Domain robust preconditioning for a staggered grid discretization of the Stokes equations

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Abstract
To our knowledge, most numerical methods for solving discretizations of the Stokes equations depend on the Schur complement and thus the convergence rate depends on the aspect ratio of the domain. We present a preconditioner for a staggered grid discretization of the Stokes equations which is proven to be independent of the aspect ratio. Finally numerical experiments confirm the robustness of the method.

Keywords: Preconditioning, Stokes equations, Staggered grid, LBB constant, Schur complement

1. Introduction
We are interested in systems of linear equations arising from the discretization of the Stokes equations

\[ L \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} -\Delta & D \\ \text{div} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega \] (1)

with velocity \( u = (u_1, \ldots, u_n)^T \), pressure \( p \) and gradient \( D = (D_1, \ldots, D_n)^T \), \( D_i = \frac{\partial}{\partial x_i} \) on the bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \). We use the standard Lebesgue and Sobolev spaces \( L^2(\Omega) \) and \( H^1(\Omega) \) with the norms \( \|u\| \) and \( \|u\|_1^2 = \|u\|^2 + \|Du\|^2 \), respectively. The inner product on \( L^2 \) is denoted by \( (\cdot, \cdot) \). Let \( H_0^1(\Omega) \) be closure of \( C_0^\infty(\Omega) \) in the usual Sobolev space \( H^1(\Omega) \) and let \( X = H_0^1(\Omega) \) be equipped with the norm \( |u|_1 = \|Du\| \). Define \( Y \) as the space of \( L^2(\Omega) \)-functions with zero average: \( Y = \{ p \in L^2(\Omega) : \int_\Omega p\, dx = 0 \} \). The norm on the dual space \( X' \) is

\[ |Dp|_{-1} = \sup_{v \in X} \frac{(\text{div} v, p)}{|v|_1}. \] (2)
The existence and uniqueness of a solution \((u, p) \in X \times Y\) of the weak problem
\[
(Dv, Du) + (Dv, Dp) = f(v) \quad \forall v \in X,
\]
\[
(div u, q) = g(q) \quad \forall q \in Y
\]
can be shown with the Lax-Milgram theorem \((f \in X', g \in Y = Y')\), if an inf-sup-condition – also named LBB condition after Ladyzhenskaya, Babuška and Brezzi – is fulfilled:
\[
\exists L(\Omega) > 0 : \forall p \in Y : L(\Omega)\|p\| \leq \sup_{v \in X} \frac{-(div v, p)}{|v|_1}.
\]
(3)

The existence of such a \(L(\Omega)\) is proved for a large class of domains in [1]. If \(\Omega\) is an \(n\)-cube and \(\Omega_a\) the cuboid obtained through stretching one direction by \(a > 1\) then from [2] we have for the LBB constant
\[
L(\Omega) \leq aL(\Omega_a) \leq 1.
\]
(4)

If the boundary of the domain is a \(C^2\)-manifold, all spectral values of the Schur complement \(S = -\text{div} R D\) (\(R\) being the inverse of \(-\Delta\): \((DRf, Dv) = f(v) \forall v \in X\)) are eigenvalues and these are bounded by \(1/L(\Omega)^2 = \lambda_{\text{min}} \leq \lambda \leq \lambda_{\text{max}} = 1\) (see [3]). Hence all numerical methods whose convergence rate depends on the Schur complement behave poorly on domains with high aspect ratio. For multigrid methods the situation is unclear. The standard proof depends on the LBB constant but in [4] it is stated that certain methods using smoothed aggregation are domain robust.

Our aim is to improve the preconditioner
\[
P_1^{-1} = \begin{pmatrix} R & -b R D \\ 0 & a \end{pmatrix} \quad a \neq 0, b \in \mathbb{R}
\]
(5)

for a staggered grid discretization so that it becomes domain robust. The solution presented in this work is to partition the domain into squares of side length \(H \approx 1\) (or rectangles with low aspect ratio) and to calculate a correction for the pressure values on these squares. For the resulting staggered grid preconditioner
\[
P_3^{-1} = \begin{pmatrix} R_h & -b R_h D \\ 0 & a(I - Q_H + R_H Q_H) \end{pmatrix}
\]
(6)

we will give analytical bounds for the operator norm of \(P_3^{-1}L\) which are independent of the aspect ratio. The continuous case for the simplified diagonal operator with \(b = 0\) has been shown in [5]. The results therein also apply to conforming discretizations of the Stokes equations but not to the staggered grid where the pressure and the components of the velocity are discretized at different points (see Fig. 1). The staggered grid method is widely used in applications for rectangular domains as all vector calculus operations can easily be calculated and the approximations are of second order (see [6]).

To keep the notation simple the calculations are done for rectangles in two dimensions but extend trivial to \(n > 2\).
The outline is as follows: In section 2 we will introduce the notation and the coarse grid, have a closer look at the negative norm and prove some norm estimates. We then show the main result that there are bounds for the operator norm independent of the aspect ratio in section 3. Then we give some numerical results in section 4 where we use the GMRES algorithm whose rate of convergence increases asymptotically as a function of the condition number $\kappa_2$. In the final section the results are summarized and discussed.

2. Notation and lemmas

2.1. The staggered grid, function spaces and differential operators

Definition 1. First of all we define the staggered grid for a closed rectangle $\Omega$ with side lengths $hN_x, hN_y$ ($N_x, N_y \in \mathbb{N}, h > 0$). Let $k,l \in \mathbb{Z}$. Then define the following sets (see Fig. 1):

$$
\mathbb{R}_h^2 = \{ (kh, lh) \mid k, l \in \mathbb{Z} \} \subset \mathbb{R}^2,$$

$$
\Omega_h = \{ (kh, lh) \in \mathbb{R}_h^2 \mid 0 \leq k \leq N_x \text{ and } 0 \leq l \leq N_y \},$$

$$
U = \left\{ \left( h \left( k + \frac{1}{2} \right), lh \right) \mid 0 \leq k < N_x \text{ and } 0 \leq l < N_y \right\},$$

$$
V = \left\{ \left( h \left( k + \frac{1}{2} \right), hl \right) \mid 0 \leq k < N_x \text{ and } 0 \leq l \leq N_y \right\},$$

$$
P = \left\{ \left( h \left( k + \frac{1}{2} \right), h \left( l + \frac{1}{2} \right) \right) \mid 0 \leq k < N_x \text{ and } 0 \leq l < N_y \right\}.
$$

We refer to the boundary of a set as $\partial \cdot = \cdot \cap \partial \Omega$ and to the interior as $\mathring{\cdot} = \cdot \setminus \partial \cdot$, for example $\Omega_h = \Omega_h \setminus \partial \Omega_h = \Omega_h \setminus (\Omega_h \cap \partial \Omega)$. The pressure values will be located on $P$, the $x$ component of the velocity on $U$ and the $y$ component on $V$.

![Figure 1: The staggered grid for mesh size $h$ with the sets $\Omega_h$, $V$ and $P$ as well as the interior $U$ and the boundary $\partial U$ of $U$.](image)

Furthermore we need the discretization of the function spaces:
**Definition 2.** For a discrete set \( A \subset \mathbb{R}^2 \) define the analogue of the \( L^2 \) norm by
\[
(u,v)_A = h^2 \sum_{\alpha \in A} u_\alpha v_\alpha, \quad \|u\|_{A}^2 = (u,u)_A \text{ for } u,v \text{ functions on } A,
\]
where \( u_\alpha = u(\alpha) \). The set of functions with finite norm \( \|u\|_{A} < \infty \) is denoted by \( L^2_{A}(A) \).

