One Hundred Years Uniform Distribution Modulo One and Recent Applications to Riemann's Zeta-Function

Jörn Steuding

Dedicated to the Memory of Professor Wolfgang Schwarz

Abstract We start with a brief account of the theory of uniform distribution modulo one founded by Weyl and others around one hundred years ago (which is neither supposed to be complete nor historically depleting the topic). We present a few classical implications to diophantine approximation. However, our main focus is on applications to the Riemann zeta-function. Following Rademacher and Hlawka, we show that the ordinates of the nontrivial zeros of the zeta-function $\zeta(s)$ are uniformly distributed modulo one. We conclude with recent investigations concerning the distribution of the roots of the equation $\zeta(s) = a$, where a is any complex number, and further questions about such uniformly distributed sequences.

1 Dense Sequences and Classical Diophantine Approximation

There are several opportunities to motivate uniform distribution modulo one. We start with a remarkable observation from the Middle Ages due to the French mathematician Nicole Oresme. In his work *De proportionibus proportionum* from around 1360 he wrote that

> ”*it is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that any [one] would be incommensurable to any [other].*”[1]

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[1] The English translation is taken from [77].

In another work entitled *Tractatus de commensurabilitate vel incommensurabilitate motuum cell* from this time ORESME considered two bodies moving on a circle with uniform but incommensurable velocities; here he claimed that

> ”*no sector of a circle is so small that two such bodies could not conjunct in it at some future time, and could not have conjuncted in it sometime [in the past].*”[2]

These sentences form part of ORESME's refutation of astrology. He considered the future as essentially unpredictable, a modern viewpoint which was pretty controversal to the standards of his contemporaries. The above quotations indicate a deep understanding of irrationality and circle rotations. In modern mathematical language ORESME's observation is that rational numbers form a negligible set (of LEBESGUE measure zero) and that the multiples of an irrational number lie dense in the unit interval; with this statement ORESME was more than half a millennium ahead of his time although his reasoning had gaps; we refer to [77] for a detailed analysis of his thinking. ORESME is also well-known for his opposition to ARISTOTLE's astronomy; indeed he thought about rotation of the Earth about two centuries before COPERNICUS. Moreover, ORESME wrote an interesting treatise on the speed of light and he invented a kind of coordinate geometry before DESCARTES, to mention just a few of his ingenious ideas.

We continue with an interesting phenomenon about irregularities in the distribution of digits in statistical data: in 1881, SIMON NEWCOMB noticed that in books consisting tabulars with values for the logarithm those pages starting with digit 1 were looking more used than others. In 1938, this phenomenon was rediscovered and popularized by the physicist FRANK BENFORD [3] who gave further examples from statistics about American towns. According to this distribution a set of numbers is said to be BENFORD *distributed* if the leading digit equals $k \in \{1,2,\ldots,9\}$ for $\log_{10}(1+\frac{1}{k})$ percent instances. Thus, slightly more than thirty percent of the numbers in a BENFORD distributed data set have leading digit 1, and only about six percent start with digit 7. BENFORD's law is supposed to hold for quite many sequences as constants in physics and stock market values. An example for which BENFORD's law is known to hold is the sequence of FIBONACCI numbers $0,1,1,2,3,5,8,13,21,34,\ldots$, however, the sequence of primes is not BENFORD distributed as was proved by JOLISSAINT [52] and DIACONIS [16]. Recent investigations show that certain stochastic processes, e.g., the geometric BROWNian motion or the $3X+1$-iteration due to COLLATZ satisfy BENFORD's law as shown by KONTOROVICH & MILLER [58].

Here is an illustrating example of a deterministic sequence which follows BENFORD's law. Considering the powers of two, we notice that among the first of those powers,

$$1,2,4,8,16,32,64,128,256,512,1024,2048,4096,8092,\ldots,$$

there are indeed more integers starting with digit 1 than with digit 3. Obviously, a power of 2 with a decimal expansion of $m+1$ digits has leading digit k if, and only

[2] The English translation is taken from [77].

if,

$$10^m k \leq 2^n < 10^m(k+1) \qquad \text{for} \quad k \in \{0,1,\ldots,9\};$$

taking the logarithm gives

$$m + \log_{10} k \leq n \log_{10} 2 < m + \log_{10}(k+1).$$

For a real number x we introduce the decomposition in its integral and fractional parts by writing $x = \lfloor x \rfloor + \{x\}$ with $\lfloor x \rfloor$ being the largest integer less than or equal to x and $\{x\} \in [0,1)$ the fractional part. Consequently, the latter inequalities transform into

$$\log_{10} k \leq \{n \log_{10} 2\} < \log_{10}(k+1).$$

Since the logarithm is concave, the interval $[\log_{10} k, \log_{10}(k+1))$ is larger for small k, so, heuristically, the chance is larger that $n \log_{10} 2$ has fractional part in such an interval as n ranges through the set of positive integers. In the next section we shall show that the seqence of numbers $\log_{10} x_n = n \log_{10} 2$ is *uniformly distributed modulo* 1 which implies that indeed the proportion of 2^n with leading digit $k \in \{1,2,3,\ldots,9\}$ equals the length of the interval $[\log_{10} k, \log_{10}(k+1))$, that is

$$\log_{10}(k+1) - \log_{10} k = \log_{10}(1 + \tfrac{1}{k}).$$

In particular, $\log_{10} 2 \approx 30.1$ percent of the powers of 2 have a decimal expansion with leading digit 1 whereas the leading digit equals 7 for only approximately 5.8 percent. On the contrary, powers of 10 have always leading digit 1 in the decimal system. This shows that the arithmetical nature of $\log_{10} 2$ is relevant for the proportion with which leading digits appear.

