlfors’ lemma in [20] (see in particular [20, Theorem 5.1]).

Thus Theorem 3.1 might be considered as a solution of the **Berger–Nirenberg problem** in the very special case that the curvature of the metric \( \lambda(z) \) \( |dz| \) is of the form \( \kappa_\lambda(z) = -4 \) \( |h(z)|^2 \) for a bounded holomorphic function \( h : \Omega \to \mathbb{C} \). Related results have been obtained for instance by Bland & Kalka [7] and Hulin & Troyanov [19]. They allow more general curvature functions \( \kappa_\lambda(z) \) (is only assumed to be Hölder continuous), but they require \( \kappa_\lambda(z) \) to be bounded below and above by negative constants near the boundary. Theorem 3.1 deals with a case of the Berger–Nirenberg problem, when the curvature is only nonpositive, i.e., \( \kappa_\lambda(z) \) is allowed to vanish arbitrarily close to the boundary.

Next, we consider the problem of uniqueness of solutions to the boundary value problem (3.4)–(3.5). This is a rather delicate matter. In general the boundary value problem (3.4)–(3.5) will have more than one solution, so in view of Remark 3 there will be more than one complete conformal Riemannian metric on \( \Omega \) having curvature \( -4 \) \( |h(z)|^2 \). This is illustrated with the following example, where the curvature comes from a singular inner function.
Example 2. Let $h : \mathbb{D} \to \mathbb{D}$ be the singular inner function

\[ \exp \left( -\frac{1 + z}{1 - z} \right). \]

Then a short calculation shows that

\[ u_1(z) = \log \left[ \frac{1}{1 - |z|^2} \frac{1}{|h(z)|} \right], \]
\[ u_2(z) = \log \left[ \frac{|h'(z)|}{1 - |h(z)|^2} \frac{1}{|h(z)|} \right], \]

are two different solutions to (3.4)–(3.5), so $e^{u_1(z)} |dz|$ and $e^{u_2(z)} |dz|$ are two complete conformal Riemannian metrics on $\mathbb{D}$ with the same curvature.

At first glance, the nonuniqueness here might have been caused by the fact that the curvature $-4 |h(z)|^2$ is not bounded from above by a negative constant. Indeed, Bland & Kalka [7], Troyanov [38], and Hulin & Troyanov [19] have shown that if the curvature is bounded below and above by negative constants near the boundary of $\Omega$, then there is at most one complete conformal Riemannian metric $\lambda(z) |dz|$ on $\Omega$ with this curvature. These uniqueness results are essentially based on Yau’s generalized maximum principle [43, 44] which seems to require an upper negative bound on the curvature at least near the boundary. Our next result, however, gives a uniqueness result when the curvature is allowed to vanish close to the boundary, even though it only deals with a very special situation.

Theorem 3.2. Let $h$ be a Blaschke product. Then there exists a unique complete regular\(^1\) conformal Riemannian metric $\lambda(z) |dz|$ on $\mathbb{D}$ with curvature $-4 |h(z)|^2$.

Theorem 3.2 and Example 2 indicate that even for the very special situation of the unit disk $\mathbb{D}$ and curvature functions of the form $-4 |h(z)|^2$ for bounded holomorphic functions $h$ on $\mathbb{D}$, the question whether there is a unique complete conformal Riemannian metric with curvature $-4 |h(z)|^2$ seems to be quite intricate. We only mention here that the uniqueness result of Theorem 3.2 can easily be extended to bounded holomorphic functions $h$ without singular inner factor in their canonical factorization (see [14, Ch. II, Corollary 5.7]), i.e.,

\[ h(z) = C B(z) S(z), \quad |C| = 1, \]

where $B$ is a Blaschke product, when we assume in addition that the outer factor

\[ S(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f^*(e^{i\theta})| d\theta \right) \]

is generated by an $L^1$–function $\log |f^*(e^{i\theta})|$ which is bounded away from $-\infty$.

The above considerations all deal with particular cases of the Berger–Nirenberg problem, when the curvature has the form $-4 |h(z)|^2$ for some holomorphic function $h$. Theses cases appear to be very restrictive from the viewpoint of differential geometry and nonlinear partial differential equations. However, from the point of view of complex analysis, they are most relevant for constructing holomorphic maps with prescribed critical points.

Theorem 3.3. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, $h : \Omega \to \mathbb{C}$ a holomorphic function, $h \neq 0$, and $u : \Omega \to \mathbb{R}$ a $C^2$–solution to $\Delta u = 4 |h(z)|^2 e^{2u}$ in $\Omega$. Then there exists a

\(^1\)We call a conformal Riemannian metric $\lambda(z) |dz|$ on a domain $\Omega \subseteq \mathbb{C}$ regular, if its density $\lambda : \Omega \to (0, +\infty)$ is of class $C^2$.
holomorphic function \( f : \Omega \to \mathbb{D} \) such that
\[
u(z) = \log \left( \frac{|f'(z)|}{1 + |f(z)|^2 |h(z)|} \right).
\]

Moreover, \( f \) is uniquely determined up to postcomposition with a conformal automorphism of the unit disk.

**Remark 4.**

(i) For the special choice \( h(z) \equiv 1 \) Theorem 3.3 is a classical result due to Liouville [23]. The function \( f \) is then locally univalent and is sometimes called developing map of \( u \).

(ii) Since \( u \) is a \( C^2 \)–solution, every holomorphic function \( f \) satisfying (3.6) has its critical points exactly at the zeros of the given holomorphic function \( h \) (counted with multiplicity).

(iii) There is a counterpart of Theorem 3.3 for conformal Riemannian metrics with nonnegative curvature: If \( h \) is a holomorphic function on a simply connected domain \( \Omega \), then every \( C^2 \)–solution to \( \Delta u = -4 |h(z)|^2 e^{2u} \) has a representation of the form
\[
u(z) = \log \left( \frac{|f'(z)|}{1 + |f(z)|^2 |h(z)|} \right),
\]
where \( f \) is a meromorphic function on \( \Omega \). Moreover, \( f \) is uniquely determined up to postcomposition with a rigid motion of the Riemann sphere. This result can be proved in exactly the same way as Theorem 3.3.