The inner product on \( \Omega_h \) has to be adjusted at the boundary to weight each point by the area of \( \Omega \) which has this grid point as nearest:
\[
(u,v)_{\Omega_h} = h^2 \sum_{\alpha \in \Omega_h} u_\alpha v_\alpha + h^2 \sum_{\alpha \in \partial \Omega_h} u_\alpha v_\alpha - \frac{h^2}{4} \sum_{\alpha \in E} u_\alpha v_\alpha,
\]
where \( E = \{(0,0),(0,hN_y),(hN_x,0),(hN_x,hN_y)\} \subset \partial \Omega_h \) is the set of the four corners of \( \Omega_h \). The norm is then given by \( \|u\|^2_{\Omega_h} = (u,u)_{\Omega_h} \).

A restriction to a subset \( \Omega' \subset \Omega \) will be denoted by a semicolon in the norm and inner product, for example \( \|u\|^2_{\Omega',\Omega} = h^2 \sum_{\alpha \in \Omega \cap \Omega'} u_\alpha^2 \).

**Definition 3 (The velocity space \( X_h \)).** Denote the approximation of \( \tilde{u}(x) \in X(\Omega) \cap C(\Omega) \) by piecewise linear functions on \( U \cup V \) by \( u \in X_h(\Omega_h) \) with \( u_\alpha = \tilde{u}(\alpha) \) for \( \alpha \in U \cup V \). In particular we have \( u_\beta = 0 \quad \forall \beta \notin (U \cup V) \), which means the approximations have zero boundary, too. The derivative of \( u \in X_h \) in \( P \cup \Omega_h \) is approximated by the derivative of the linear approximation:
\[
(D_xu)_\alpha = \frac{u_{\alpha+(0,h/2)} - u_{\alpha-(0,h/2)}}{h}, \quad \alpha \in (P \cup \Omega_h),
\]
\[
(D_yu)_\alpha = \frac{u_{\alpha+(h/2,0)} - u_{\alpha-(h/2,0)}}{h}, \quad \alpha \in (P \cup \Omega_h),
\]
\[
(D_xu)_\alpha = \frac{2}{h} u_{\alpha+(h/2,0)}, \quad \alpha \in \{(l,0) \mid 0 \leq l \leq N_y \},
\]
\[
(D_yu)_\alpha = \frac{2}{h} u_{\alpha-(h/2,0)}, \quad \alpha \in \{(0,l) \mid 0 \leq l \leq N_x \},
\]
\[
(\text{div}_h u)_\alpha = (D_xu)_\alpha + (D_yu)_\alpha, \quad \alpha \in P,
\]
\[
(-\Delta_h u)_\alpha = -(D_x(D_xu))_\alpha - (D_y(D_yu))_\alpha, \quad \alpha \in (U \cup V).
\]

Thus, the linear approximation is also used at the boundary. This gives us second order consistency in the interior (see [6]) and first order for the first derivatives at the boundary. The definition of \( D_y \) at the upper and lower boundary is analogous. The inverse of \( -\Delta_h \) is denoted by \( R_h \). Furthermore, we introduce the gradient \( D = (D_x,D_y)^T \). The \( x \) and the \( y \) component \( u|_U \) and \( u|_V \) of \( u \) are written as \( u_x \) and \( u_y \), respectively. The space of the derivatives of functions of \( X_h(\Omega_h) \) is \( X_h'(\Omega_h) \) and its elements are located on the set \( P \cup \Omega_h \). Finally, for \( u,v \in X_h(\Omega_h) \) the inner products
\[
(u,v)_{X_h} = h^2 \sum_{\alpha \in U \cup V} u_\alpha v_\alpha,
\]
\[
(Du,Dv)_{X_h} = (D_xu_x,D_xv_x)_P + (D_yu_x,D_yv_x)_{\Omega_h} + (D_xu_y,D_xv_y)_{\Omega_h} + (D_yu_y,D_yv_y)_P,
\]
and the norm $|u|_{1,h}^2 = (Du, Du)_{X_h}$
\[= \|D_x u_x\|_P^2 + \|D_y u_x\|_{\Omega_h}^2 + \|D_x u_y\|_{\Omega_h}^2 + \|D_y u_y\|_P^2\]
are defined.

**Definition 4 (The pressure space $Y_h$).** Let $Y_h(\Omega_h)$ denote the discretization of $Y(\Omega)$ on $P$ using piecewise constant functions with the norm
\[|Dq|_{-1,h} = \sup_{\phi \in X_h(\Omega_h) \setminus \{0\}} -\langle \text{div}_h \phi, q \rangle_{Y_h^*}, \tag{7}\]
The inner product for $p, q \in Y_h(\Omega_h)$ is
\[(p, q)_{Y_h} = h^2 \sum_{\alpha \in P} p_{\alpha} q_{\alpha}, \quad \|p\|_{Y_h}^2 = (p, p)_{Y_h}. \tag{8}\]
The derivatives are once again given by
\[(D_x p)_{\alpha} = p_{\alpha + (h/2,0)} - p_{\alpha - (h/2,0)}, \quad \alpha \in \tilde{U},
(D_y p)_{\alpha} = p_{\alpha + (0,h/2)} - p_{\alpha - (0,h/2)}, \quad \alpha \in \tilde{V}.
\]

Theorem 2 below shows that using this definition we have integration by parts in the form of $(v, Dp)_{X_h} = -\langle \text{div}_h v, p \rangle_{Y_h}$.

The product space $X_h \times Y_h$ has the norm $\|(u, p)^T\|_{X_h \times Y_h}^2 = |u|_{1,h}^2 + |Dp|_{-1,h}^2$.

### 2.2. Summation by parts

Similar to the continuous case we have summation by parts, which can be verified using direct calculation. In the proof, we only have to pay attention to properly include the zero boundary condition in the different treatment of the spatial directions.

**Lemma 1.** For $u \in X_h, p \in Y_h$ and $\omega \in L^2(\Omega_h)$ we have
\[(u_x, -D_x p)_U = (D_x u_x, p)_{Y_h}, \quad (u_y, -D_x \omega)_V = (D_x u_y, \omega)_{\Omega_h},
(u_x, -D_y \omega)_U = (D_y u_x, \omega)_{\Omega_h}, \quad (u_y, -D_y p)_V = (D_y u_y, p)_{Y_h}.
\]

With this result we can show the following summation by parts for the vector calculus operators:

**Theorem 2.** For $p \in Y_h$ and $u, v \in X_h$ we have
\[ (v, Dp)_{X_h} = -\langle \text{div}_h v, p \rangle_{Y_h}, \]
\[ (v, -\Delta_h u)_{X_h} = (Dv, Du)_{X_h^*}, \]
\[ (\nabla_h \times u, \nabla_h \times v)_{\Omega_h} = \left( u, \left( \frac{D_y}{-D_x} \right) \nabla_h \times v \right)_{X_h} \]
with the two dimensional curl $\nabla_h \times u = D_x u_y - D_y u_x$.