For the first we shall generalize this problem to powers of some positive integer a with respect to expansions to an arbitrary base b. Here both, a and b are positive integers at least two. We shall use classical inhomogeneous DIOPHANTine approximation in order to show that any possible digit will appear as leading digit of a power of a if, and only if, $\log_b a$ is irrational. A theorem of LEOPOLD KRONECKER [60] from 1884 states that *given* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\beta \in \mathbb{R}$, *for any* $N \in \mathbb{N}$ *and any* $\varepsilon > 0$, *there exist integers* $n > N$ *and* m *such that*

$$|n\alpha - m - \beta| < \varepsilon.$$

There are many proofs of this theorem; see [43] for a collection of such proofs. An elementary proof could start with the observation that, if $\alpha \notin \mathbb{Q}$, then there exist integers k, ℓ such that $|k\alpha - \ell| < \varepsilon$. Hence, the sequence $\{k\alpha\}, \{2k\alpha\}, \ldots$ provides a chain of points across $[0,1)$ where the distance between consecutive points is less than ε. Applying KRONECKER's approximation theorem, we find that for any fixed $k \in \{1,\ldots,b-1\}$, for any $\beta \in (\log_b k, \log_b(k+1))$, and any $\varepsilon > 0$, there do exist integers m, n such that

$$|n \log_b a - m - \beta| < \varepsilon$$

provided $\log_b a$ is irrational. Thus, for sufficiently small ε, the number a^n has leading digit k with respect to its expansion in base b. Otherwise, if $\log_b a$ is rational, the sequence $\{n \log_b a\}$ is periodic and the distribution of leading digits of a^n differs from BENFORD's law. We may interpret KRONECKER's approximation theorem as follows: *Given an irrational α, the sequence $\{n\alpha\}$ lies dense in the unit interval $[0,1)$ as n ranges through $\mathbb{N}$.* This is nothing but ORESME's statement from the beginning! In the next section we shall strengthen this approximation theorem significantly.

2 Uniform Distribution Modulo One

Given a dense sequence in the unit interval, e.g., the fractional parts of the numbers $n\alpha$ with some explicit irrational real number α, it is natural to ask how this sequence is distributed: *are there subintervals that contain* only a few *elements of this sequence? How soon does a sequence meet a given subinterval?* The elaborated study of such dense sequences was started around 1909 by three mathematicians independently.

The Latvian mathematician PIERS BOHL [7] was the first to succeed with a quantitative improvement of KRONECKER's denseness theorem. He came across the following DIOPHANTine result:

> ”*We consider a onesided unlimited yarn with an infinite number of knots in such a way that the first knot is at the end of the yarn and the other knots follow equally spaced with subsequent distance $r > 0$. (...) On a circle of circumference 1 we take a line segment AB of length s $(0 < s < 1)$ and wrap the yarn starting from some arbitrary point on the circumference. The number of the first n knots which after wrapping the yarn fall into the line segment AB is denoted by $\varphi(n)$. (...) If r is an irrational number, then (...)* $\lim_{n\to\infty} \frac{\varphi(n)}{n} = s$.”[3]

BOHL's reasoning was of geometrical nature and rather complicated; his motivation originated from astronomical questions.

In order to formulate his result in modern language we begin with a crucial definition: a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo one* (resp. *equidistributed*) if for all α, β with $0 \leq \alpha < \beta \leq 1$ the proportion of the fractional parts of the x_n in the interval $[\alpha, \beta)$ corresponds to its length in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : \{x_n\} \in [\alpha, \beta)\} = \beta - \alpha.$$

[3] This is the author's free translation of the original German text: ”*Wir denken uns nun einen einseitig unbegrenzten Faden und nehmen auf demselben eine unbegrenzte Zahl von Knoten in der Weise an, daß der erste Knoten mit dem Fadenende zusammenfällt, während die übrigen im Abstände $r > 0$ der Reihe nach aufeinander folgen. (...) Auf einem Kreise vom Umfang 1 nehmen wir (...) eine Strecke AB von der Länge s $(0 < s < 1)$ an und wickeln den Faden von irgendeinem Punkte ausgehend auf die Peripherie auf. Die Anzahl derjenigen unter den n ersten Knoten, welche bei der Aufwickelung auf der Strecke AB liegen, bezeichnen wir mit $\varphi(n)$. (...) Ist r eine Irrationalzahl, so folgt (...)* $\lim_{n\to\infty} \frac{\varphi(n)}{n} = s$.”

Obviously, it suffices to consider only intervals of the form $[0,\beta)$ with arbitrary $\beta \in (0,1)$.

Theorem 1. *Given a real number α, the sequence $(n\alpha)_{n\in\mathbb{N}}$ is uniformly distributed modulo one if, and only if, α is irrational.*

This theorem is due to BOHL [7] and it provides an immediate solution of the problem concerning the powers of two from the previous section. Since $\log_{10} 2$ is irrational, an application of Theorem 1 shows that the proportion of positive integers n for which the inequalities $\log_{10} k \leq \{n \log_{10} 2\} < \log_{10}(k+1)$ hold equals the length of the interval, that is $\log_{10}(1+\frac{1}{k})$, as predicted by BENFORD's law. We shall give an elegant and short proof of BOHL's theorem below.