There are many proofs of Liouville’s theorem (the case \( h(z) \equiv 1 \) of Theorem 3.3) scattered throughout the literature (see for instance Bieberbach [5], Nitsche [29] and Minda [26]) and they continue to appear (see [9]). They basically use the fact that for \( h(z) \equiv 1 \) the Schwarzian derivative of the conformal Riemannian metric \( e^{u(z)} |dz| \), i.e.,
\[
\frac{\partial u^2}{\partial z^2}(z) - \left( \frac{\partial u}{\partial z}(z) \right)^2,
\]
is a holomorphic function \( A(z) \) in \( \Omega \) and \( f \) can be found among the solutions to the Schwarzian differential equation
\[
\left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = 2 A(z).
\]
Since \( A(z) \) is holomorphic in \( \Omega \) every solution to this differential equation is meromorphic in \( \Omega \). The main difficulty in proving the more general Theorem 3.3 is that now \( A(z) \) is meromorphic in \( \Omega \) and indeed has poles of order 2 exactly at the zeros of \( h(z) \). In the theory of Schwarzian differential equations (see Laine [22]) this is known to be the most complicated situation. Nevertheless, it turns out that the associated Schwarzian differential equation still has only meromorphic solutions, but this requires a considerable amount of work (cf. Section 4.4, in particular Lemma 4.3).

An immediate consequence of Theorem 3.3 is that the equation \( \Delta u = 4 |h(z)|^2 e^{2u} \) has no \( C^2 \)–solution \( u : \mathbb{C} \to \mathbb{R} \) if \( h(z) \) is an entire function. For the case \( h(z) \equiv 1 \) this observation is due to Wittich [42], Nitsche [29] and Warnecke [41].
4. Proofs

4.1. Proof of Theorem 2.1

Let \( \{z_j\} \) be a Blaschke sequence in \( \mathbb{D} \). Then the corresponding Blaschke product

\[
B(z) := \prod_{j=1}^{\infty} \frac{z - z_j}{\overline{z}_j - z}
\]

converges locally uniformly in \( \mathbb{D} \), so \( B : \mathbb{D} \to \mathbb{D} \) is a holomorphic function. We employ Theorem 3.1 with \( h = B \) and get a \( C^2 \)-solution \( u : \mathbb{D} \to \mathbb{R} \) of \( \Delta u = 4 |B(z)|^2 e^{2u} \) in \( \mathbb{D} \) with \( u = +\infty \) on \( \partial \mathbb{D} \). Theorem 3.3 gives us a holomorphic function \( f : \mathbb{D} \to \mathbb{D} \) with

\[
u(z) = \log \left( \frac{|f'(z)|}{1 - |f(z)|^2} \right).
\]

Notice that the critical points of \( f \) are exactly the zeros of \( B \), i.e., the points \( z_j, j = 1, 2, \ldots \). We claim that \( f \) is an inner function. This is not difficult to see by considering the sets

\[
A := \left\{ \xi \in \partial \mathbb{D} \mid f(\xi) := \angle \lim_{z \to \xi} f(z) \text{ exists and } f(\xi) \in \partial \mathbb{D} \right\},
\]

\[
A' := \left\{ \xi \in \partial \mathbb{D} \mid f(\xi) := \angle \lim_{z \to \xi} f(z) \text{ exists and } f(\xi) \in \mathbb{D} \right\}.
\]

By Fatou’s theorem [14, Chapter I.5] the union \( A \cup A' \) has (one dimensional) Lebesgue measure \( 2\pi \). If \( \xi \in A' \), then

\[
\angle \lim_{z \to \xi} \frac{|f'(z)|}{|B(z)|} = \angle \lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} \frac{1}{|B(z)|} (1 - |f(z)|^2) = \angle \lim_{z \to \xi} \left[ e^{\nu(z)} (1 - |f(z)|^2) \right] = +\infty
\]

by construction. Thus the holomorphic function \( B/f' \) has angular limit 0 at every point of the set \( A' \). By Privalov’s theorem [14, p. 94] and noting that \( B/f' \) does not vanish identically, \( A' \) is a nullset, so \( A \) has measure \( 2\pi \) and \( f \) is an inner function. Frostman’s theorem (see [14, Ch. II, Theorem 6.4]) now guarantees that

\[
f_{\alpha}(z) = \frac{f(z) - \alpha}{1 - \overline{\alpha} f(z)}
\]

is a Blaschke product for all \( \alpha \in \mathbb{D} \) except for a set of capacity zero. In particular, \( f_{\alpha} \) is a Blaschke product for some \( |\alpha| < 1 \). Since

\[
f_{\alpha}'(z) = \frac{1 - |\alpha|^2}{(1 - \overline{\alpha} f(z))^2} f'(z),
\]

\( f_{\alpha} \) is a Blaschke product with critical points \( \{z_j\} \) and no others. \( \square \)

Remark 5. If \( \{z_j\} \) is a finite sequence of \( n \) points in \( \mathbb{D} \), then \( B(z) \) in the above proof is a finite Blaschke product, so

\[
\angle \lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = +\infty \quad \text{for every } \xi \in \partial \mathbb{D}
\]

by construction. Corollary 1.10 in [20] (which improves upon an earlier result of Heins [17]) then implies that \( f \) is a finite Blaschke product. The degree of this Blaschke product must be \( n+1 \) in view of the Riemann–Hurwitz formula and the existence–part of Theorem A follows.
4.2. Tools from PDE and conformal geometry

In this paragraph we collect some well–known facts from conformal geometry and about the Gaussian curvature equation which we shall need in the sequel.