From this, we immediately get $\|\text{div}_h u\|_{Y_h}^2 + \|\nabla_h \times u\|_{\Omega_h}^2 = |u|_{1,h}^2$ and conclude

**Corollary 3.** $\|\text{div}_h u\|_{Y_h} \leq |u|_{1,h}$ for $u \in X_h$.  


2.3. The coarse grid

In order to obtain a domain robust preconditioner we correct the preconditioner $P_1$ on a coarse grid. Therefore let $\Pi_H$ be a partition of $\Omega$ into closed rectangles

$$\Lambda_H = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x - x_0 \leq H_x, 0 \leq y - y_0 \leq H_y\}$$

with lower left vertex at $(x_0, y_0)$ and side lengths $H_x, H_y \simeq 1$, which are without much loss of generality integer multiples of $h$. As the rectangles are closed, the intersection of two is either empty or consists of an edge or a vertex. (Interior edges belong to both neighbouring rectangles.) The maximum of both side lengths is denoted by $H = \max\{H_x, H_y\}$.

The space of piecewise constant functions on $\{\Lambda_H\}$ is called $Y_H \subset Y_h$. We define the projection $Q_H : Y_h \rightarrow Y_H$ by the averaging

$$(Q_H q)(x) = \frac{1}{N} \sum_{\beta \in P \cap \Lambda_H} q_\beta \quad \text{for } x \in \Lambda_H \text{ with } N = \sum_{\beta \in P \cap \Lambda_H} 1.$$  

(10)

Be $S_H = \{\Gamma_H\}$ the set of the interior edges. Then define

$$[q_H]\Gamma_H = q_H|_{\Lambda_2} - q_H|_{\Lambda_1},$$

(11)

for two neighbouring elements $\Lambda_1, \Lambda_2$ with one common edge $\Gamma_H \in S_H$ with $x_2 \geq x_1$ or $y_2 \geq y_1 \forall (x_1, y_1) \in \Lambda_1, (x_2, y_2) \in \Lambda_2$.

**Definition 5.** Now let us define the inner product $Y_H \times Y_H \rightarrow \mathbb{R}$ and the norm on $Y_H$ by

$$(q_H, p_H)_{1,H} = \sum_{\Gamma_H \in S_H} \mu(\Gamma_H) [q_H]|_{\Gamma_H} [p_H]|_{\Gamma_H}, \quad \text{with } \mu(\Gamma_H) = h \sum_{\alpha \in \Gamma_H \cap (U \cup V)} 1,$$

$$|q_H|^2_{1,H} = (q_H, q_H)_{1,H}.$$  

Then the operator $-\Delta_H : Y_H \rightarrow Y_H$ is given by $(-\Delta_H q_H, p_H)_{Y_h} = (q_H, p_H)_{1,H}$ which is the usual five point stencil with natural boundary conditions. Its inverse is denoted by $R_H$.

2.4. Trace theorem and Poincaré inequality

Similar to the continuous case a trace theorem and a Poincaré inequality can be derived. At first define the boundary of a rectangle $\Theta$ in $\Omega$ with side lengths $hM_x, hM_y$. Without loss of generality the lower left corner will be taken as $(0,0)$. Then the left and the upper boundary of $\Theta$ are given by

$$\Gamma_w(\Theta) = \{(0, (k + 1/2) h) \mid k \in \mathbb{N}_0, k < M_y\},$$

$$\Gamma_n(\Theta) = \{((l + 1/2) h, hM_y) \mid l \in \mathbb{N}_0, l < M_x\}.$$  

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The right and lower boundary \((\Gamma_r(\Theta), \Gamma_s(\Theta))\) are defined similarly. The whole boundary is denoted by \(\Gamma(\Theta) = \bigcup_{r \in \{w, e, s, n\}} \Gamma_r(\Theta)\). For \(v \in X_h\) we define the norms

\[
\|v\|^2_{\Gamma_r(\Theta), h} = h \sum_{\alpha \in \Gamma_r(\Theta)} v_{\alpha}^2,
\]

for \(r \in \{w, e, s, n\}\),

\[
\|v\|^2_{\Gamma(\Theta), h} = h \sum_{\alpha \in \Gamma(\Theta)} v_{\alpha}^2,
\]

and

\[
\|v\|^2_{S_H} = \sum_{\Gamma_H \in S_H} \|v\|^2_{\Gamma_H, h} = h \sum_{\Gamma_H \in S_H} \sum_{\alpha \in \Gamma_H \cap \Gamma(\Theta)} v_{\alpha}^2.
\]

**Theorem 4** (Trace theorem). Let \(\Theta_G\) be a rectangle in \(\Omega\), for example an element of the coarse grid, with side lengths \(G_x = hM_x, G_y = hM_y\). Then for \(u \in X_h(\Omega_h)\) we have

\[
\|u\|^2_{\Gamma_w(\Theta_G), h} + \|u\|^2_{\Gamma_s(\Theta_G), h} \leq c(G)(\|u\|^2_{1, h; \Theta_G} + \|u\|^2_{X_h; \Theta_G}),
\]

and therefore

\[
\|u\|^2_{S_H} = \sum_{\Gamma_H \in S_H} \|u\|^2_{\Gamma_H, h} \leq c(H)(\|u\|^2_{1, h} + \|u\|^2_{X_h})
\]

with a constant \(c(F) = 2 \max\{F_x, F_y, 1/F_x, 1/F_y\}\).

**Proof.** Recall Young's inequality \(2ab \leq a^2 + b^2\) and the generalization for \(a_1, \ldots, a_N \in \mathbb{R}\)

\[
\left(\sum_{i=1}^{N} a_i \right)^2 \leq 2 \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i^2 + a_j^2) = N \sum_{i=1}^{N} a_i^2. \tag{12}
\]

Let us start with the left boundary \(\Gamma_w(\Theta_G)\). For fixed \(k_0\) in \(\{k \in \mathbb{N}| k \leq M_x\}\) and arbitrary \(\tau\) in \(T = \{h(t + 1/2)| t \in \mathbb{N}_0, t < M_y\}\), squaring a telescoping sum and using Young's inequality we get

\[
\frac{1}{2} |u(hk, \tau)|^2 \leq h^2 M_x \sum_{k=1}^{M_x} \left| \frac{u(3k, \tau) - u(h(k-1), \tau)}{h} \right|^2 + |u(hk_0, \tau)|^2.
\]

Now summing both sides over \(k_0\) from 1 to \(M_x\) and over all \(\tau \in T\) we arrive at

\[
\frac{1}{2} \|u\|^2_{\Gamma_w(\Theta_G), h} \leq G_x \|D_x u_x\|^2_{X_h; \Theta_G} + \frac{1}{G_x} \|u_x\|^2_{X_h; \Gamma(\Theta)}.
\]

Similar results can be found for the other edges and then can be combined to get the proposition.

