Rational vs. Irrational Numbers and Applications

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Rational Numbers

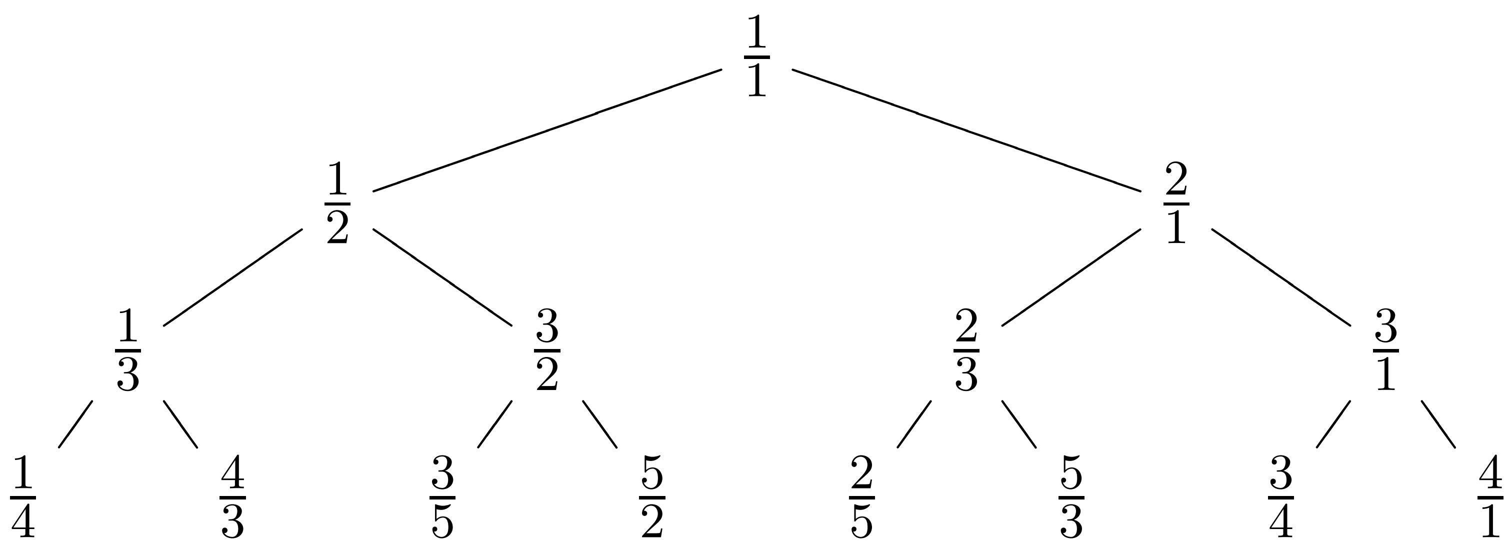
A number x is said to be rational if there exist integers a, b such that $x = \frac{a}{b}$ (e.g., 13, $\frac{1}{2}$, $-\frac{10}{15}$ etc.).

The Calkin-Wilf Tree

is growing according to the iteration

$$\frac{a}{b} \quad \mapsto \quad \frac{a}{a+b}\,, \quad \frac{a+b}{b},$$

out of the root $\frac{1}{1}$. The first iterations provide



Calkin & Wilf (2000): Every positive rational appears in the tree once and only once, namely as a reduced fraction.

How to design a good calendar?

A tropical year lasts

$$365 \text{ days } 5 \text{ hours } 48 \text{ minutes and } 45.8 \text{ seconds}$$
$$\approx \quad 365 + \frac{\mathbf{419}}{\mathbf{1730}} \text{ days.}$$

Unfortunately, this is not an integer!

Using the euclidean algorithm we find

$$\begin{aligned} 1730 &= 4 \cdot 419 + 54, \\ 419 &= 7 \cdot 54 + 41, \\ 54 &= 1 \cdot 41 + 13, \\ &\dots \end{aligned}$$

A First Approximation

and deduce

$$365 + \frac{\mathbf{419}}{\mathbf{1730}} = 365 + \left(\frac{\mathbf{1730}}{\mathbf{419}}\right)^{-1} \approx \; 365 + \frac{1}{\mathbf{4}}.$$

This is the Julian Calender (named after Julius Caesar, 45 B.C.; the design is due to the Greek astronomer Sosigenes of Alexandria): a leap day all four years!