The basic example of a conformal Riemannian metric is the Poincaré metric $\lambda_\Omega(z) \, |dz|$ on the unit disk. Its density is given by

$$\lambda_\Omega(z) = \frac{1}{1 - |z|^2}.$$  

When the domain $\Omega \subset \mathbb{C}$ has at least two boundary points, usually dubbed hyperbolic domain, and $-a$ is a negative constant, then $\Omega$ carries a unique complete regular conformal Riemannian metric with constant curvature $-a$. This metric $\lambda(z) \, |dz|$ is obtained from the Poincaré metric $\lambda_\Omega(z) \, |dz|$ by means of a universal cover projection $\pi : \mathbb{D} \to \Omega$ from

$$\lambda(\pi(z)) \, |\pi'(z)| = \frac{2}{\sqrt{-a}} \lambda_\Omega(z) = \frac{2}{\sqrt{-a}} \frac{1}{1 - |z|^2}.$$  

We call $\lambda(z) \, |dz|$ the hyperbolic metric on $\Omega$ with curvature $-a$. Unless explicitly stated otherwise we usually take the normalization $a = 4$ and call the corresponding metric the hyperbolic metric of $\Omega$ (with constant curvature $-4$). This metric is denoted by $\lambda_\Omega(z) \, |dz|$.

An important result about conformal Riemannian metrics is the Ahlfors–Yau lemma (see Yau [4]). It states that the hyperbolic metric $\lambda_\Omega(z) \, |dz|$ is maximal in the sense that for every regular conformal Riemannian metric $\lambda(z) \, |dz|$ with curvature $\leq -4$ the inequality $\lambda(z) \leq \lambda_\Omega(z)$ holds for every point $z \in \Omega$. This is even true for regular conformal pseudo–metrics $\lambda(z) \, |dz|$, i.e., the density $\lambda$ is not necessarily strictly positive, but is only assumed to be nonnegative, and $\lambda$ is of class $C^2$ off its zero set. On the other hand, for every complete regular conformal Riemannian metric on $\Omega$ with curvature bounded below by $-4$ the estimate $\lambda(w) \geq \lambda_\Omega(w)$ is valid in $\Omega$. We refer to this fact as the Ahlfors–Yau lemma (see Yau [43, 44]). There are boundary versions of these results (cf. Bland [6], Troyanov [38], and Kraus, Roth & Ruscheweyh [20]).

We next compile a number of facts about the Gaussian curvature equation (3.4). We adopt standard notation, so $C(\overline{\Omega})$ is the set of real–valued continuous functions on the set $\overline{\Omega} \subset \mathbb{C}$ and $C^k(\Omega)$ is the set of real–valued functions having all derivatives of order $\leq k$ continuous in the open set $\Omega \subseteq \mathbb{C}$.

**Lemma 4.1.** Let $\Omega \subseteq \mathbb{C}$ be a bounded regular domain and $h : \Omega \to \mathbb{C}$ a bounded holomorphic function.

(a) (Comparison principle)

If $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ are two solutions to (3.4) and if $u_1 \leq u_2$ on the boundary $\partial \Omega$, then $u_1 \leq u_2$ in $\Omega$.

(b) Let $g_\Omega(z, \zeta) \geq 0$ denote Green’s function of the regular domain $\Omega$. If $u$ is a bounded and integrable function on $\Omega$, then

$$v(z) := -\frac{1}{2\pi} \int_\Omega g_\Omega(z, \zeta) \left\{ 4 |h(\zeta)|^2 e^{2u(\zeta)} d\sigma_\zeta \right\} \zeta \in \Omega,$$

where $\sigma_\zeta$ denotes two–dimensional Lebesgue measure, belongs to $C^1(\Omega) \cap C(\overline{\Omega})$ and $v \equiv 0$ on $\partial\Omega$. If, in addition, $u$ is locally Hölder continuous with exponent $\beta$, $0 < \beta \leq 1$, then $v \in C^2(\Omega)$ and $\Delta v = 4 |h(z)|^2 e^{2u}$ in $\Omega$.

(c) If $u : \Omega \to \mathbb{R}$ is a $C^2$–solution to (3.4) and $u$ is continuous on the closure $\overline{\Omega}$, then the integral formula

$$u(z) = H(z) - \frac{1}{2\pi} \int_\Omega g_\Omega(z, \zeta) \left\{ 4 |h(\zeta)|^2 e^{2u(\zeta)} d\sigma_\zeta \right\}$$  

(4.1)
holds for every \( z \in \Omega \). Here, \( H \) is harmonic in \( \Omega \) and continuous on \( \overline{\Omega} \) with boundary values \( u \), i.e.,

\[
H|_{\partial \Omega} \equiv u|_{\partial \Omega}.
\]

Conversely, if \( u \) is a locally integrable and bounded function on the regular domain \( \Omega \) satisfying (4.1) for some harmonic function \( H \) in \( \Omega \) which is continuous on \( \overline{\Omega} \), then \( u \) belongs to \( C^2(\Omega) \cap C(\overline{\Omega}) \) and is a solution to \( \Delta u = 4|h(z)|^2 e^{2u} \) in \( \Omega \) with \( u \equiv H \) on \( \partial \Omega \).

(d) If \( \phi : \partial \Omega \to \mathbb{R} \) is a continuous function, then there exists a unique \( C^2 \)-solution \( u : \Omega \to \mathbb{R} \) continuous on \( \overline{\Omega} \) to the boundary value problem

\[
\Delta u = 4|h(z)|^2 e^{2u} \quad \text{in } \Omega \\
u = \phi \quad \text{on } \partial \Omega.
\]

For the comparison principle (Lemma 4.1 (a)) see [15, Theorem 10.1]. Part (b) of Lemma 4.1 is established in [15, p. 54/55] and [11, p. 241], and part (c) follows by combining (a) and (b). Lemma 4.1 (d) finally might be found in [15], in particular Theorem 12.5 and the remarks on p. 308/309.