**Theorem 5** (Poincaré inequality). Denote the side lengths of \(\Omega_h\) in \(x\) and \(y\) direction by \(L_x = hN_x, L_y = hN_y\). Then for \(u \in X_h\) we have

\[
\|u\|_{X_h} \leq \min\{L_x, L_y\} |u|_{1, h}.
\]

We want to emphasize that the constant does not depend on the aspect ratio of the domain.
Proof. It is sufficient to show the inequalities $\|u_x\|_{L_x} \leq L_x \|D_x u_x\|_{\Omega_h}$ and $\|u_y\|_{L_x} \leq L_x \|D_x u_y\|_{\Omega_h}$. For the latter we start again with the telescoping sum ($\tau \in T = \{lh | l \in \mathbb{N}_0, l \leq N_y\}$ and $k_0 \in \mathbb{N}_0, k_0 < N_x$)

$$u((k_0+1/2)h, \tau) = \frac{h}{2} (D_x u_y)(0,\tau) + \sum_{k=1}^{k_0} (D_x u_y)(kh,\tau),$$

which is to be squared using Young’s inequality (12) and then summed over $k_0$ from 0 to $N_x - 1$ and $\tau \in T$ to yield the inequality. Similarly the first one can be derived keeping in mind the zero boundary condition. \[\square\]

2.5. The negative norm on the staggered grid

We have to have a closer look at the $-1$ norm. Using Corollary 3 and the Cauchy Schwarz inequality we find for all $q \in Y_h$ (as in the continuous case)

$$|Dq|_{-1,h} = -\sup_{\varphi \in X_h \setminus \{0\}} \frac{(\text{div}_h \varphi, q)_{Y_h}}{||\varphi||_{1,h}} \leq \sup_{\varphi \in X_h \setminus \{0\}} \frac{||q||_{Y_h} \text{div}_h \varphi ||_{Y_h}}{||\varphi||_{1,h}} \leq ||q||_{Y_h}. \quad (13)$$

Now, we want to show that the supremum in the $-1$ norm is actually attained at $w = R_h Dp$.

**Theorem 6.** Let $p \in Y_h$. Then we have for $w = R_h Dp \in X_h$ that $|Dp|_{-1,h} = |w|_{1,h}$. \[\square\]

**Proof.** At first we show that $|w|_{1,h}$ is an upper bound of $|Dp|_{-1,h}$: For arbitrary $\phi \in X_h$ we have

$$-(\text{div}_h \phi, p)_{Y_h} = (\phi, -\Delta_h R_h Dp)_{X_h} = (D\phi, D R_h Dp)_{X_h} \leq |\phi|_{1,h} |w|_{1,h}. $$

Finally we show that the supremum is really attained in $w$:

$$|w|_{1,h}^2 = (R_h Dp, -\Delta_h R_h Dp)_{X_h} = (w, Dp)_{X_h} = -(\text{div}_h w, p)_{Y_h}. \quad \square$$

**Theorem 7** (LBB condition for the staggered grid). There is a positive constant $L_h(\Omega_h) > 0$ such that the inf-sup condition

$$|Dp|_{-1,h} = \sup_{u \in X_h \setminus \{0\}} \frac{(\text{div}_h u, p)_{Y_h}}{|u|_{1,h}} \geq L_h(\Omega_h) ||p||_{Y_h}$$

is satisfied for all $p \in Y_h$. Furthermore there is a $c > 0$ such that the following inequality between the continuous LBB constant $L(\Omega)$ and the one for the staggered grid holds:

$$L_h(\Omega_h) \geq \frac{L(\Omega)}{c}.$$ 

**Proof.** The proof can be found in [7]. \[\square\]

This result together with equation (13) gives the equivalence of the norms $\| \cdot \|_{Y_h}$ and $| \cdot |_{-1,h}$. The following lemma shows the superadditivity of the $-1$-norm on subsets.

\[8\]
Lemma 8. Let $\Omega_{h,i}, i = 1, \ldots, I$ be disjoint rectangular subsets of $\Omega_h$. Then for all $q \in Y_h$ we have
\[ \sum_{i=1}^{I} |Dq|_{-1,h;\Omega_{h,i}}^2 \leq |Dq|_{-1,h;\Omega_h}^2, \]
hence for $\{\Omega_{h,i}\} = \Pi_H$ ($\Lambda_H$ are not disjoint)
\[ \sum_{\Lambda_H \in \Pi_H} |Dq|_{-1,h;\Lambda_H}^2 \leq 4|Dq|_{-1,h;\Omega_h}^2. \]

Proof. Let $w \in X_h(\Omega_h)$ and $w_i \in X_h(\Omega_{h,i})$ be the solutions of
\[ (Dw, D\phi)_{X_h^\prime} = (-\text{div}_h \phi, q)_{Y_h} \quad \forall \phi \in X_h(\Omega_h), \]
\[ (Dw_i, D\phi)_{X_{h,i}^\prime} = (-\text{div}_h \phi, q)_{Y_{h,i}} \quad \forall \phi \in X_h(\Omega_{h,i}). \]

Then from Theorem 6 we have $|Dq|_{-1,h} = |w|_{1,h}$. Outside of $\Omega_{h,i}$ we define $w_i$ to be zero so that $w_i \in X_h(\Omega_h)$.

We start with two remarks on the additivity of the $Y_h$ inner product and the 1-norm. The outward normal vector of length $h/2$ is called $\delta$, e.g. $\delta(\Gamma_w) = (-h/2, 0)$. For $w_i \in X_h(\Omega_{h,i})$ as defined above and an arbitrary $p \in Y_h$ we have
\[ (p, \text{div}_h w_i)_{Y_h} = (p, \text{div}_h w_i)_{Y_{h,i}} \quad (14) \]
because of $w_i = 0$ on $\partial \Omega_{h,i}$. Paying attention to the definition of the norm and the derivative at the boundary we get for the 1-norm
\[ |w_i|_{1,h;\Omega_{h,i}}^2 = |w_i|_{1,h;\hat{\Omega}_{h,i}}^2 + \frac{h^2}{2} \sum_{\alpha \in \Gamma(\Omega_{h,i})} \left( \frac{2w_{i,\alpha} - \delta}{h} \right)^2 \]
\[ = |w_i|_{1,h;\hat{\Omega}_{h,i}}^2 + 2h^2 \sum_{\alpha \in \Gamma(\Omega_{h,i})} \left( \frac{w_{i,\alpha} - \delta - 0}{h} \right)^2 \quad (15) \]
\[ = |w_i|_{1,h}^2 + \frac{h^2}{2} \sum_{\alpha \in \Gamma(\Omega_{h,i})} \left( \frac{w_{i,\alpha} - \delta - 0}{h} \right)^2 \geq |w_i|_{1,h}^2. \]

As in every point at least one of the functions $Dw_i, Dw_j, i \neq j$ vanishes, it follows from (15) that
\[ \sum_{i=1}^{I} |w_i|_{1,h}^2 = \sum_{i=1}^{I} (Dw_i, Dw_i)_{X_h^\prime} = \sum_{i=1}^{I} \sum_{j=1}^{I} (Dw_i, Dw_j)_{X_h^\prime} \]
\[ = \sum_{i=1}^{I} (Dw_i, Dw_i)_{X_h^\prime} = \sum_{i=1}^{I} |w_i|_{1,h;\Omega_{h,i}}^2 \leq \sum_{i=1}^{I} |w_i|_{1,h;\Omega_{h,i}}^2. \quad (16) \]
Using (14), (16) and $\|a\| \leq \|(a, b)\| \leq \frac{1}{2} \|a\|^2 + \frac{1}{2} \|b\|^2$ we finally get the inequality
\[
\sum_{i=1}^I |Dq|_{-1,h;\Omega_h,i}^2 = \sum_{i=1}^I |w|_{1,h;\Omega_h,i}^2 = - \sum_{i=1}^I (q, \text{div}_h w_i)_{\Omega_h} = (Dw, \sum_{i=1}^I D w_i)_{X_h}^*
\]
\[
\leq \frac{1}{2} |w|_{1,h}^2 + \frac{1}{2} \sum_{i=1}^I |w_i|_{1,h}^2 \leq \frac{1}{2} |w|_{1,h}^2 + \frac{1}{2} \sum_{i=1}^I |w_i|_{-1,h;\Omega_h,i}^2
\]
\[
= \frac{1}{2} |Dq|_{-1,h}^2 + \frac{1}{2} \sum_{i=1}^I |Dq|_{-1,h;\Omega_h,i}^2.
\]

### 2.6. Norm estimates on the staggered grid

In this section we are going to prove some norm estimates which are important to study the operator norm of $P_3$.