A Continued Fraction

For the tropical year we find the representation

$$365 + \frac{\mathbf{419}}{\mathbf{1730}} = 365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{2}}}}}}.$$

Using this without the last fraction we obtain the better approximation

$$365 + \frac{194}{801} \approx 365 + \frac{\mathbf{419}}{\mathbf{1730}},$$

wich represents the Gregorian Calendar (Pope Gregor XIII, 1582; designed by Aloysius Lilius and Pietro Pitati): in 800 years 6 $(= 200 - 194)$ leap years are omitted.

Irrationalities

A number is called irrational if it is not rational (e.g., $\sqrt{2}$, π, $\sum_{n\geq 1} n^{-3}$ etc.).

Georg Cantor (1845-1918)

There are by far more irrational numbers than rationals.

Georg Cantor (1873): You cannot make a list of all real numbers.

In particular, you cannot make a list of all irrational numbers!

Din A-Format

What is the proportion of length and width of a Din A-size sheet of paper?

The proportion in question equals

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}$$

as an infinite continued fraction!

But $\sqrt{2}$ is irrational, so how to realize that in practice?

Use the approximation $\frac{29{,}7\,\mathrm{cm}}{21\,\mathrm{cm}} = \frac{99}{70}$, i.e., the fifth convergent to the continued fraction representing $\sqrt{2}$; then the error is small: $|\sqrt{2} - \frac{99}{70}| < 0.00007\ldots$.

A Glimpse to 'Higher' Mathematics

A Question due to König & Szücs from 1913

Tübingen students playing billiards, early 19th century, Städtische Sammlungen Tübingen

(Source: R.A. Müller, Geschichte der Universität, 1990

Billiards on a Square

Describe the trajectory of a billiard ball on a quadratic table! Under which circumstances is it periodic?

The ball is assumed to be one point moving on straight lines and without friction; it is reflected on the boundary according to the laws of physics, and is supposed not to hit a corner!

Theorem: If the tangent of the incoming angle is rational, then the trajectory is periodic.

The trajectory of the ball in the xy-plane can be described by

$$y = y_0 + \alpha x.$$

The slope of this straight line is equal to the tangent of the incoming angle or its reciprocal.

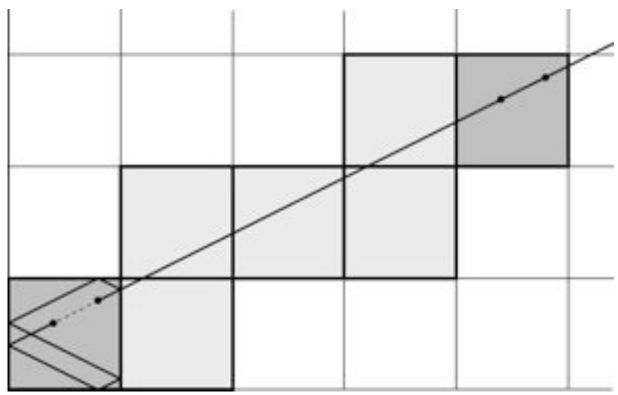
Periodic Trajectories

We define a lattice in the xy-plane by the straight lines

$$x = k, \quad y = j \qquad \text{for} \quad k, j \in \tfrac{1}{2}\{\ldots, -2, , -1, 0, 1, 2, \ldots\}$$