We now can also give an example which illustrates that Theorem 2.2 does not remain valid in general, when the domain \( \Omega \) is not simply connected, see Remark 1 (b).

**Example 3.** Let \( \Omega \) be the annulus \( \{ z \in \mathbb{C} : 1/4 < |z| < 1/2 \} \) and let \( \phi : \partial \Omega \to \mathbb{R} \) be the continuous function

\[
\phi(\xi) = \begin{cases} 
4/3 & \text{if } |\xi| = 1/4, \\
\sqrt{2} & \text{if } |\xi| = 1/2.
\end{cases}
\]

Now assume there exists a locally univalent holomorphic function \( f : \Omega \to \mathbb{D} \) such that

\[
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi) \quad \text{for every } \xi \in \partial \Omega.
\]

Note,

\[
\tilde{\lambda}(z) = \frac{1}{2\sqrt{|z|}(1 - |z|)}
\]

is the density of a conformal Riemannian metric of constant curvature \(-4\) in \( \Omega \) and \( \tilde{\lambda}(\xi) = \phi(\xi) \) for \( \xi \in \partial \Omega \). Hence, by the uniqueness part of Lemma 4.1 (d), we see that

\[
\tilde{\lambda}(z) = \frac{|f'(z)|}{1 - |f(z)|^2}, \quad z \in \Omega.
\]

On the other hand, in the simply connected domain \( D = \Omega \setminus (-1/4, -1/2) \), we have

\[
\tilde{\lambda}(z) = \frac{|g'(z)|}{1 - |g(z)|^2}
\]

for the holomorphic function \( g : D \to \mathbb{D} \), \( g(z) = \sqrt{z} \). Applying Theorem 3.3, we deduce \( g = T \circ f \) in \( D \) for some automorphism \( T \) of \( \mathbb{D} \). Thus \( g \) has an analytic extension to \( \Omega \) given by \( T \circ f \) which is absurd.
4.3. Proofs of Theorems 3.1 and Theorem 3.2

Proof of Theorem 3.1. In order to prove Theorem 3.1, we consider for each integer \( n \geq 1 \) the unique real-valued solution \( u_n \in C^2(\Omega) \cap C(\overline{\Omega}) \) to the boundary value problem

\[
\Delta u = 4 |h(z)|^2 e^{2u} \quad \text{in } \Omega
\]

\[
u = n \quad \text{on } \partial \Omega,
\]

see Lemma 4.1 (d). By the comparison principle (Lemma 4.1 (a)) we get a monotonically increasing sequence \( \{u_n\} \). Our task is to show that \( u_n \) converges to a solution of the boundary value problem (3.4)–(3.5). We proceed in a series of steps.

(i) For each fixed \( n \geq 1 \) consider the conformal pseudo-metric

\[
\lambda_n(z) = |h(z)| e^{u_n(z)} |dz|.
\]

The Gaussian curvature of \( \lambda_n(z) |dz| \) is found to be

\[
\kappa_{\lambda_n}(z) = -\frac{\Delta u_n(z)}{|h(z)|^2 e^{2u_n(z)}} = -4.
\]

Thus, by Ahlfors’ lemma,

\[
\lambda_n(z) \leq \lambda_0(z), \quad z \in \Omega, \quad (4.2)
\]

so we get a uniform bound

\[
u_n(z) \leq \log \lambda_0(z) - \log |h(z)|, \quad z \in \Omega. \quad (4.3)
\]

In particular, \( \{u_n\} \) converges monotonically to some limit function \( u : \Omega \to \mathbb{R} \cup \{+\infty\} \).

Note that \( u(z) \) is certainly finite when \( z \) is not a zero of \( h \). We need to show that \( u \in C^2(\Omega) \), \( \Delta u = 4 |h(z)|^2 e^{2u} \) in \( \Omega \) and \( u(z) \to +\infty \) whenever \( z \to \xi \in \partial \Omega \).

(ii) Let \( C \) be a regular domain, which is compactly contained in \( \Omega \). Then by Lemma 4.1 (c),

\[
u_n(z) = H_n(z) - \frac{1}{2\pi} \int_C g_C(z, \zeta) 4 |h(\zeta)|^2 e^{2u_n(\zeta)} d\sigma_{\zeta} \quad (4.4)
\]

with a harmonic function \( H_n : C \to \mathbb{R} \) which is continuous on \( \overline{C} \) and \( H_n \equiv u_n \) on \( \partial C \). Note that the second term on the right-hand side is uniformly bounded in \( C \). This follows from

\[
0 \leq 4 \int_C g_C(z, \zeta) 4 |h(\zeta)|^2 e^{2u_n(\zeta)} d\sigma_{\zeta} - \frac{1}{2\pi} \int_C g_C(z, \zeta) 4 \lambda_n(\zeta)^2 d\sigma_{\zeta}
\]

\[
\leq \frac{1}{2\pi} \int_C g_C(z, \zeta) 4 \lambda_0(\zeta)^2 d\sigma_{\zeta} \leq C_1(C) \cdot \frac{1}{2\pi} \int_C g_C(z, \zeta) d\sigma_{\zeta} \leq C(C) < \infty,
\]

where we have used the facts that the hyperbolic density \( \lambda_0 \) is clearly bounded on \( C \) and

\[
v(z) := \frac{1}{2\pi} \int_C g_C(z, \zeta) d\sigma_{\zeta}
\]

is the solution to the boundary value problem \( \Delta v = -1 \) in \( C \) and \( v = 0 \) on \( \partial C \), so \( v \in C(\overline{C}) \) is in particular bounded on \( C \).