- First we show $|Dq_H|_{-1,H} \leq c|q_H|_{1,H}$ for $q_H \in Y_H$ and then
- the converse $|Q_H q|_{1,H} \leq c|Dq|_{-1,h}$ for $q \in Y_h$;
- this gives us finally the equivalence of the norms $\|q - Q_H q\|_{Y_h}^2 + |Q_H q|_{1,H}^2)^{1/2}$ and $|Dq|_{-1,h}$ on $Y_h$.

All constants depend only on the short side of the domain $\Omega_h$ (min\{L_x, L_y\}) and local properties of the coarse grid $\Pi_H$ (side length, ratio of area and circumference as well as the local LBB constant $L_H$ on $\Lambda_H$). We start with a useful lemma:

**Lemma 9.** There is a positive constant $c$ independent of the aspect ratio of $\Omega_h$ such that for $v \in X_h$ and $q_H \in Y_H$ we have

$$\langle \text{div}_h v, q_H \rangle_{\Omega_h} \leq c|v|_{1,h} |q_H|_{1,H}.$$ 

**Proof.** With the trace theorem 4 and the Poincaré inequality 5 we have for the norm on the interior vertices of $S_H$

$$\|v\|_{S_H} \leq \sqrt{c(H)}(|v|_{1,h} + \|v\|_{X_h}) \leq c|v|_{1,h}, \tag{17}$$

with the constant $c = \sqrt{2 \max \{H_x, H_y, 1/H_x, 1/H_y\}(1 + \min\{L_x, L_y\})}$. $c$ only depends on local properties of the coarse grid and one side length of $\Omega_h$ but not on the aspect ratio.
As \( Dq_H = 0 \) in the interior of the coarse grid elements and \( v = 0 \) on \( \partial(U \cup V) \), we find
\[
(\text{div}_h v, q_H)_Y_h = -(v, Dq_H)_X_h = -h \sum_{\Gamma_H \in \mathcal{S}_H} \sum_{\alpha \in \Gamma_H \cap (U \cup V)} v_\alpha [q_H]_\alpha
\]
\[
\leq \left( h \sum_{\Gamma_H \in \mathcal{S}_H} \sum_{\alpha \in \Gamma_H \cap (U \cup V)} v_\alpha^2 \right)^{1/2} \left( \sum_{\Gamma_H \in \mathcal{S}_H} \sum_{\alpha \in \Gamma_H \cap (U \cup V)} h [q_H]_\alpha^2 \right)^{1/2}
\]
\[
= ||v||_{S_H} \left( \sum_{\Gamma_H \in \mathcal{S}_H} [q_H]_{1_H}^2 \left( \sum_{\alpha \in \Gamma_H \cap (U \cup V)} h \right) \right)^{1/2}
\]
using Definition 5 of \( |\cdot|_{1,H} \) this is together with (17)
\[
= ||v||_{S_H} |q_H|_{1,H} \leq c |v|_{1,h} |q_H|_{1,H}.
\]

**Corollary 10.** For \( q_H \in Y_H \) we have \( |Dq_H|_{-1,h} \leq c |q_H|_{1,H} \).

**Proof.**
\[
|Dq_H|_{-1,h} = \sup_{v \in X_h \setminus \{0\}} \frac{- (\text{div}_h v, q_H)_Y_h}{|v|_{1,h}} \leq \sup_{v \in X_h \setminus \{0\}} \frac{c |v|_{1,h} |q_H|_{1,H}}{|v|_{1,h}} = c |q_H|_{1,H}.
\]

**Lemma 11.** For \( q \in Y_h \) we have \( |Q_H q|_{1,H} \leq c |Dq|_{-1,h} \).

**Proof.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be two neighbouring elements of the coarse grid \( \Pi_H \) with areas \( \mu_1, \mu_2 \). Define the number of points of \( P \) in \( \Lambda_k \) as \( n_k = \sum_{\alpha \in (\Lambda_k \cap P)} 1 \); thus we have \( \mu_k = h^2 n_k \). Furthermore let \( q_k = \sum_{\alpha \in (\Lambda_k \cap P)} q_\alpha \) and \( \bar{q} = (q_1 + q_2)/(n_1 + n_2) \) be the average. Then we have the following relations:
\[
\sum_{\alpha \in (\Lambda_1 \cup \Lambda_2) \cap P} |\bar{q}\|^2 = (n_1 + n_2) |\bar{q}\|^2 = \frac{|q_1 + q_2|^2}{n_1 + n_2},
\]
\[
\sum_{\alpha \in (\Lambda_1 \cup \Lambda_2) \cap P} 2 q_\alpha \bar{q} = 2 \bar{q} \sum_{\alpha \in (\Lambda_1 \cup \Lambda_2) \cap P} q_\alpha = 2 \bar{q}(q_1 + q_2) = 2 \frac{|q_1 + q_2|^2}{n_1 + n_2},
\]
\[
|q_k|^2 = \left| \sum_{\alpha \in \Lambda_k \cap P} q_\alpha \right|^2 \leq \left( \sum_{\alpha \in \Lambda_k \cap P} |q_\alpha| \right)^2 \leq n_k \sum_{\alpha \in \Lambda_k \cap P} |q_\alpha|^2
\]
\[
|Q_H q|_{L_H}^2 = (Q_H q|_{\Lambda_2} - Q_H q|_{\Lambda_1})^2 = \left| \frac{q_2}{n_2} - \frac{q_1}{n_1} \right|^2.
\]
If we sum over all $\Gamma$

\begin{align*}
\|q - \bar{q}\|^2_{Y_h;\Omega(\Gamma_H)} &= h^2 \sum_{\alpha \in (\Lambda_1 \cup \Lambda_2) \cap P} (|q_\alpha|^2 - 2q_\alpha \bar{q} + |\bar{q}|^2) \\
&= h^2 \left( \sum_{\alpha \in (\Lambda_1 \cup \Lambda_2) \cap P} |q_\alpha|^2 - \frac{|q_1 + q_2|^2}{n_1 + n_2} \right) \\
&\geq h^2 \left( \frac{|q_1|^2}{n_1} + \frac{|q_2|^2}{n_2} - \frac{|q_1 + q_2|^2}{n_1 + n_2} \right) \\
&= h^2 \frac{n_1 n_2}{n_1 + n_2} \left( \frac{q_1}{n_1} \right)^2 + \left( \frac{q_2}{n_2} \right)^2 - 2 \frac{q_1 q_2}{n_1 n_2} \\
&= \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left| \frac{q_2}{n_2} - \frac{q_1}{n_1} \right|^2 = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} [Q_H q]^2_{1, H} \geq m \mu(\Gamma_H) |Q_H q|^2_{1, H},
\end{align*}

where the last estimate holds with a constant $m$, because $H \simeq 1$ and hence $\mu_k \simeq 1, \mu(\Gamma_H) \simeq 1$. Together with the LBB condition on $\Omega(\Gamma_H)$ with LBB constant $L_H(\Gamma_H)$ and the estimate by the continuous LBB constant $L(\Gamma_H)$ (Theorem 7) we find

$$
\mu(\Gamma_H) |Q_H q|^2_{1, H} \leq \frac{1}{m} \|q - \bar{q}\|^2_{Y_h;\Omega(\Gamma_H)} \leq \frac{1}{m L_H(\Gamma_H)} |D(q - \bar{q})|^2_{1, h;\Omega(\Gamma_H)} = \frac{1}{m L(\Gamma_H)} |D q|^2_{1, h;\Omega(\Gamma_H)} \leq \frac{c}{m L(\Gamma_H)} |D q|^2_{1, h;\Omega(\Gamma_H)}.
$$

