The Thue-Morse Sequence and How to Create An Endless Game of Chess

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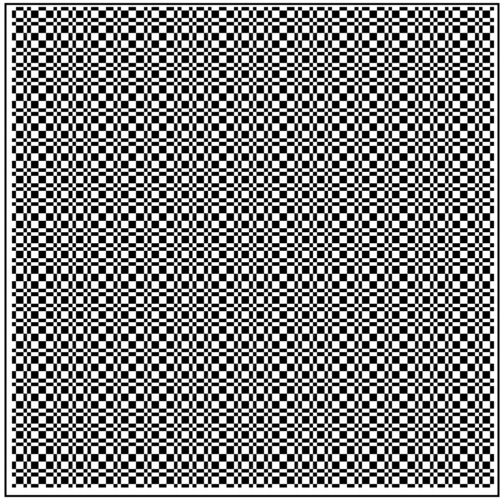
ANSBACH – 12 FEBRUARY 2015



From 1906-1912 the Norwegian mathematician **Axel Thue** investigated...

... the following sequence of 0s and 1s

$$t = 0\ 1\ 10\ 1001\ 10010110\ \ldots$$

The digits of $t = t_0 t_1 t_2 \ldots$ are defined by

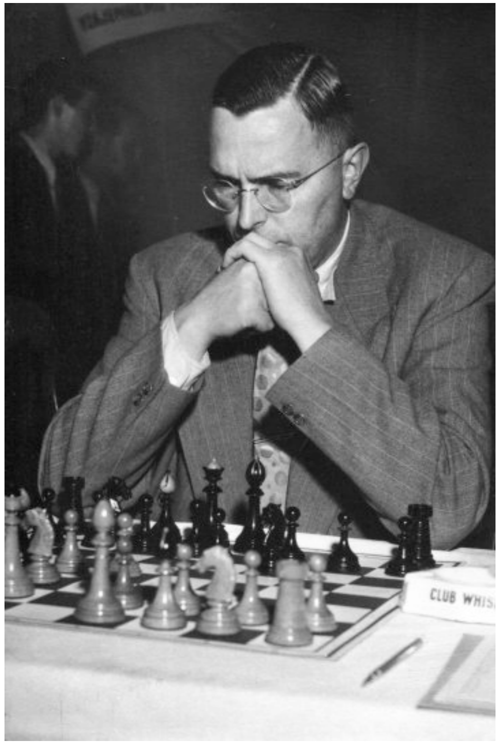
$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n.$$

t is a fixed point of the transformation $0 \mapsto 01, 1 \mapsto 10$

$$\begin{aligned} t &= \ 0 \ \ 1 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 1 \ \ 1 \ \ldots \\ &= 01 \ 10 \ 10 \ 01 \ 10 \ 01 \ 01 \ 10 \ 10 \ \ldots \ ; \end{aligned}$$

Thue showed that t is not periodic and
does not contain any subword three times in a row!

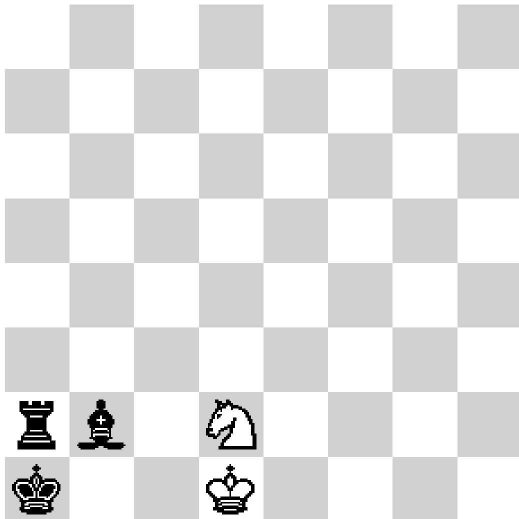
In 1921 the U.S.American mathematician **Harold Marston Morse** studied the sequence with respect to Riemann surfaces and deduced that on surfaces with negative curvature there exist recurrent non-periodic geodesics.



In 1929 the Dutch mathematician and chess player **Machgielis (Max) Euwe** rediscovered the sequence once more with respect to never-ending chess games...

The German Rule

finished a chess game with a draw ('remis'), if the same sequence of moves – with all figures at the same positions – appears three times in a row.



Euwe noticed that nevertheless never-ending chess games were possible despite the German rule!

Encoding the Thue-Morse sequence via

0 : Sb1-c3, Sb8-c6; Sc3-b1, Sc6-b8

1 : Sg1-f3, Sg8-f6; Sf3-g1, Sf6-g8

into an appropriate chess position one could play endlessly according to the rules of 1929!

As a consequence the German Rule was disregarded and **Euwe** became chess world champion in 1935-1937 **:-)**

Three times the same arrangement on a chess board would be a sufficient criterion to finish any chess game!