(iii) We next show that the sequence \( \{H_n\} \) of harmonic functions in \( C \) converges locally uniformly in \( C \). Since \( H_n \equiv u_n \) on \( \partial C \) and \( \{u_n\} \) is monotonically increasing on \( C \), it suffices by Harnack’s theorem to show that \( \{H_n(z_0)\} \) is bounded above at some point \( z_0 \in C \). But this follows immediately from equation (4.4), by what we have proved in part (ii) and noting that (4.3) implies \( \{u_n(z_0)\} \) is bounded above if \( h(z_0) \neq 0 \).
From (iii) we deduce that \( \{ H_n \} \) converges locally uniformly in \( G \) to some harmonic function \( H : G \to \mathbb{R} \). Since we have proved in part (i) that \( \{ u_n \} \) converges monotonically in \( \Omega \) to some limit function \( u \), we thus see from part (ii) and formula (4.4) that \( u(z) \) is finite for every \( z \in G \) and

\[
  u(z) = H(z) - \frac{1}{2\pi} \int_G g_G(z, \zeta) \frac{4 |h(\zeta)|^2 e^{2u(\zeta)}}{d\sigma_\zeta}, \quad z \in G
\]

by Lebesgue’s theorem on monotone convergence. Now Lemma 4.1 (v) shows that \( u \) belongs to \( C^2(G) \cap C(\overline{G}) \) and solves (3.4) in \( G \). Since \( G \) is an arbitrary regular domain compactly contained in \( \Omega \), we see that \( u \in C^2(\Omega) \) solves (3.4) in \( \Omega \).

(v) It is now very easy to verify the boundary condition (3.5). Assume to the contrary that we can find a sequence \( \{ z_j \} \subset \Omega \) which converges to \( \xi \in \partial \Omega \) and a constant \( 0 < C_2 < \infty \) such that \( u(z_j) < C_2 \) for all \( j \). Now choose an integer \( m > C_2 \). Since \( u_m(z_j) \to m \) as \( j \to \infty \), there is an integer \( J \) such that \( u_m(z_j) > C_2 \) for all \( j > J \). But then the monotonicity of \( \{ u_n \} \) yields \( u(z_j) \geq C_2 \) for all \( j > J \) and the contradiction is apparent. Thus

\[
  \lim_{z \to \xi} u(z) = +\infty
\]

for every \( \xi \in \partial \Omega \) as desired.

Proof of Theorem 3.2. The existence–part of Theorem 3.2 follows from Theorem 3.1 and Remark 3, so we need only show the uniqueness statement. Let \( \lambda(z) \ |dz| \) be a complete regular conformal metric on \( \mathbb{D} \) with curvature \(-4 \ |h(z)|^2 \), where \( h \) is a Blaschke product, so its curvature is bounded from below by \(-4 \). The Ahlfors–Yau lemma then implies that its density \( \lambda(z) \) is bounded below by \( 1/(1 - |z|^2) \), the density of the hyperbolic metric on \( \mathbb{D} \) with constant curvature \(-4 \). Thus \( u(z) := \log \lambda(z) \) is a \( C^2 \)–solution to the boundary value problem (3.4)–(3.5). Let \( \phi_1 \) and \( \phi_2 \) be two such solutions. We need to show \( u_1 \equiv u_2 \). Consider the auxiliary function

\[
  v(z) := \max\{ u_1(z), u_2(z) \} - u_2(z).
\]

We first notice that \( v(z) \) is subharmonic in \( \mathbb{D} \). In fact, if \( v(z_0) > 0 \) at some point \( z_0 \in \mathbb{D} \), then \( v(z) = u_1(z) - u_2(z) > 0 \) in a neighborhood of \( z_0 \). Thus

\[
  \Delta v(z) = \Delta u_1(z) - \Delta u_2(z) = 4 |h(z)|^2 \left( e^{2u_1(z)} - e^{2u_2(z)} \right) \geq 0
\]

there, i.e., \( v \) is subharmonic in this neighborhood. If \( v(z_0) = 0 \) for some point \( z_0 \in \mathbb{D} \), then \( v \) satisfies the submean inequality

\[
  v(z_0) = 0 \leq \frac{1}{2\pi} \int_0^{2\pi} v \left( z_0 + r e^{i\theta} \right) \, dt
\]

for all \( r \) small enough. Hence \( v \) is subharmonic in \( \mathbb{D} \).

As we have already noted above, we have

\[
  u_2(z) \geq \log \frac{1}{1 - |z|^2} \quad (4.5)
\]

from the Ahlfors–Yau lemma. An inequality in the opposite direction follows from Theorem 3.3 and the Schwarz–Pick lemma. Namely, Theorem 3.3 gives us holomorphic functions \( f_j : \mathbb{D} \to \mathbb{D} \) such that

\[
  u_j(z) = \log \left( \frac{|f_j(z)|}{1 - |f_j(z)|^2 |h(z)|} \cdot \frac{1}{|h(z)|} \right), \quad j = 1, 2,
\]

so the Schwarz–Pick lemma implies

\[
  u_j(z) \leq \log \left( \frac{1}{1 - |z|^2} \cdot \frac{1}{|h(z)|} \right), \quad j = 1, 2. \quad (4.6)
\]
In particular, following Laine \[ \text{22} \]

This, however, forces \( v(z) \equiv 0 \), because the integral means \( \int_0^{2\pi} v(re^{it}) \, dt \) are monotonically increasing for \( r \in (0, 1) \) and

\[
0 \leq \int_0^{2\pi} v(re^{it}) \, dt \leq \lim_{r \to 1} \int_0^{2\pi} |h(re^{it})| \, dt = 0
\]

since \( h \) is a Blaschke product (see \[ 14, \text{Ch. II, Theorem 2.4} \]). Thus \( v(z) \equiv 0 \) and \( u_1 \equiv u_2 \). □

4.4. Proof of Theorem 3.3

Before turning to the proof of Theorem 3.3, we wish to point out explicitly that every \( C^2 \)-solution \( u : \Omega \to \mathbb{R} \) to \( \Delta u = 4|h(z)|^2e^{2u} \) in \( \Omega \) is in fact real analytic there. This follows from standard elliptic regularity results for the Poisson equation and Bernstein's positive solution to Hilbert's Problem 19: A priori the right-hand side of \( \Delta u = 4|h(z)|^2e^{2u} \) is in \( C^2(\Omega) \), so \( u \) certainly belongs to \( C^3(\Omega) \) (cf. \[ 15, \text{Theorem 6.19} \]). By Bernstein's analyticity theorem, \( u \) is actually real analytic in \( \Omega \). This additional information will be a crucial ingredient in the proof of Theorem 3.3, which we now turn towards.