If we sum over all $\Gamma_H \in S_H$, every element $\Lambda_H$ is summed up four times, hence, with Lemma 8 we get

$$
|Q_H q|^2_{1, H} = \sum_{\Gamma_H \in S_H} \mu(\Gamma_H) |Q_H q|^2_{1, H} \leq \sum_{\Gamma_H \in S_H} \frac{c}{m L(\Gamma_H)} |D q|^2_{1, h;\Omega(\Gamma_H)} \leq \frac{c}{m L^2} \sum_{\Gamma_H \in S_H} |D q|^2_{1, h;\Omega(\Gamma_H)} \leq \frac{12 c}{m L^2} |D q|^2_{1, h},
$$

with $L = \min_{\Gamma_H \in S_H} L(\Gamma_H)$.

If none of the coarse grid elements is degenerate, $s = \max\{H_x, H_y\}/\min\{L_x, L_y\}$ is moderate and so is $L$. Hence, the proof is completed with a constant $c$ only depending on $\mu(\Lambda_H)/\mu(\Gamma_H)$ and the LBB constant on a coarse grid square. $\square$

**Lemma 12.** The norms $(\|q - Q_H q\|^2_{1, H} + |Q_H q|^2_{1, H})^{1/2}$ and $|D q|_{-1, h}$ are equivalent on $Y_h$ with constants depending only on the short side $\min\{L_x, L_y\}$ and local properties.

**Proof.** From (13) and Lemma 10 we have

$$
|D q|_{-1, h} \leq |D(q - Q_H q)|_{-1, h} + |D Q_H q|_{-1, h} \leq \|q - Q_H q\|_{Y_h} + |D Q_H q|_{-1, h} \leq \|q - Q_H q\|_{Y_h} + c |Q_H q|_{1, H},
$$

and
For the other way we have with the LBB condition and Lemma 8
\[ \|q - Q_H q\|_{Y_h}^2 = \sum_{\Lambda_H \in \Pi_H} \|q - Q_H q\|_{Y_h; \Lambda_H}^2 \]
\[ \leq \sum_{\Lambda_H \in \Pi_H} \frac{1}{L_H(\Lambda_H)^2} |D(q - Q_H q)|_{1, h; \Lambda_H}^2 \]
\[ = \frac{1}{L_H^2} \sum_{\Lambda_H \in \Pi_H} |Dq|_{1, h; \Lambda_H}^2 \leq \frac{4}{L_H^2} |Dq|_{1, h}^2 \]
with \( \frac{1}{L_H^2} = \max_{\Lambda_H \in \Pi_H} \frac{1}{L_H(\Lambda_H)^2} \).

The relation with the continuous LBB constant is the same as in the proof of Lemma 11. Furthermore, from Lemma 11 we have
\[ |Q_H q|_{1, H} \leq \sqrt{\frac{6}{mL^2}} |Dq|_{1, h} \]
which completes the proof. \( \Box \)

3. The operator norm of the domain robust preconditioner

Having done these preparations we are now ready to proof the main result.

**Theorem 13.** There are positive constants \( c_1, c_2 \), depending only on the local shape of the subdivision \( \Pi_H \), the Poincaré constant \( \min\{L_x, L_y\} \) and the parameters \( a, b \), such that
\[ c_1 \leq \|P_3^{-1} L\|_{X_h \times Y_h \rightarrow X_h \times Y_h} \leq c_2 \]
with \( a, b \in \mathbb{R}, a \neq 0 \), \( P_3^{-1} = \begin{pmatrix} R_h & -b R_h D \\ 0 & a(I - Q_H + R_H Q_H) \end{pmatrix} \).

**Proof.** In this proof \( c \) is a generic constant which only depends on the local properties of the coarse grid \( m \) (from the proof of Lemma 11), \( L_H \) and \( H \) as well as \( \min\{L_x, L_y\} \). Let
\[ -\Delta_H v = -\Delta_H u + D(p - b \text{ div}_h u) , \]
\[ q_\perp = \text{div}_h u - Q_H \text{div}_h u , \]
\[ -\Delta_H q_H = Q_H \text{div}_h u . \]

Then we have
\[ P_3^{-1} L \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} u + R_h D(p - b \text{ div}_h u) \\ a(\text{div}_h u - Q_H \text{div}_h u + R_H Q_H \text{div}_h u) \end{pmatrix} = \begin{pmatrix} v \\ a(q_\perp + q_H) \end{pmatrix} . \]

**Upper bound:** \( |v|_{1, h}^2 + a^2 |D(q_\perp + q_H)|_{1, h}^2 \leq c_2 (|u|_{1, h}^2 + |Dp|_{1, h}^2) \). With Lemma 12 we have
\[ |v|_{1, h}^2 + a^2 |D(q_\perp + q_H)|_{1, h}^2 \leq |v|_{1, h}^2 + ca^2 (|q_\perp|_{Y_h}^2 + |q_H|_{Y_h}^2) \] (24)

\[ 13 \]
for the left hand side. At first we estimate \( |v|_{1,h}^2 \). Therefore, (21) is multiplied with \( v \):

\[
|v|_{1,h}^2 = (-\Delta_h v, v)_{X_h} = (-\Delta_h u, v)_{X_h} + (D(p - b \operatorname{div}_h u), v)_{X_h}
\]

\[
\leq |u|_{1,h} |v|_{1,h} - (\operatorname{div}_h v, p)_{Y_h} + b(\operatorname{div}_h v, \operatorname{div}_h u)_{Y_h}
\]

\[
\leq |u|_{1,h} |v|_{1,h} - (\operatorname{div}_h v, p)_{Y_h} + b|u|_{1,h} |v|_{1,h},
\]

where Theorem 2 was used. Applying the definition of the \(-1\)-norm we get

\[
|v|_{1,h} \leq (1 + b)|u|_{1,h} + |Dp|_{-1,h}.
\]

For the second term in (24) we have with Corollary 3

\[
\|q_\perp\|_{Y_h} = \|(I - Q_H) \operatorname{div}_h u\|_{Y_h} \leq \|\operatorname{div}_h u\|_{Y_h} \leq |u|_{1,h}.
\]

Finally for the third term we use Definition 5, (23) and Lemma 9:

\[
|q_H|_{1,h}^2 = (q_H, q_H)_{-1,h} = (-\Delta_H q_H, q_H)_{Y_h} = (Q_H \operatorname{div}_h u, q_H)_{Y_h}
\]

\[
= (\operatorname{div}_h u, q_H)_{Y_h} \leq c|u|_{1,h} |q_H|_{1,h}.
\]

Together with the inequality (24) the assertion

\[
|v|_{1,h}^2 + a^2 |D(q_\perp + q_H)|_{-1,h}^2 \leq |v|_{1,h}^2 + ca^2(\|q_\perp\|_{Y_h}^2 + |q_H|_{1,h}^2)
\]

\[
\leq (1 + b)^2 (|u|_{1,h} + |Dp|_{-1,h})^2 + ca^2 |u|_{1,h}^2
\]

\[
\leq (2(1 + b)^2 + ca^2) |u|_{1,h}^2 + 2(1 + b)^2 |Dp|_{-1,h}^2
\]

follows with the constant \( c_2 = 2(1 + b)^2 + ca^2 \).