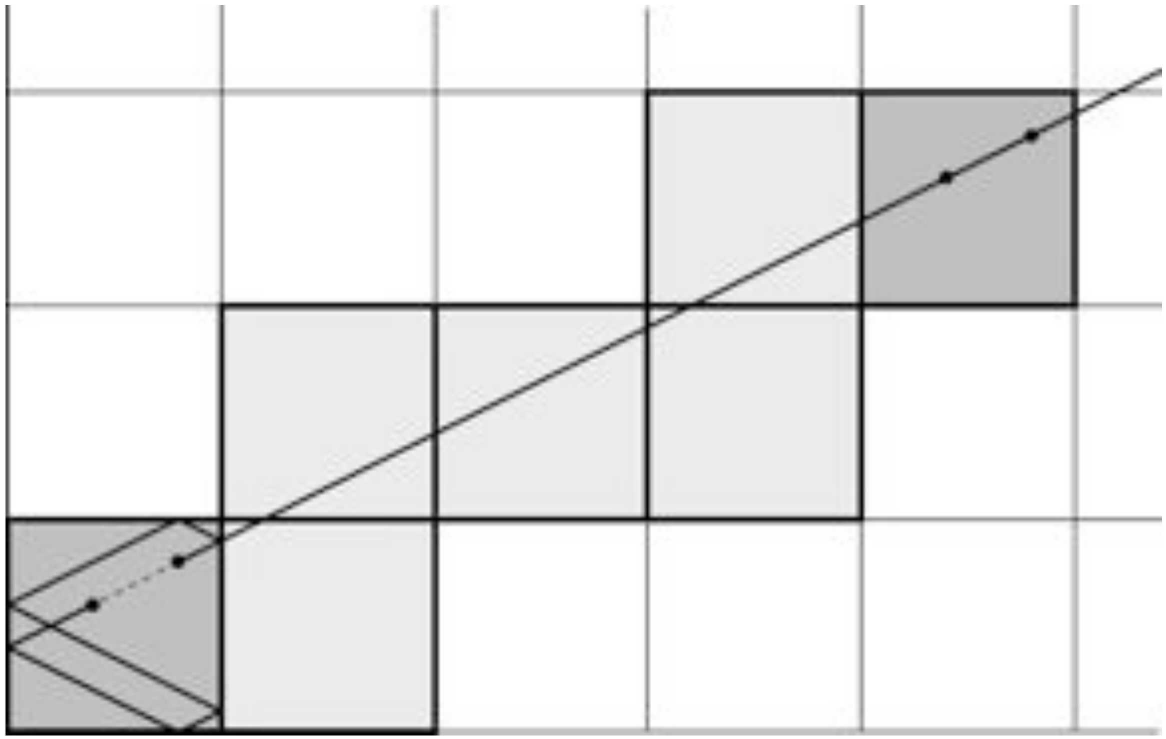
and consider the quadratic billard table as the square with vertices $(0,0), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})$.

Instead of reflecting the trajectory of the ball on the boundary of the table we reflect the table when the ball hits the boundary...



Periodic Trajectories

appear if and only if there exists an integer translation of the form $(x, y) \mapsto (x + q, y + p)$;



in this case $\alpha = \frac{p}{q}$.
Hence, the trajectory is periodic if and only if α is rational.

Non-Periodic Trajectories

What can be said if the trajectory is not periodic? that means, if α is irrational?

Theorem: If the tangent of the incoming angle is irrational, then the trajectory visits every neighbourhood of every point of the table.

The proof uses the following result from the theory of diophantine approximation:

Bohl's Theorem (1909): Let α be irrational. For every x with $0 < x < 1$ and every $\epsilon > 0$ there exists a positive integer n such that

$$\left| n\alpha - \lfloor n\alpha \rfloor - x \right| < \epsilon.$$

Hence, some fractional parts of $n\alpha$ lie arbitrarily close to any given point $x \in [0, 1)$; this is false for rational α.

A Question...

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... and this can indeed be proved by methods from the theory of uniform distribution;

Uniform distribution is a simple form of ergodicity...