Around the same time WACŁAW SIERPIŃKSI [86, 87] gave an independent proof of this result; his motivation was of pure arithmetical nature. Finally, there is to mention HERMANN WEYL [98, 99] who at the same time was investigating GIBB's phenomenon in FOURIER analysis; he was faced with essentially the same arithmetical question as BOHL and SIERPINSKI. In view of these rather different motivations uniform distribution was indeed a *hot topic* around 1909/10. A little later, FELIX BERNSTEIN [4] observed the similarities in the papers of BOHL, SIERPINKSI, and WEYL. Interestingly, his approach is based on LEBESGUE theory, a modern tool in that time which turned out to be not appropriate with respect to uniform distribution modulo one (as follows from Theorem 2 below). The paper of BERNSTEIN was rather influential; it stands at the beginning of further investigations of illustrous mathematicians.

Once HARALD BOHR said "*To illustrate to what extent Hardy and Littlewood in the course of the years came to be considered as the leaders of recent English mathematical research, I may report what an excellent colleague once jokingly said: 'Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy-Littlewood.*" (cf. [69]) In 1912, the third of these three English mathematicians, HARDY & LITTLEWOOD [40, 41] started his (their) research on uniform distribution and succeeded to prove

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(\pi i n^k \alpha) = 0 \tag{1}$$

for fixed $k \neq 0$ and irrational α. Their aim were applications to the RIEMANN zeta-function, however, of different nature than our later applications. The same can be said about the work of BOHR & COURANT [9]. We refer to BINDER & HLAWKA [5] for a detailed historical account of their work and the very beginnings of uniform distribution theory. We remark that besides Cambridge, where HARDY & LITTLEWOOD were doing their work, Göttingen was the place of location giving the impetus on uniform distribution theory. Here BERNSTEIN was working as a professor, WEYL as young docent, and BOHR was visiting COURANT.

Another impact of BERNSTEIN's paper [4] was the new awakening of WEYL's old interest in questions on rational approximation. The following quote is from a late work of WEYL [105]:

"When the problem and Bohl's paper were pointed out to me by Felix Bernstein in 1913, it started me on my investigations on Diophantine approximations..."

Indeed, WEYL starts his pathbreaking article [103] with almost the same words as the above quotation of BOHL's theorem. We continue with presenting the main results from WEYL's papers [101, 102, 103].

Theorem 2. *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo one if, and only if, for any* RIEMANN *integrable function $f:[0,1]\to\mathbb{C}$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}x. \tag{2}$$

By this criterion uniform distribution modulo one can be characterized by a certain property in some class of functions. This point of view is completely different from previous approaches and might be seen as starting point of any deeper study of uniform distribution. Moreover, WEYL's first theorem may be interpreted as a forerunner of the celebrated BIRKHOFF pointwise ergodic theorem [6]; nowadays, any treatise on ergodic theory with applications in number theory as, for instance [19], includes uniform distribution modulo one and, in particular, WEYL's theorem as motivation for the concept of ergodicity. And indeed, in 1913/14, ROSENTHAL [81, 82] showed the impossibility of the *strong* ergodicity hypothesis from statistical mechanics and how a *weak* ergodicity hypothesis can be used as substitute; for the latter purpose he used ideas similar to those of BOHR and WEYL.

Proof. Given $\alpha,\beta\in[0,1]$, denote by $\chi_{[\alpha,\beta)}$ the indicator function of the intervall $[\alpha,\beta)$, i.e.,

$$\chi_{[\alpha,\beta)}(x) = \begin{cases} 1 & \text{if } \alpha\le x<\beta, \\ 0 & \text{otherwise.}\end{cases}$$

Obviously,

$$\int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x = \beta-\alpha.$$

Therefore, the sequence (x_n) is uniformly distributed modulo 1 if, and only if, for any pair $\alpha,\beta\in[0,1)$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{[\alpha,\beta)}(\{x_n\}) = \int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x.$$

Assuming the asymptotic formula (2) for any RIEMANN integrable function f, it follows that (x_n) is indeed uniformly distributed modulo one.

In order to show the converse implication we suppose that (x_n) is uniformly distributed modulo 1. Then (2) holds for $f=\chi_{[\alpha,\beta)}$ and, consequently, for any linear combination of such indicator functions. In particular, we may deduce that (2) is true for any step function. For any real-valued RIEMANN integrable function f and any $\varepsilon>0$, we can find step functions t_-,t_+ such that

$$t_-(x) \le f(x) \le t_+(x) \qquad \text{for all} \quad x \in [0,1],$$

and

$$\int_0^1 (t_+(x) - t_-(x))\,\mathrm{d}x < \varepsilon.$$

Hence,

$$\int_0^1 f(x)\,\mathrm{d}x \ge \int_0^1 t_-(x)\,\mathrm{d}x > \int_0^1 t_+(x)\,\mathrm{d}x - \varepsilon,$$

and

$$\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)\,\mathrm{d}x \le \frac{1}{N}\sum_{n=1}^{N} t_+(\{x_n\}) - \int_0^1 t_+(x)\,\mathrm{d}x + \varepsilon,$$

which is less than 2ε for all sufficiently large N. Analogously, we obtain a similar lower bound. Consequently, (2) holds for all real-valued RIEMANN integrable functions f. The case of complex-valued RIEMANN integrable functions can be deduced from the real case by treating the real and imaginary part of f separately. •