It is convenient to make use of the \( \partial/\partial z^- \) and \( \partial/\partial z^+ \)-operators,

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy.
\]

In particular,

\[
\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}. \quad (4.8)
\]

We start off with the following simple, but important observation.

**Lemma 4.2.** Let \( \Omega \subseteq \mathbb{C} \) be an open set, \( h : \Omega \to \mathbb{C} \) a holomorphic function, \( h \not\equiv 0 \), and \( u : \Omega \to \mathbb{R} \) a \( C^2 \)-solution to \( \Delta u = 4|h(z)|^2e^{2u} \) in \( \Omega \). Then the function

\[
B_u(z) := \frac{\partial^2 u}{\partial z^2}(z) - \left( \frac{\partial u}{\partial \bar{z}}(z) \right)^2 - \frac{h'(z)}{h(z)} \frac{\partial u}{\partial z}(z)
\]

is holomorphic in \( \Omega \) with the exception of possible simple poles at the zeros of \( h \).

**Proof.** Since we know that \( u \) is of class \( C^\infty \), we may certainly differentiate the PDE \( \Delta u = 4|h(z)|^2e^{2u} \) once with respect to \( z \), and using (4.8), we get

\[
\frac{\partial^3 u}{\partial z^3 \partial \bar{z}} = \frac{\partial^2 u}{\partial z \partial \bar{z}} \frac{h'}{h} + \frac{\partial}{\partial \bar{z}} \left[ \left( \frac{\partial u}{\partial \bar{z}} \right)^2 \right],
\]

so \( \partial B_u / \partial \bar{z} = 0 \), which means \( B_u \) is meromorphic in \( \Omega \). □

In other words, \( \frac{\partial}{\partial \bar{z}} \) is a formal (non holomorphic) solution of the Riccati equation

\[
w'(z) - w(z)^2 - \frac{h'(z)}{h(z)} w(z) = B_u(z).
\]

Following Laine \[ 22, \text{p. 165} \] we transfer this Riccati equation via

\[
w(z) = v(z) - \frac{1}{2} \frac{h'(z)}{h(z)}
\]
to normal form:

\[ v' - v^2 = A_u(z) , \tag{4.9} \]

where

\[ A_u(z) := B_u(z) + \frac{1}{2} \left( \frac{h'(z)}{h(z)} \right)' - \frac{1}{4} \left( \frac{h'(z)}{h(z)} \right)^2 . \]

This function \( A_u \) is holomorphic at every point \( z_0 \in \Omega \) except when \( h(z_0) = 0 \). If \( h \) has a zero of order \( n \in \mathbb{N} \) at \( z = z_0 \), then \( A_u \) has a pole of order 2 there with Laurent expansion

\[ A_u(z) = \frac{b_0}{(z - z_0)^2} + \frac{b_1}{z - z_0} + b_2 + \cdots , \tag{4.10} \]

where

\[ b_0 = \frac{1 - (n + 1)^2}{4} . \tag{4.11} \]

Our next goal is to show that every local solution of the Riccati equation (4.9) admits a meromorphic extension to the whole of \( \Omega \) provided \( \Omega \) is a simply connected domain. According to Laine [22, Theorem 9.1.7] this is the case if and only if the first \( n + 2 \) coefficients \( b_0, \ldots, b_{n+1} \) in the Laurent expansion of \( A_u \) satisfy a certain very complicated nonlinear relation (see formula (6.18) in [22]), which appears to be difficult to verify directly. We therefore choose a different path to establish:

**Lemma 4.3.** Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain, \( h : \Omega \to \mathbb{C} \) a holomorphic function, \( h \neq 0 \), and \( u : \Omega \to \mathbb{R} \) a \( C^2 \)-solution to \( \Delta u = 4|h(z)|^2 e^{2u} \) in \( \Omega \). Then every local meromorphic solution to the Riccati equation \( v' - v^2 = A_u(z) \) admits a meromorphic continuation to all of \( \Omega \).

**Remark 6.** If \( h(z) \equiv 1 \) then Lemma 4.3 reduces to the elementary fact that if \( A_u(z) \) is holomorphic, every solution to the Riccati equation \( v' - v^2 = A_u(z) \) is meromorphic. Thus Lemma 4.3 is the essential step from Liouville’s classical theorem for \( \Delta u = 4 e^{2u} \) to the more general Theorem 3.3.

**Proof.** Let \( z_0 \in \Omega \) be a pole of \( A_u \), i.e., a zero of order \( n \) of \( h \), so \( A_u \) has an expansion of the form (4.10)–(4.11) at \( z_0 \). We need only show that every local meromorphic solution to the Riccati equation \( v' - v^2 = A_u(z) \) admits a meromorphic continuation to a neighborhood of \( z_0 \).

In order to simplify notation we take without loss of generality \( z_0 = 0 \).

(i) We first show that \( v' - v^2 = A_u(z) \) has at least one meromorphic solution in a neighborhood of the origin with residue \(-(n + 2)/2\) at \( z_0 \). We substitute

\[ v_1(z) = \frac{-n+2 + \omega(z)}{2} , \]

and find after some manipulation that \( v'_1 - v_1^2 = A_u(z) \) if and only if

\[ z \omega'(z) = -(n + 1) \omega(z) + \omega(z)^2 + A_u(z) z^2 - \frac{1 - (n + 1)^2}{4} . \]

This is a Briot–Bouquet differential equation, which has a unique holomorphic solution \( \omega(z) \) in a neighborhood of 0 such that \( \omega(0) = 0 \), see for instance [18, Theorem 11.1.1], because \(-(n + 1) \) is not a positive integer. Therefore, the Riccati equation \( v' - v^2 = A_u(z) \) has at least one meromorphic solution \( v_1 \) with a simple pole at 0 and such that the residue of \( 2v_1 \) at \( z_0 \) is an integer.