Background

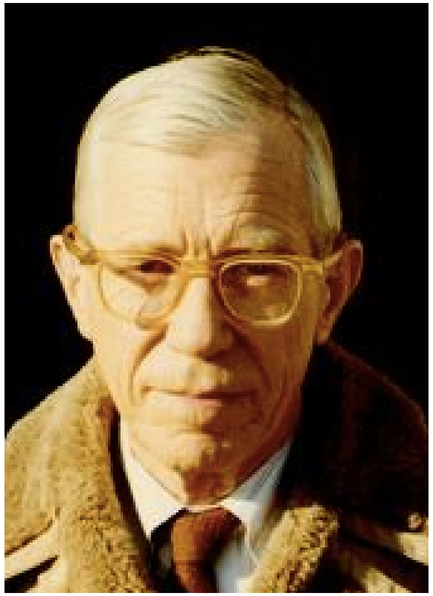
As already shown by **Thue**, the Thue-Morse sequence $t = t_0 t_1 t_2 \ldots$ does not contain any cube (a subword appearing three times in a row):

$$\ldots BBB \ldots \not\subset t \qquad \text{for all} \quad B \in \{0,1\}^n.$$

A theorem due to **Morse** and **Hedlund** even implies the impossibility of a subword of the form BBb where b is the leading digit of B.

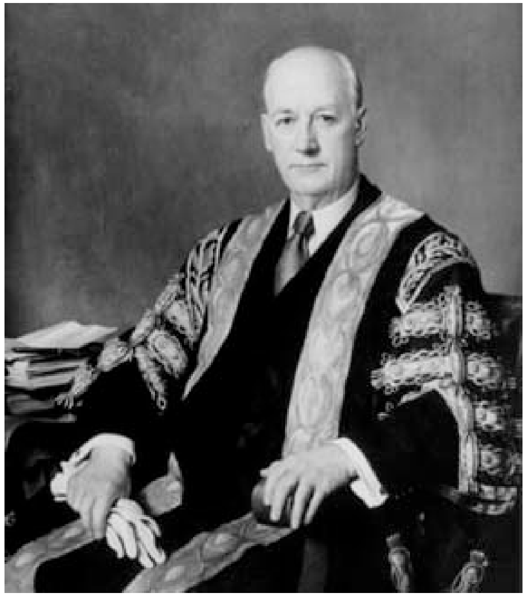
Sketch of proof: Induction with respect to the number $\sharp B$ of digits of B. Since t is built from subwords 01 and 10, there cannot be subwords of the form 000 or 111. Moreover, anything like 010101 or 101010 would correspond via the transformation $0 \mapsto 01, 1 \mapsto 10$ to 000 and 111, so they are impossible as well...

A New Discipline



In a series of papers in the 1940s **Morse** and his former doctoral student **Gustav Arnold Hedlund** founded the theory of Symbolic Dynamics (also Combinatorics on Words). In the sequel we shall have a look on another example of their theory...

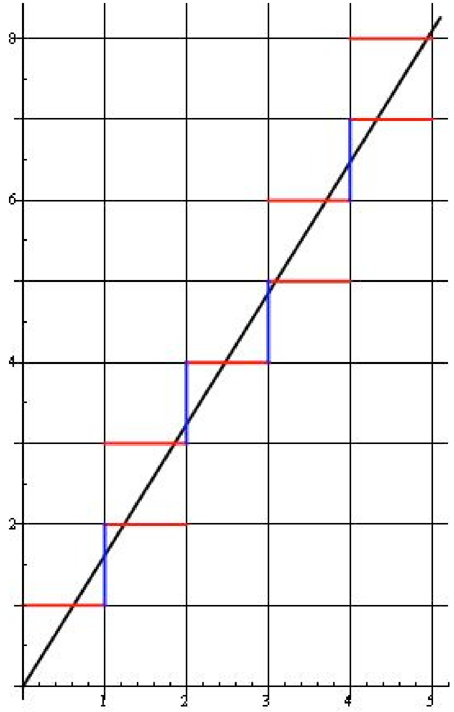
New Protagonists



enter the stage: the 19th centruy mathematician Swiss **Jacques Charles François Sturm** (on the left) and the Canadian 20th mathematician **Samuel Beatty** (on the right).

Sturmian Words

A straight line with irrational slope through the origin – e.g., $y = \frac{1}{2}(\sqrt{5}+1)x$ – does not contain any rational point except the origin.