We shall illustrate Theorem 2 with an example of a sequence which is not uniformly distributed modulo one. For this purpose we consider the fractional parts of the numbers $x_n = \log n$ and the function defined by $f(u) = \exp(2\pi i u)$. An easy computation shows

$$\sum_{n=1}^{N} f(\log n) = \sum_{n=1}^{N} n^{2\pi i} = \sum_{n=1}^{N} \left(\frac{n}{N}\right)^{2\pi i} N^{2\pi i} \sim N^{1+2\pi i}\int_0^1 u^{2\pi i}\,\mathrm{d}u = \frac{N^{1+2\pi i}}{1+2\pi i}$$

which is not $o(N)$. Hence, the sequence $(\log n)_n$ is not uniformly distributed modulo 1. Actually, this is the reason why we have been surprised by BENFORD's law: if (x_n) is uniformly distributed modulo one, then $(\log x_n)$ is BENFORD distributed. As a matter of fact, the BENFORD distribution is nothing else than the probability law of the mantissa with respect to the basis.

The converse of WEYL's Theorem was found by DE BRUIJN & POST [11]: *given a function $f : [0,1) \to \mathbb{C}$ with the property that for any uniformly distributed sequence (x_n) the limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$$

exists, then f is RIEMANN *integrable.* It is interesting that here the RIEMANN integral is superior to the LEBESGUE integral (different from ergodic theory where it is vice versa). In fact, Theorem 2 does not hold for LEBESGUE integrable functions f in general since f might vanish at each point $\{x_n\}$ but have a non-vanishing integral. This subtle difference is related to a rather important application of uniformly distributed sequences, namely so-called Monte-Carlo methods and their use in numerical integration: if N points are *uniformly distributed* in the square $[-1,1]^2$ in the EUCLIDean plane and the number M counts those points which lie inside the unit circle centered at the origin, then the quotient M/N is a good guess for the area π of the unit disk. In view of this idea uniformly distributed sequences can be used to evaluate numerically certain integrals for which there is no elementary method, e.g. the GAUSSian integral $\int \exp(-x^2)\,\mathrm{d}x$. More on this topic can be found in [47].

Our next aim is another characterization of uniform distribution modulo one, also due to WEYL. For abbreviation, we write $\mathrm{e}(\xi) = \exp(2\pi i \xi)$ for $\xi \in \mathbb{R}$ which translates the $2\pi i$-periodicity of the exponential function to 1-periodicity: $\mathrm{e}(\xi) = \mathrm{e}(\xi + \mathbb{Z})$.

Theorem 3. *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo one if, and only if, for any integer $m \neq 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(mx_n) = 0. \tag{3}$$

Proof. Suppose the sequence (x_n) is uniformly distributed modulo one, then Theorem 2 applied with $f(x) = \mathrm{e}(mx)$ shows

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(mx_n) = \int_0^1 \mathrm{e}(mx)\,\mathrm{d}x.$$

For any integer $m \neq 0$ the right-hand side equals zero which gives (3).

For the converse implication suppose (3) for all integers $m \neq 0$. Starting with a trigonometric polynomial

$$P(x) = \sum_{m=-M}^{+M} a_m\,\mathrm{e}(mx) \qquad \text{with} \quad a_m \in \mathbb{C},$$

we compute

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(\{x_n\}) = \sum_{m=-M}^{+M} a_m \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(mx_n) = a_0 = \int_0^1 P(x)\,\mathrm{d}x. \tag{4}$$

Recall WEIERSTRASS' approximation theorem which claims that, for any continuous 1-periodic function f and any $\varepsilon > 0$, there exists a trigonometric polynomial P such that

$$|f(x) - P(x)| < \varepsilon \qquad \text{for} \quad 0 \leq x < 1 \tag{5}$$

(this can be proved with FOURIER analysis). Using this approximating polynomial, we deduce

$$\begin{aligned} &\left| \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)\,\mathrm{d}x \right| \\ &\leq \left| \frac{1}{N} \sum_{n=1}^{N} \big(f(\{x_n\}) - P(\{x_n\})\big) \right| + \left| \frac{1}{N} \sum_{n=1}^{N} P(\{x_n\}) - \int_0^1 P(x)\,\mathrm{d}x \right| \\ &\quad + \left| \int_0^1 (P(x) - f(x))\,\mathrm{d}x \right|. \end{aligned}$$

The first and the third term on the right are less than ε thanks to (5); the second term is small by (4). Hence, formula (2) holds for all continuous, 1-periodic functions f. Denoting by $\chi_{[\alpha,\beta)}$ the indicator function of the interval $[\alpha,\beta)$ (as in the proof of the previous theorem), for any $\varepsilon > 0$, there exist continuous 1-periodic functions f_-, f_+ satisfying

$$f_-(x) \le \chi_{[\alpha,\beta)}(x) \le f_+(x) \qquad \text{for all} \quad 0 \le x < 1,$$

and

$$\int_0^1 (f_+(x) - f_-(x))\,\mathrm{d}x < \varepsilon.$$

This leads to

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}) = \int_0^1 \chi_{[\alpha,\beta)}(x)\,\mathrm{d}x.$$

Hence, the sequence (x_n) is uniformly distributed modulo one. •

A probabilistic proof can be found in ELLIOTT's monography [21].