**Lower bound:** Again with Lemma 12 we have to show the inequality

\[
|u|_{1,h}^2 + |Dp|_{-1,h}^2 \leq c(|v|_{1,h}^2 + \|q_\perp\|_{Y_h}^2 + |q_H|_{1,h}^2).
\]

First of all we look at the inner product

\[
(\operatorname{div}_h u, p - b \operatorname{div}_h u)_{Y_h} = (\operatorname{div}_h u, p - b \operatorname{div}_h u - Q_H(p - b \operatorname{div}_h u))_{Y_h}
\]

\[
+ (\operatorname{div}_h u, Q_H(p - b \operatorname{div}_h u))_{Y_h}.
\]

An estimate for the first term can be found using the local LBB condition on the coarse grid as in the proof of Lemma 12:

\[
(\operatorname{div}_h u, p - b \operatorname{div}_h u - Q_H(p - b \operatorname{div}_h u))_{Y_h}
\]

\[
= (q_\perp + Q_H \operatorname{div}_h u, p - b \operatorname{div}_h u - Q_H(p - b \operatorname{div}_h u))_{Y_h}
\]

\[
= (q_\perp, p - b \operatorname{div}_h u - Q_H(p - b \operatorname{div}_h u))_{Y_h}
\]

\[
\leq \|q_\perp\|_{Y_h} |p - b \operatorname{div}_h u - Q_H(p - b \operatorname{div}_h u)|_{Y_h}
\]

\[
\leq \frac{1}{L_H} \|q_\perp\|_{Y_h} |D(p - b \operatorname{div}_h u)|_{-1,h} + \frac{1}{L_H} \max_{H \in \mathcal{H}} \frac{4}{L_H} = \frac{1}{L_H} \|q_\perp\|_{Y_h} |D(p - b \operatorname{div}_h u)|_{-1,h}
\]

\[
\leq \frac{1}{2L_H^2} \|q_\perp\|_{Y_h}^2 + \frac{1}{2} \|D(p - b \operatorname{div}_h u)|_{-1,h}^2
\]

\[
\forall \varepsilon > 0.
\]
For the second term we have with (23) and Lemma 11

$$(\text{div}_h u, Q_H (p - b \text{div}_h u)) Y_h = (Q_H \text{div}_h u, Q_H (p - b \text{div}_h u)) Y_h$$

$$= (-\Delta q_h, Q_H (p - b \text{div}_h u)) Y_h$$

$$= (q_H, Q_H (p - b \text{div}_h u))_{1,H}$$

$$\leq |q_H|_{1,H} |Q_H (p - b \text{div}_h u)|_{1,H}$$

$$\leq c |q_H|_{1,H} |D(p - b \text{div}_h u)|_{-1,h}$$

$$\leq \frac{c}{2 \varepsilon} |q_H|^2_{1,H} + \frac{1}{2} \varepsilon |D(p - b \text{div}_h u)|^2_{-1,h}$$

for an arbitrary $\varepsilon > 0$. Adding both terms together yields

$$(\text{div}_h u, p - b \text{div}_h u) Y_h \leq \frac{c}{2 \varepsilon} (\|q_L\|^2_{Y_h} + |q_H|^2_{1,H}) + \varepsilon |D(p - b \text{div}_h u)|^2_{-1,h}. \quad (27)$$

To find an estimate for $|u|_{1,h}$ we multiply (21) by $u$ and rewrite the product similar to (25). Hence, with (27) we get

$$|u|^2_{1,h} \leq |u|_{1,h} |v|_{1,h} + (p - b \text{div}_h u, \text{div}_h u) Y_h$$

$$\leq \frac{1}{2} |u|^2_{1,h} + \frac{1}{2} |v|^2_{1,h} + (p - b \text{div}_h u, \text{div}_h u) Y_h$$

$$\Rightarrow |u|^2_{1,h} \leq |v|^2_{1,h} + 2(\text{div}_h u, p - b \text{div}_h u) Y_h$$

$$\leq |v|^2_{1,h} + \frac{c}{\varepsilon} (\|q_L\|^2_{Y_h} + |q_H|^2_{1,H}) + 2 \varepsilon |D(p - b \text{div}_h u)|^2_{-1,h}. \quad (28)$$

To find a bound for the last term in (28), we apply Theorem 6 to $w = R_h(D(p - b \text{div}_h u))$ and use (21), which yields

$$|D(p - b \text{div}_h u)|^2_{-1,h} = |w|^2_{1,h} = (Dw, D R_h(D(p - b \text{div}_h u))) X_h$$

$$= (w, D(p - b \text{div}_h u)) X_h$$

$$= (w, \Delta_h u - \Delta_h v) X_h$$

$$= (Dw, Du) X_h$$

$$\leq |D(p - b \text{div}_h u)|_{-1,h} (|u|_{1,h} + |v|_{1,h})$$

$$\Rightarrow |D(p - b \text{div}_h u)|_{-1,h} \leq |u|_{1,h} + |v|_{1,h}$$

$$\Rightarrow |D(p - b \text{div}_h u)|^2_{-1,h} \leq 2 |u|^2_{1,h} + 2 |v|^2_{1,h}. \quad (29)$$

Using (29) in (28) we get for an $0 < \varepsilon < 1/4$

$$|u|^2_{1,h} \leq c (|v|^2_{1,h} + \|q_L\|^2_{Y_h} + |q_H|^2_{1,H}), \quad (30)$$

where the constant $c$ depends only on $m$, $L_H$ and the choice of $\varepsilon$. The minimal value of $\max\{3, 16c\}$ ($c$ from (27)) is attained for $\varepsilon = 1/8$.

Finally, we need an estimate for $|Dp|_{-1,h}$. This is similar to the above but with $u' = R_h(Dp)$. Using the summation by parts we find with Corollary 3
that

\[ |Dp|^{2}_{-1,h} = |w'|^{2}_{1,h} = (\Delta_h u + b \text{div}_h u - \Delta_h v, w')_{X_h} \]
\[ = (\Delta_h u - \Delta_h v, w')_{X_h} + (b \text{div}_h u, w')_{X_h} \]
\[ = (Dv - Du, Dw')_{X_h} - b(\text{div}_h u, \text{div}_h w')_{Y_h} \]
\[ \leq |w'|_{1,h}(|u|_{1,h} + |v|_{1,h}) + b\|\text{div}_h u\|_{Y_h} \|\text{div}_h w'\|_{Y_h} \]
\[ \leq |w'|_{1,h}(1 + b)|u|_{1,h} + |v|_{1,h} \]

\[ \Rightarrow |Dp|^{2}_{-1,h} \leq 2(1 + b)^2|u|^{2}_{1,h} + 2|v|^{2}_{1,h}. \quad (31) \]

The desired inequality follows with (26) using (31), (30) and Lemma 12:

\[ |u|^{2}_{1,h} + |Dp|^{2}_{-1,h} \leq (2(1 + b)^2 + 1)|u|^{2}_{1,h} + 2|v|^{2}_{1,h} \]
\[ \leq c(2(1 + b)^2 + 1)(|v|^{2}_{1,h} + \|q\|^{2}_{Y_h} + |q|^{2}_{H,1,H}) + 2|v|^{2}_{1,h} \]
\[ \leq \frac{1}{c_1}(|v|^{2}_{1,h} + a^2|Dq|^{2}_{2,1,h}). \]

All the constants in the proof of Theorem 13 only depend on the parameters

- the LBB constant \(L_H\) on the elements of the coarse grid; but these are (close to) squares, so \(L_H\) is bounded,
- the size \(H \simeq 1\) of the coarse mesh,
- the ratio \(\mu(\Gamma_H)/\mu(\Lambda_H)\), which is moderate (as \(H \simeq 1\)) and
- the shorter side length of the domain \(\min\{L_x, L_y\}\).