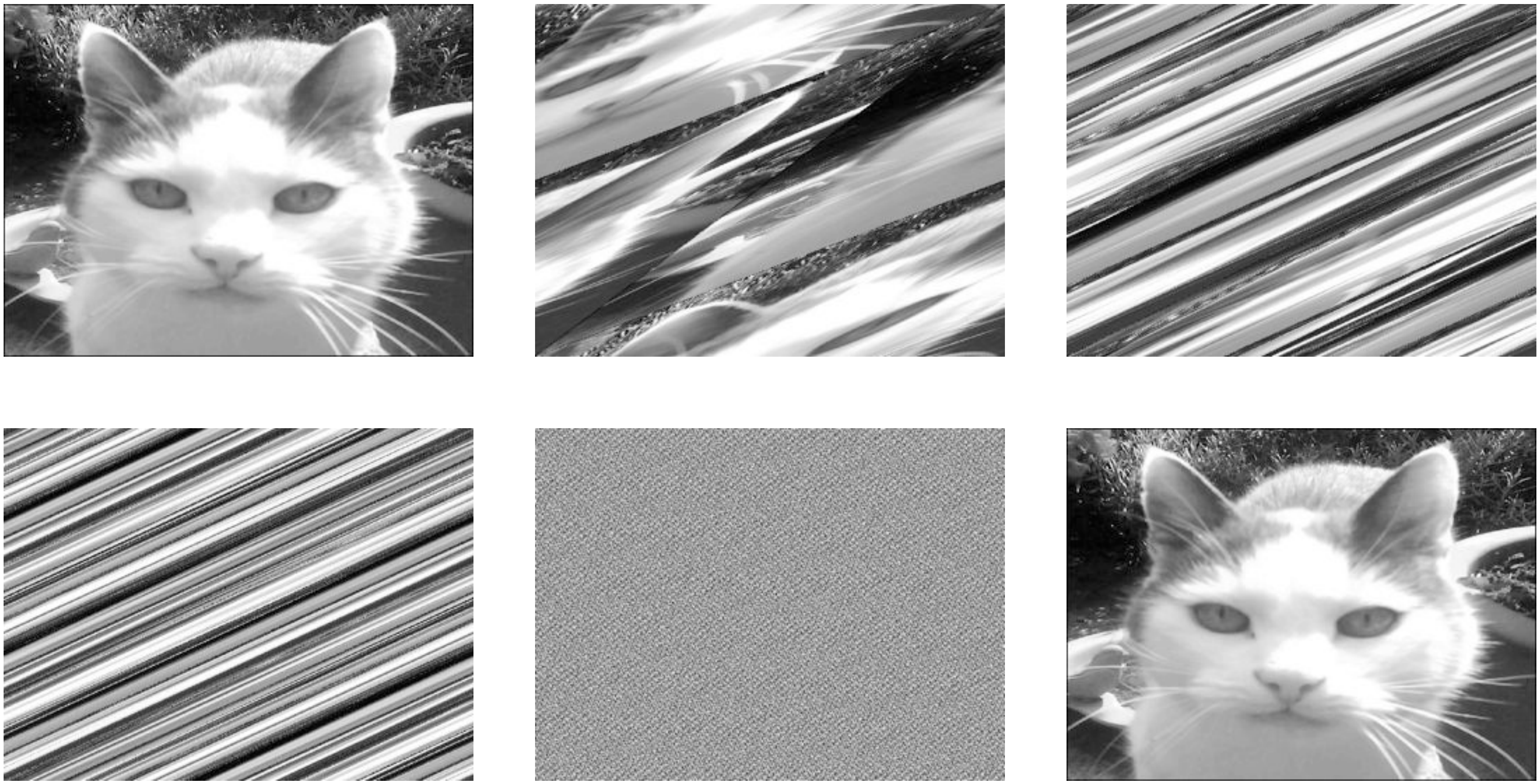
Ergodicity

appears for the mapping $n \mapsto n\alpha - \lfloor n\alpha \rfloor$ if α is irrational;

ergodic mappings show a chaotic and unpredictable behaviour.

The physicists **Boltzmann (1871)** and **Maxwell (1879)** asked questions and posed conjectures concerning the trajectories of physical particles in closed systems (one consequence is the main theorem of thermodynamics); the mathematicians **Poincaré (1890), Weyl (1914), Neumann, Birkhoff (1931)** and **Khintchine (1933)** treated these problems by use of methods from probability theory and analysis ...
There are plenty of number-theoretical applications!