As an easy application of the latter criterion we shall deduce BOHL's Theorem 1: if α is irrational, then $\mathrm{e}(m\alpha) \neq 1$ for any $0 \neq m \in \mathbb{Z}$ and the formula for the finite geometric series yields

$$\sum_{n=1}^{N} \mathrm{e}(mn\alpha) = \mathrm{e}(m\alpha)\frac{1-\mathrm{e}(mN\alpha)}{1-\mathrm{e}(m\alpha)}$$

for all integers $m \neq 0$. Since this quantity is bounded independently of N, it follows that (3) holds with $x_n = n\alpha$. Otherwise, $\alpha = \frac{a}{b}$ for some integers a, b with $b \neq 0$; in this case the limit is different from zero for all integer multiples m of b and Theorem 3 implies the assertion. For an elementary proof see MIKLAVC [73].

WEYL [103] gave the following polynomial generalization of BOHL's Theorem 1 extending the result on the uniform distribution of αn^k implied from (1) significantly: *If $P = a_dX^d + \ldots + a_1X + a_0$ is a polynomial with real coefficients, where at least one coefficient a_j with $j \neq 0$ is irrational, then the values $P(n)$ are uniformly distributed modulo one as n ranges through $\mathbb{N}$.*

In the brief introduction to uniform distribution modulo one above we have closely followed WEYL [103]. It should be noticed that Theorem 3 was already known to WEYL as early as Summer 1913 previous to Theorem 2; in [103] he wrote about BOHL's theorem:

> ”*The claim, that this sequence is everywhere dense, is the content of a famous approximation theorem due to Kronecker. The present stronger theorem has been presented first by myself in a talk at the Göttingen Mathematical Society in Summer 1913 and it had been proved in a similar way as here.*”[4]

[4] This is the author's free translation of the original German text: ”*Die Behauptung, daß diese Punktfolge überall dicht liegt, ist der Inhalt eines berühmten Approximationssatzes von Kronecker. Das vorliegende viel schärfere Theorem ist zuerst im Sommer 1913 von mir in einem Vortrag in der Göttinger Mathematischen Gesellschaft aufgestellt und ähnliche Weise wie hier bewiesen worden.*”

However, after its presentation at the meeting of the *Göttingen Mathematical Society* in July 1913 WEYL did not intend to publish this criterion immediately since at that time he was much impressed by BOHR's approach to related problems (see [101]). The first publication of WEYL's elegant characterization of uniform distribution is [102]. In view of HARDY & LITTLEWOOD's estimate (1) it follows from Theorem 3 that for an arbitrary positive integer k the sequence $n^k\alpha$ is uniformly distributed modulo one if α is irrational. It might have been that WEYL was inspired by (1) to consider exponential sums with respect to uniform distribution and his extension to more general polynomials changed his reservation to publish his results.

The year 1913 must have been a very important year for HERMANN WEYL for various reasons. Not only that he gave birth to the theory of uniform distribution modulo one, in the same year WEYL married HELENE JOSEPH, a student of the philosopher HUSSERL, he left Göttingen for Zurich where he became full professor at the Polytechnic Zurich,[5] and he published his famous treatise on RIEMANN surfaces [100]. At that time WEYL was 27 years old. His later works to mathematics include his important contributions to the theory of group representations, mathematical physics, and philosophy of mathematics. In 1919, WEYL adopted BROUWER's ideas about intuitionism and in particular WEYL's approach to uniform distribution modulo one was based on non-constructive mathematics. This is neither curious nor tragic since WEYL was discussing this type of questions with a certain gingerliness. As TASCHNER [90] showed, WEYL's reasoning can be constructivised.

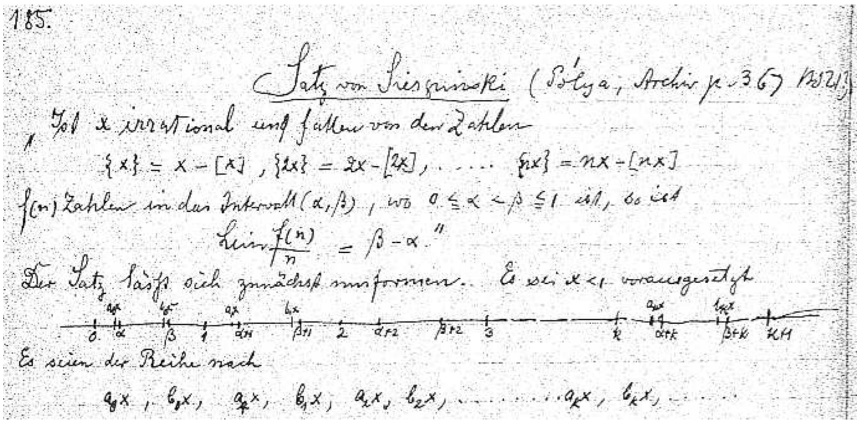

Fig. 1 Adolf Hurwitz, Mathematische Tagebücher, No. 26, page 185: Sierpiński's theorem. As follows from a handwritten note [78] Hurwitz's protégé Pólya gave the inspiration.