(ii) By Lemma 9.1.4 in [22], we conclude that every local meromorphic solution to the Riccati equation \( v' - v^2 = A_u(z) \) admits a meromorphic continuation to 0 provided that we can exhibit a second meromorphic solution in a neighborhood of 0.
(iii) Recall that
\[ \frac{\partial u}{\partial z}(z) = \frac{1}{2} \frac{h'(z)}{h(z)} \]
is a “formal” solution to \( v' - v^2 = A_u(z) \) with “residue” \( n/2 \) at \( z = 0 \). We extract an actual (meromorphic) solution \( v_2 \) from this formal solution as follows. As \( h \) has a zero of order \( n \) at \( z = 0 \), we have \( h(z) = z^n h_1(z) \) for a function \( h_1 \) holomorphic at \( z = 0 \) with \( h_1(0) \neq 0 \). A quick calculation shows that the function
\[ \nu(z) := u(z) + \log |h_1(z)| \]
satisfies
\[ A_u(z) = \frac{1 - (n + 1)^2}{4z^2} + \frac{\partial^2 \nu}{\partial z^2}(z) - \frac{n}{z} \frac{\partial \nu}{\partial z}(z) - \left( \frac{\partial \nu}{\partial z}(z) \right)^2. \]  
(4.12)
Recall that \( u \) and hence also \( \nu \) is a real analytic function. This allows us to expand \( \nu \) in a power series in \( z \) and \( \bar{z} \) in a neighborhood \( U \) of \( z = 0 \). We thus obtain for \( z, \bar{z} \in U \)
\[ \nu(z, \bar{z}) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{jk} z^j \right) \bar{z}^k = \sum_{j=0}^{\infty} a_{j0} z^j + \sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{jk} z^j \right) \bar{z}^k, \]
that is
\[ \nu(z, \bar{z}) = g(z) + \Lambda(z, \bar{z}), \]
if we set
\[ g(z) = \sum_{j=0}^{\infty} a_{j0} z^j \quad \text{and} \quad \Lambda(z, \bar{z}) = \sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{jk} z^j \right) \bar{z}^k. \]
Clearly, \( g(z) \) is holomorphic and \( \Lambda(z, \bar{z}) \) is real analytic in \( U \).
Replacing \( \nu \) by \( g + \Lambda \) in (4.12) yields
\[ A_u(z) = \frac{1 - (n + 1)^2}{4z^2} - \frac{n}{z} g'(z) - g'(z)^2 + g''(z) + H(z, \bar{z}) \]  
(4.13)
with
\[ H(z, \bar{z}) = A_{zz}(z, \bar{z}) - \frac{n}{z} A_z(z, \bar{z}) - 2 g'(z) \Lambda_z(z, \bar{z}) - (\Lambda_z(z, \bar{z}))^2. \]
Since \( z H(z, \bar{z}) \) is real analytic in \( U \), \( H(z, \bar{z}) \) can be written as
\[ H(z, \bar{z}) = \frac{1}{z} \left( \sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} b_{jk} z^j \right) \bar{z}^k \right). \]
Now, as \( A_u \) has a pole of order 2 at \( z = 0 \), identity (4.13) shows that \( z^2 H(z, \bar{z}) \) is holomorphic there:
\[ 0 = (z^2 H(z, \bar{z}))(z) = \sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} b_{jk} z^{j+1} \right) k \bar{z}^{k-1}. \]
This implies \( b_{jk} = 0 \) for all \( j \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \), and consequently \( H(z, \bar{z}) \equiv 0 \).

We thus obtain
\[ A_u(z) = \frac{1 - (n + 1)^2}{4z^2} - \frac{n}{z} g'(z) - g'(z)^2 + g''(z) \quad \text{for} \quad z \in U, \]
and a glance at the right hand side of this equation shows
\[ A_u(z) = \left( \frac{n}{2z} + g'(z) \right)' - \left( \frac{n}{2z} + g'(z) \right)^2. \]
Therefore the function
\[ v_2(z) = \frac{n}{2z} + g'(z) \]
is a meromorphic solution of the Riccati differential equation \( v' - v^2 = A_u(z) \) in \( U \).

**Proof of Theorem 3.3.** Let \( u \) be a \( C^2 \)-solution to \( \Delta u = 4|h(z)|^2 e^{2u} \) in \( \Omega \) and \( z_0 \) not a zero of \( h(z) \). Then
\[ u_1(z) := u(z) + \log |h(z)| \]
is a \( C^2 \)-solution to Liouville’s equation \( \Delta u_1 = 4e^{2u_1} \) in a neighborhood of \( h(z) \). By Liouville’s theorem, we get
\[ u_1(z) = \log \frac{|f'(z)|}{1 - |f(z)|^2} \]
for a holomorphic function \( f : V \to \mathbb{D} \), i.e. (3.6) holds for \( z \in V \). A straightforward calculation now shows that the Schwarzian of \( f \)
\[ S_f(z) := \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \]
satisfies
\[ S_f(z) = 2A_u(z) \] \hspace{1cm} (4.14)
in \( V \subseteq \Omega \). By Lemma 4.3, the associated Riccati equation \( v' - v^2 = A_u(z) \) has only meromorphic solutions in \( \Omega \), so the Schwarzian differential equation (4.14) only has meromorphic solutions in \( \Omega \) as well. This follows from Theorem 9.1.7 and Corollary 6.8 in [22]. In particular, \( f : V \to \mathbb{D} \) has a meromorphic continuation to \( \Omega \) which we continue to call \( f \). We finally note that \( |f(z)| < 1 \) in all of \( \Omega \). In fact, let \( \Omega' \) be the component of \( \{ z \in \Omega : |f(z)| < 1 \} \) which contains \( V \). Then \( \Omega' \) is clearly open in \( \Omega \), but also closed in \( \Omega \). This is immediate from the fact that (3.6) holds for all \( z \in \Omega' \).