Arnold’s Cat Map

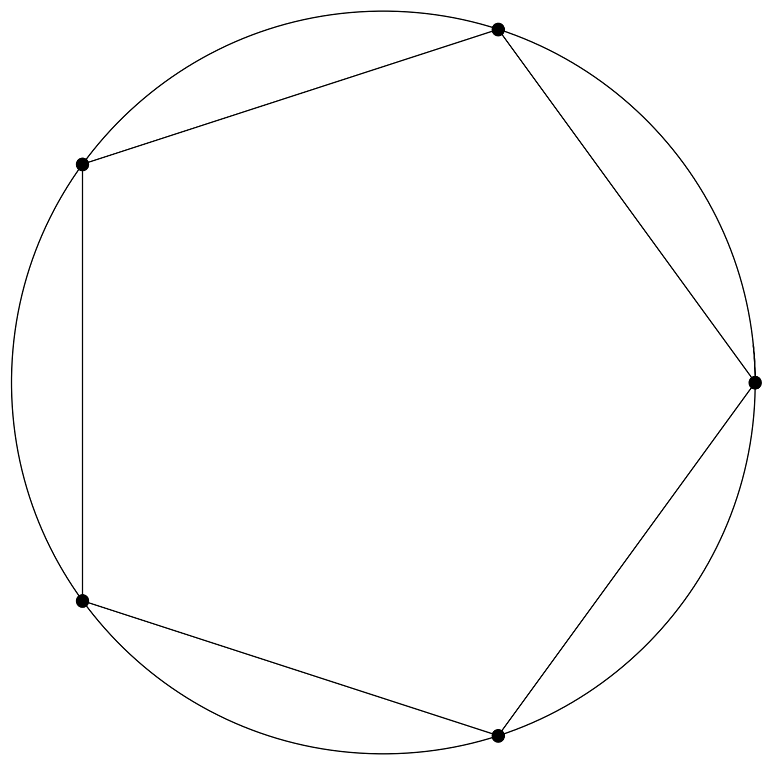


”Arnold’s cat map” is the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x+y \\ x+y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1$.

A Puzzle: Circle Billiards

Under which circumstances is the trajectory of a billiard ball inside a circle periodic?

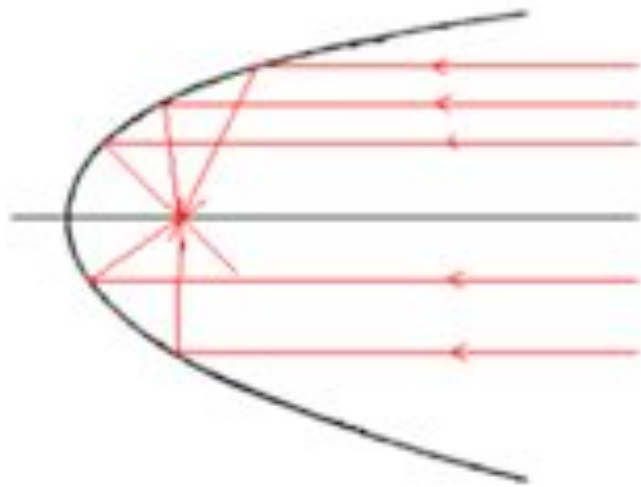
A periodic example with the incoming angle $\frac{\pi}{5} = 36^\circ$



Billiard in Parabolas

Parabolic antenna use the geometry of parabola.

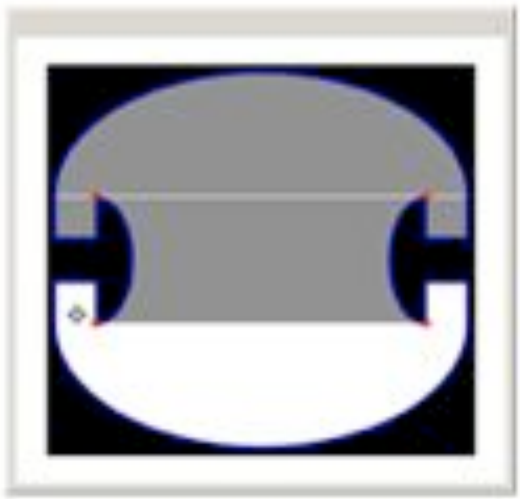
Every ray is reflected such that it goes through the inner focal point of the parabola!



(The pictures are taken from the wikipedia-website.)

Can Every Room with Mirror Walls...

... be completely illuminated by just one candle? asked **Straus** 195?.
In 1958 **Roger Penrose** gave a first counter-example of a not completely illuminated room.



On the left a picture due to Wolfram showing such a room; on the right a photograph of **Penrose** on his famous tiling; in the 1980s chemists discovered quasi-crystalls with this symmetry! More on this topic can be found at http://en.wikipedia.org/wiki/Penrose_tiling

Thank you!

You can find these slides on my webpage

http://www.mathematik.uni-wuerzburg.de/∼steuding/,

as well you can download course notes on ergodic theory at

http://www.mathematik.uni-wuerzburg.de/∼steuding/ergod.htm