It might be interesting to notice that in the mathematical diaries of Adolf Hurwitz [48] one can find an entry from April 1914[6] dealing with Sierpiński's theorem [86,

[5] *Eidgenössische Hochschule Zürich* (ETH)

[6] A precise date is impossible because this entry is without date, however, comparing with other dates entries one may deduce that Hurwitz wrote this in between April 2 and April 30.

87] claiming that, for any $a \in [0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m\leq n} \{m\alpha + a\} = \tfrac{1}{2}$$

if, and only if, α is irrational. At that time young WEYL and the established HURWITZ were colleagues at the ETH Zurich but it seems that the elder did not know about the younger's work on this topic beyond the papers [98, 99] from 1909/10. Actually, SIERPIŃSKI used a result of HURWITZ in his second paper [87] and we may guess that this started HURWITZ's interest on this topic. In his diary HURWITZ proves a generalization of SIERPIŃSKI's theorem which is very close to Weyl's Theorem 2, namely: *if f is* RIEMANN *integrable on* $[0,1]$ *and α irrational, then*

$$\lim_{n\to\infty} \frac{1}{n}\left(f(\{\alpha\}) + f(\{2\alpha\}) + \ldots + f(\{n\alpha\})\right) = \int_0^1 f(x)\,\mathrm{d}x$$

(by slight modification of his notation from the diary). It is also mentioned that this holds with replacing the left hand-side by $\lim_{n\to\infty} \frac{1}{n}(f(\{c+\alpha\}) + f(\{c+2\alpha\}) + \ldots + f(\{c+n\alpha\}))$, where c is an arbitrary real number.

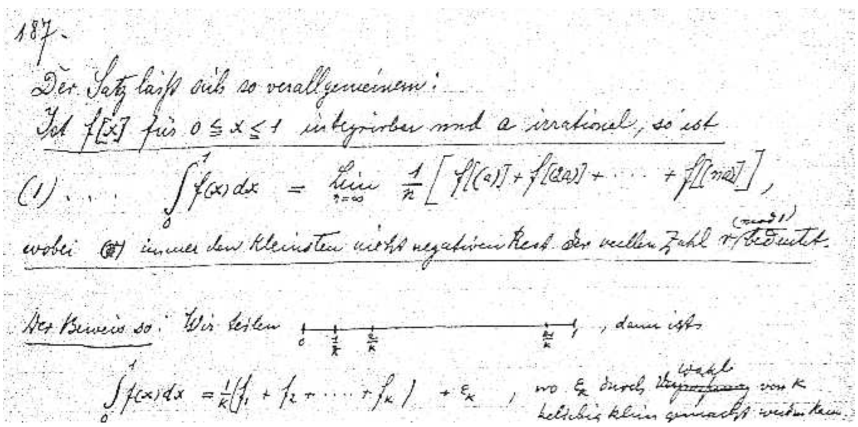

Fig. 2 Adolf Hurwitz, Mathematische Tagebücher, No. 26, page 187: Hurwitz's generalization.

The strong criterion, WEYL's Theorem 3, can be applied and extended in various ways giving generalizations beyond BOHL's theorem. He himself strengthened results of HARDY & LITTLEWOOD on sequences of the form $n^k\alpha$ (as already mentioned above), mathematical billiards, and the three body problem (see [103]). One of the most spectacular results is due to I.M. VINOGRADOV [94] being the main ingredient in his proof of the *ternary* GOLDBACH conjecture that any sufficiently large odd integer can be represented as a sum of three primes. For this purpose he found a non-trivial estimate for the exponential sum $\sum_{p_n\leq N} \mathrm{e}(p_n\alpha)$, where p_n denotes the nth prime (in ascending order). VINOGRADOV proved that for irrational α

the sequence $(p_n\alpha)$ is uniformly distributed modulo 1. In order to get an impression on the depth of this result one may notice that in case of rational α this question is intimately related to the distribution of primes in arithmetic progressions.[7] The *binary* GOLDBACH conjecture that any even integer larger than two is representable as sum of two primes is wide open.

We conclude this section with another open question. It is not known whether the sequence of powers $(\frac{3}{2})^n$ or the numbers $\exp(n)$ are uniformly distributed modulo one. KOKSMA [56] showed that almost all sequences (α^n) with $\alpha > 1$ are uniformly distributed, however, there is no single α with this property explicitly known. On the contrary, if α is a SALEM number, i.e., all algebraic conjugates of α (except α) have absolute value less than one, then the sequence (α^n) is not uniformly distributed. An excellent reading on the beautiful theory of uniform distribution modulo one are the monographs [12] and [61] by BUGEAUD and KUIPERS & NIEDERREITER, respectively.

3 Basic Theory of the Riemann Zeta-Function

Prime numbers are the fascinating multiplicative atoms from which the integers are built. It was the young GAUSS who was the first to conjecture the true order of growth for the number $\pi(x)$ of primes $p \le x$. In a letter to ENCKE from Christmas 1849 GAUSS wrote

> ”*You have reminded me of my own pursuit of the same subect, whose first beginnings occurred a very long time ago, in 1792 or 1793, when I had procured for myself Lambert's supplement to the table of logarithms. Before I had occupied myself with the finer investigations of higher arithmetic, one of my first projects was to direct my attention to the decreasing frequency of prime numbers, to which end I counted them up in several chiliads and recorded the results on one of the affixed white sheets. I soon recognized, that under all variations of this frequency, on average, it is nearly inversely proportional to the logarithm, so that the number of all prime numbers under a given boundary n were nearly expressed through the integral*
>
> $$\int \frac{\mathrm{d}n}{\log n},$$
>
> *if the integral is understood hyperbolic.*” (see [92]).[8]

---

[7] Recently, H. HELFGOTT published an article ’Major arcs for Goldbach's theorem’ (see arXiv:1305.2897) and another article ’Numerical Verification of the Ternary Goldbach Conjecture up to 8.875e30’ (see arXiv:1305.3062) which is joint work with D.J. PLATT; both pieces together imply the full ternary GOLDBACH conjecture provided that there is no serious gap in their reasoning.