Golden ratio: $\frac{1}{2}(\sqrt{5}+1) \to 010010100100\ldots$

The Fibonacci Word f

is the fixed point of the transformation $0 \mapsto 01, 1 \mapsto 0$

$$\begin{aligned} f &= 0 \;\; 1\,0 \;\; 0 \;\; 1\,0 \;\; 1\,0 \;\; 0 \ldots \\ &= 01\;0\;01\;01\;0\;01\;0\;01\;01\; \ldots\; ; \end{aligned}$$

it is **non-periodic** and limit of the recursion

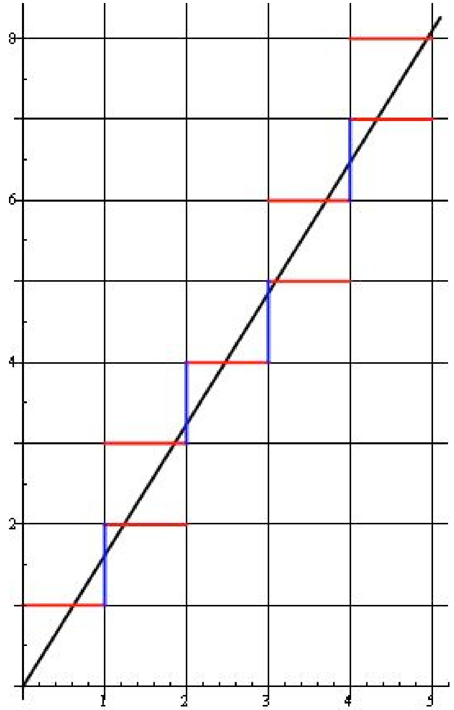
$$f_0 := 0,\; f_1 := 01 \qquad \text{and} \qquad f_{n+1} = f_n f_{n-1}.$$

The related Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ are recursively defined by $F_0 = 0$, $F_1 = 1$ und $F_{n+1} = F_n + F_{n-1}$; the quotients of consecutive Fibonacci numbers converge to the golden ratio:

$$1,\; 2,\; \tfrac{3}{2} = 1,5,\; \tfrac{5}{3} = 1,\overline{6},\; \tfrac{8}{5} = 1,6,\; \tfrac{13}{8} = 1,625, \ldots \to \tfrac{\sqrt{5}+1}{2} = 1,618\ldots$$

And Once More

we intersect the line $y = \frac{\sqrt{5}+1}{2}x$ with the lattice:



For $n = 1, 2, 3, \ldots$ we obtain the numbers $\lfloor n\frac{\sqrt{5}+1}{2} \rfloor$

$$1, 3, 4, 6, 8, \ldots$$

Beatty Sequences

If $\alpha > 1$ is irrational, then

$$B(\alpha) = \{\lfloor n\alpha \rfloor \,:\, n = 1, 2, \ldots\}$$

is the associated Beatty sequence.

Define β by $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then β is irrational too, and **Beatty**'s theorem claims that

$$B(\alpha) \cup B(\beta) = 1, 2, 3, \ldots \quad \text{and} \quad B(\alpha) \cap B(\beta) = \emptyset;$$

that is, both Beatty sequences provide a partition of the set of positive integers!

In case of the golden ratio $\frac{\sqrt{5}+1}{2}$ we obtain

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$$

If you want to read more on this topic:

• G.A. HEDLUND, J. MARSTON MORSE, Unending chess, symbolic dynamics and a problem in semi-groups, *Duke Math. J.* **11** (1944), 1-7

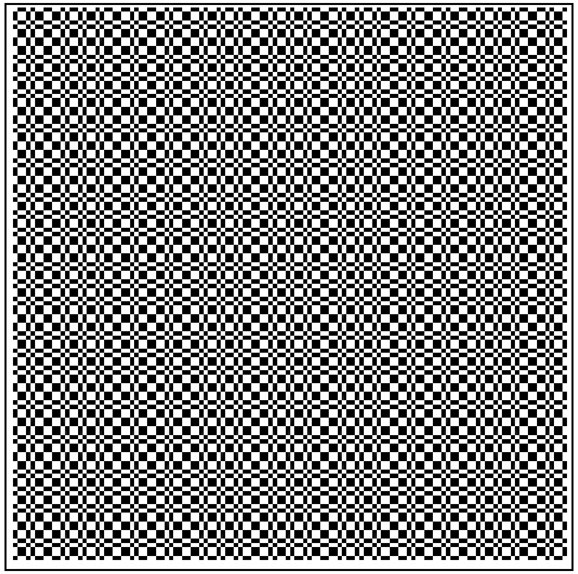
contains the proof of the impossibility of BBb in t

• K.B. STOLARSKY, Beatty sequences, continued fractions, and certain shift operators, *Canadian Math. Bull.* **19** (1976), 473-482

interesting with respect to the second part

• J.-P. ALLOUCHE, J. SHALLIT, The ubiquitous Prouhet-Thue-Morse sequence, in *Sequences and their applications*, Proceedings of SETA 98, Springer 1999, 1-16

good reading, contains much more than what we discussed here!



Thank you!

You can find the slides at

http://www.mathematik.uni-wuerzburg.de/∼steuding/