In order to prove the uniqueness statement, let \( f, g : \Omega \to \mathbb{D} \) be two holomorphic maps such that
\[ u(z) = \log \left( \frac{|f'(z)|}{1 - |f(z)|^2} \frac{1}{|h(z)|} \right) = \log \left( \frac{|g'(z)|}{1 - |g(z)|^2} \frac{1}{|h(z)|} \right) \] \hspace{1cm} (4.15)
In particular, \( f \) and \( g \) are nonconstant and as above \( S_f(z) = S_g(z) \) in \( \Omega \), so \( f = T \circ g \) for some Möbius transformation \( T \). Thus (4.15) shows
\[ \frac{|T'(w)|}{1 - |T'(w)|^2} = \frac{1}{1 - |w|^2} \]
first for all points \( w \) in the open set \( g(\Omega) \subseteq \mathbb{D} \) and then clearly for every \( w \in \mathbb{D} \). Hence \( T(\mathbb{D}) \subseteq \mathbb{D} \) and the Schwarz–Pick lemma implies that \( T \) is a conformal disk automorphism.

**Remark 7.** The above proof of Theorem 3.3 uses Liouville’s theorem (i.e., the special case \( h(z) \equiv 1 \)). This can be avoided by showing directly as in [26] or [29] that the solution \( f \) to the initial value problem
\[ S_f(z) = 2A_u(z), \quad f(z_0) = 0, \quad f'(z_0) = e^{u_1(z_0)}, \quad f''(z_0) = 2e^{u_1(z_0)} \frac{\partial u_1}{\partial z}(z_0), \]
which is meromorphic in all of \( \Omega \) by Lemma 4.3, fulfills (3.6) in a neighborhood of \( z_0 \). In particular, the equation (3.6) can be solved for \( f \) constructively.
4.5. Proof of Theorem 2.2 and Theorem 2.3

Proof of Theorem 2.2. Let \( p \) be a polynomial with zeros \( z_j \) (counted with multiplicities). In view of Lemma 4.1 (d) there exists a uniquely determined conformal Riemannian metric \( \lambda(z) \, |dz| \) in \( \Omega \) with curvature \( -4 \, |p(z)|^2 \) and boundary values \( \phi(\xi)/|p(\xi)| \). From Theorem 3.3 we deduce

\[
\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2 \, |p(z)|}, \quad z \in \Omega,
\]

for some holomorphic function \( f : \Omega \to \mathbb{D} \). Thus \( \{z_j\} \) is the set of critical points of \( f \), and the boundary condition (2.3) is fulfilled, because

\[
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \lim_{z \to \xi} \lambda(z) \, |p(z)| = \phi(\xi).
\]

If \( g \) is another holomorphic function \( g : \Omega \to \mathbb{D} \) with the properties stated in Theorem 2.2, then

\[
\tilde{\lambda}(z) := \frac{|g'(z)|}{1 - |g(z)|^2 \, |p(z)|} \quad \text{for} \quad z \in \Omega.
\]

is the density of a regular conformal Riemannian metric in \( \Omega \) of curvature \( -4 \, |p(z)|^2 \) in \( \Omega \) and boundary values \( \phi/|p| \). From the uniqueness statement of Lemma 4.1 (d) we infer \( \lambda = \lambda \) in \( \Omega \), that is

\[
\frac{|g'(z)|}{1 - |g(z)|^2 \, |p(z)|} = \frac{|f'(z)|}{1 - |f(z)|^2 \, |p(z)|} \quad \text{for} \quad z \in \Omega.
\]

Hence, applying Theorem 3.3, we see that \( g = T \circ f \) for some conformal automorphism \( T \) of \( \mathbb{D} \).

Proof of Theorem 2.3. We only prove the existence part. Let \( B \) be a Blaschke product with zeros \( z_j \) (counted with multiplicities). Note,

\[
\lim_{r \to 1} |B(r\xi)| = 1 \quad \text{for} \quad \text{a.e.} \ \xi \in \partial \mathbb{D}.
\]

Next, let \( v \) be the harmonic function in \( \mathbb{D} \) with boundary values \( \log \phi \in L^\infty(\partial\mathbb{D}) \), so \( |v(z)| \leq M \) in \( \mathbb{D} \) for some constant \( M > 0 \) and

\[
\lim_{z \to \xi} v(z) = \log \phi(\xi) \quad \text{for} \quad \text{a.e.} \ \xi \in \partial \mathbb{D}.
\]

Then \( v(z) = \text{Re} \log g(z) \) for some nonvanishing holomorphic function \( g : \mathbb{D} \to \mathbb{C} \).

By Lemma 4.1 (d) there exists a unique conformal Riemannian metric \( \mu(z) \, |dz| \) with curvature \( -4 \, |B(z) \, g(z)|^2 \) in \( \mathbb{D} \) and \( \mu(\xi) = 1 \) for \( \xi \in \partial \mathbb{D} \). Thus we can apply Theorem 3.3 and get a holomorphic function \( f : \mathbb{D} \to \mathbb{D} \) with critical points \( z_j \) (counted with multiplicities) and no others and

\[
\mu(z) = \frac{|f'(z)|}{1 - |f(z)|^2 \, |B(z)| \, |g(z)|} \quad \text{for} \quad z \in \mathbb{D}.
\]

By construction,

\[
\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \lim_{z \to \xi} \mu(z) \, |B(z)| \, |g(z)| = \phi(\xi) \quad \text{for} \quad \text{a.e.} \ \xi \in \partial \mathbb{D}
\]

and

\[
\sup_{z \in \mathbb{D}} \frac{|f'(z)|}{1 - |f(z)|^2} < \infty.
\]
References

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