[8] ”*Die gütige Mittheilung Ihrer Bemerkungen über die Frequenz der Primzahlen ist mir in mehr als einer Beziehung interessant gewesen. Sie haben mir meine eigenen Beschäftigungen mit demselben Gegenstande in Erinnerung gebracht, deren erste Anfänge in eine sehr entfernte Zeit fallen, ins Jahr 1792 oder 1793, wo ich mir die Lambertschen Supplemente zu den Logarithmentafeln angeschafft hatte. Es war noch ehe ich mit feineren Untersuchungend er höheren Arithmetik mich befasst hatte eines meiner ersten Geschäfte, meine Aufmerksamkeit auf die abnehmende Frequenz der Primzahlen zu richten, zu welchem Zweck ich die einzelnen Chiliaden abzählte, und die Resul-*

The appearing integral is the logarithmic integral; if the upper limit equals x, then it is asymptotically equal to $\frac{x}{\log x}$, hence GAUSS' conjecture can be made precise in writing

$$\pi(x) \sim \mathrm{li}\,(x) := \int_2^x \frac{\mathrm{d}u}{\log u}\,, \qquad \text{as} \quad x \to \infty, \tag{6}$$

using a modified logarithmic integral.

In 1859, RIEMANN [80] figured out how the distribution of prime numbers can be studied by means of analysis; in contrast to previous work of EULER on the zeta-function RIEMANN had the stronger tools of complex analysis at hand. In the following we shall briefly survey his remarkable insights in the close relation between primes and the zeta-function.

For $\mathrm{Re}\, s > 1$, the RIEMANN *zeta-function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}; \tag{7}$$

here the product is taken over all prime numbers p. The identity between the series and the product is an analytic version of the unique prime factorization of the integers as becomes obvious by expanding each factor of the product into a geometric series. This type of series is called a DIRICHLET series and a product over primes as above is referred to as EULER product. It is not difficult to show that both, the series and the product in (7) converge absolutely for all complex numbers s with $\mathrm{Re}\, s > 1$. We need an analytic continuation of the zeta-function to the left of this half-plane of absolute convergence. Following RIEMANN [80] we substitute $u = \pi n^2 x$ in EULER's representation of the Gamma-function,

$$\Gamma(u) = \int_0^{\infty} u^{z-1} \exp(-u)\,\mathrm{d}u,$$

and obtain

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} \exp(-\pi n^2 x)\,\mathrm{d}x. \tag{8}$$

Summing up over all $n \in \mathbb{N}$ yields

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} \exp(-\pi n^2 x)\,\mathrm{d}x.$$

On the left-hand side we find the Dirichlet series defining $\zeta(s)$; in view of its convergence, the latter formula is valid for $\mathrm{Re}\, s > 1$. On the right-hand side we may interchange summation and integration, justified by absolute convergence. Thus we obtain

tate auf einem der angehefteten weissen Blätter verzeichnete. Ich erkannte bald, dass unter allen Schwankungen diese Frequenz durchschnittlich nahe dem Logarithmen verkehrt proportional sei, so dass die Anzahl aller Primzahlen unter einer gegebenen Grenze n nahe durch das Integral $\int \frac{\mathrm{d}n}{\log n}$ *ausgedrückt werde, wenn der hyperbolische Logarithm. verstanden werde.*"

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty \exp(-\pi n^2 x)\,\mathrm{d}x.$$

We split the integral at $x = 1$ and get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \left\{\int_0^1 + \int_1^\infty\right\} x^{\frac{s}{2}-1}\omega(x)\,\mathrm{d}x, \tag{9}$$

where the series $\omega(x)$ is given in terms of the theta-function of JACOBI:

$$\omega(x) := \tfrac{1}{2}(\vartheta(x)-1) \qquad \text{with} \quad \vartheta(x) := \sum_{n=-\infty}^{+\infty}\exp(-\pi n^2 x).$$

In view of the functional equation for the theta-function, we have

$$\omega\left(\frac{1}{x}\right) = \frac{1}{2}\left(\theta\left(\frac{1}{x}\right)-1\right) = \sqrt{x}\omega(x) + \frac{1}{2}(\sqrt{x}-1),$$

which can be deduced from POISSON's summation formula. By the substitution $x \mapsto \frac{1}{x}$ it turns out that the first integral in (9) equals

$$\int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\mathrm{d}x = \int_1^\infty x^{-\frac{s+1}{2}}\omega(x)\,\mathrm{d}x + \frac{1}{s-1} - \frac{1}{s}.$$

Using this in (9) yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-\frac{s+1}{2}} + x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x. \tag{10}$$