Hence, this preconditioned Stokes operator is independent of the aspect ratio of the domain.

4. Numerical experiments

Analytical results for \(P_1\): To compare our results we first look at the non-domain robust preconditioner \(P_1\) from (5) (or similarly \(P_2^{-1} = \begin{pmatrix} R & 0 \\ -b \text{div} R & a \end{pmatrix}\), \(a \neq 0, b \in \mathbb{R}\)). From [3] it is known that for domains with smooth boundary all spectral values are eigenvalues and that \(L(\Omega)^2 \leq \mu \leq 1\). In the continuous case (which also applies to conforming discretizations) the eigenvalues of \(P_{1,2}^{-1}L\) can be calculated analytically. Simply starting from \(P_{1,2}^{-1}L(u,p)^T = \lambda(u,p)^T\) an equivalence of the eigenvalues \(\lambda\) of \(P_{1,2}^{-1}L\) and \(\mu\) of the Schur complement \(S = -\text{div} R D\) can be established. For both \(P_1\) and \(P_2\) the equivalence is given by

\[ \lambda = \frac{1}{2} \left( 1 + b\mu \pm \sqrt{(1 + b\mu)^2 - 4a\mu} \right). \quad (32) \]
Under the condition that the eigenvalues have to be positive real we can find bounds to the parameters $a$ and $b$: $a$ has to be strictly positive and either $a \leq b$ or $\frac{1}{4}(b^2 + 2b + 1) \geq a \geq b \land -1 < b \leq 1$.

To find the smallest and the largest possible eigenvalues which satisfy these constraints the monotony of $\lambda$ is studied by directly calculating the partial derivatives $\partial \lambda_{\pm}/\partial \mu$. $\lambda_{-}$ is always monotonically increasing whereas $\lambda_{+}$ is monotonically increasing for $a \leq b$ and strictly monotonic decreasing for $a > b$. As the convergence rate depends on the condition number $\kappa = \lambda_{\max}/\lambda_{\min}$, we then minimize a Taylor expansion of $\kappa$ with respect to $a, b$ and get for the optimal parameters $a = b \geq 1$ as leading term

$$\kappa = \frac{\mu_{\max}}{\mu_{\min}} = \frac{1}{\mu_{\min}},$$

which is the limiting case in [8], too.

**Numerical method:** As numerical examples the staggered grid discretization of the Stokes equations defined in section 2 was solved with the GMRES algorithm (see [9]). We start with $x^0 = 0$ and calculate the norm of the residual $r$ (of the non-preconditioned system) before the first and after the $k$-th iteration. The root of the quotient

$$\rho_k = \sqrt{\|r^k\|_2 / \|r^0\|_2}$$

(34)

can be taken as measure of the convergence rate, even though GMRES is not a linear method.

**Choice of parameters:** In the case of $P_3$ an analytical treatment in the same way as for $P_{1,2}$ is not possible because we only have bounds for the operator norm instead of an explicit formula for the eigenvalues. Therefore, the optimal parameters were determined numerically: For $a, b = 0.05, \ldots, 2$ and $h = 1/32$ the Stokes equations were solved on the rectangle $1 \times 20$. The resulting convergence rate is plotted in Fig. 2(a). As optimal choice for the parameter we found

$$a = b = 0.84.$$ (35)

These values were used for the other calculations.

**Independence of aspect ratio:** To see the independence of the aspect ratio the preconditioned system $P_3^{-1}L$ was solved using a staggered grid discretization with a grid constant of $h = 1/64$ on a rectangle of size $1 \times s, s = 10, \ldots, 90$.

In Fig. 2(b) the convergence rate of $P_3$ stabilizes below a level of $\rho = 0.585$. This is in contrast to the non-domain robust operator $P_1$ (which is identically to $P_3$ on squares), whose convergence rate asymptotically approaches 1 for increasing aspect ratio (see Fig. 2(b)). The slightly higher value of the optimum compared to Fig. 2(a) is due to the larger system.

Finally, the preconditioner $P_3$ was compared to the domain robust finite element preconditioner $C = P_3(a = 1/4, b = 0)$ from [5] (rectangles $1 \times s$, $h = 1/64$). To account for the different discretizations the convergence rate of the system $P_3^{-1}(a = 1/4, b = 0)L$ was determined on the staggered grid, too (in
the case $b = 0$ the analytical optimum is attained for $a = 1/4$). The results are compiled in Table 1.

5. Result and discussion

In this paper we proved that the operator norm of the preconditioned staggered grid approximation of the Stokes equations $P_3^{-1}L$ is independent of the
aspect ratio of the domain. Hence, all methods depending on the condition number converge independent of the shape of the domain. This could be confirmed numerically solving examples with GMRES.

The cost of the pressure correction on the coarse grid is small because we only have to solve a tridiagonal system whose size is the number of coarse grid elements. Furthermore it is advisable not to restrict the operator to $P_3(a, b = 0)$ because $P_3(a, b \neq 0)$ has not much computational overhead, basically one additional gradient, but a significantly higher convergence rate (smaller $\rho$). In comparison with the finite element preconditioner $C$ we see that the staggered grid is indeed advantageous for rectangles. Combining these benefits we could substantially improve the convergence rate at very low cost.

<table>
<thead>
<tr>
<th>Aspect ratio $s$</th>
<th>1</th>
<th>4</th>
<th>16</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{20}(C^{-1}L)$ from [5]</td>
<td>0.633</td>
<td>0.702</td>
<td>0.737</td>
<td>0.730</td>
</tr>
<tr>
<td>$\rho_{20}(P_3(\frac{1}{4}, 0)^{-1}L)$</td>
<td>0.405</td>
<td>0.685</td>
<td>0.691</td>
<td>0.693</td>
</tr>
<tr>
<td>$\rho_{20}(P_3(0.84, 0.84)^{-1}L)$</td>
<td>0.163</td>
<td>0.327</td>
<td>0.566</td>
<td>0.585</td>
</tr>
</tbody>
</table>

Table 1: Convergence rate $\rho_{20}$ of the preconditioned system. $C$ is the finite element version of $P_3(1/4, 0)$ from [5], whereas the second and third row contain $P_3$ on the staggered grid for the optimal parameters $(a = 1/4, b = 0)$ and $(a = 0.84, b = 0.84)$, respectively. For $s = 1$ $P_3$ is the same as $P_1$ from (5).

References