Since $\omega(x) \ll \exp(-\pi x)$, the integral converges for all values of s, and thus (10) holds, by analytic continuation, throughout the complex plane. The right-hand side remains unchanged by $s \mapsto 1-s$. This proves

Theorem 4. *The zeta-function $\zeta(s)$ has an analytic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole at $s = 1$ and satisfies*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{11}$$

RIEMANN's functional equation (11) in combination with the EULER product disclose important information about the analytic behaviour of the zeta-function. In view of the EULER product (7) it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\mathrm{Re}\, s > 1$. It follows from the functional equation (11) and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\mathrm{Re}\, s < 0$ exactly at the so-called *trivial zeros* $s = -2n$ with $n \in \mathbb{N}$. All other zeros of $\zeta(s)$ are said to be *nontrivial*, and we denote them by $\rho = \beta + i\gamma$. Obviously, they lie inside the so-called *critical strip* $0 \leq \mathrm{Re}\, s \leq 1$, and they are non-real. The functional equation (11) and the identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ show some symmetries of $\zeta(s)$. In particular, the nontrivial zeros of

$\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\mathrm{Re}\, s = \frac{1}{2}$. It was RIEMANN's ingenious contribution to number theory to point out how the distribution of these nontrivial zeros is linked to the distribution of prime numbers. RIEMANN conjectured the asymptotics for the number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$ (counted according multiplicities). This conjecture was proved in 1895 by VON MANGOLDT [70, 71] who found more precisely, as $T \to \infty$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{12}$$

Here and elsewhere the nontrivial zeros are counted according to their multiplicity. There is no multiple zero known, however, one cannot exclude their existence so far; it follows from (12) that the multiplicity of a nontrivial zero $\rho = \beta + i\gamma$ is bounded by $O(\log |\gamma|)$. Since there are no zeros on the real line except the trivial ones, and nontrivial zeros are symmetrically distributed with respect to the real axis, it suffices to study the distribution of zeros in the upper half-plane.

RIEMANN worked with the function $t \mapsto \zeta(\frac{1}{2} + it)$ and wrote that *very likely all roots t are real,*[9] i.e., all nontrivial zeros lie on the so-called *critical line* $\mathrm{Re}\, s = \frac{1}{2}$. This is the famous, yet unproved RIEMANN hypothesis which we rewrite equivalently as

Riemann's hypothesis. $\zeta(s) \neq 0$ *for* $\mathrm{Re}\, s > \frac{1}{2}$.

In support of his conjecture, RIEMANN calculated some zeros; the first one with positive imaginary part is $\rho = \frac{1}{2} + i14.134\ldots$. Furthermore, he conjectured that there exist constants A and B such that

$$\tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \exp(A + Bs)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)\exp\left(\frac{s}{\rho}\right), \tag{13}$$

where the product on the right is taken over all nontrivial zeros (the trivial zeta zeros are cancelled by the poles of the Gamma-factor). This latter conjecture was shown by HADAMARD [37] in 1893 (on behalf of his theory of product representations of entire functions). Finally, RIEMANN conjectured the so-called *explicit formula* which states that

$$\pi(x) + \sum_{n=2}^{\infty} \frac{\pi(x^{\frac{1}{n}})}{n} = \mathrm{li}(x) - \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \left(\mathrm{li}(x^{\rho}) + \mathrm{li}(x^{1-\rho})\right) + \int_x^{\infty} \frac{\mathrm{d}u}{u(u^2-1)\log u} - \log 2 \tag{14}$$

[9] The original German text is: "*und es ist wahrscheinlich, daß alle Wurzeln den Realteil* $1/2$ *haben: Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.*"

for any $x \geq 2$ not being a prime power (otherwise a term $\frac{1}{2k}$ has to be added on the left-hand side where k stems from $x = p^k$). The appearing *modified* integral logarithm is defined by

$$\mathrm{li}(x^{\beta+i\gamma}) = \int_{(-\infty+i\gamma)\log x}^{(\beta+i\gamma)\log x} \frac{\exp(z)}{z+\delta i\gamma}\,\mathrm{d}z,$$

where $\delta = +1$ if $\gamma > 0$ and $\delta = -1$ otherwise. The explicit formula was proved by VON MANGOLDT [70] in 1895 as a consequence of both product representations for $\zeta(s)$, the EULER product (7) and the HADAMARD product (13). Building on these ideas, HADAMARD [38] and DE LA VALLÉE-POUSSIN [93] gave (independently) in 1896 the first proof of GAUSS' conjecture (6), the celebrated prime number theorem. For technical reasons it is of advantage to work with the logarithmic derivative of $\zeta(s)$ which is for $\mathrm{Re}\, s > 1$ given by

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where the VON MANGOLDT Λ-function is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Important information concerning the prime counting function $\pi(x)$ can be recovered from information about

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \log p + O\left(x^{\frac{1}{2}} \log x\right).$$

Partial summation gives $\pi(x) \sim \frac{\psi(x)}{\log x}$. First of all, we shall express $\psi(x)$ in terms of the zeta-function. If c is a positive constant, then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\,\mathrm{d}s = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

This yields the so-called PERRON formula: for $x \notin \mathbb{Z}$ and $c > 1$,

$$\psi(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\,\mathrm{d}s. \tag{16}$$

Moving the path of integration to the left, we find that the latter expression is equal to the corresponding sum of residues, that are the residues of the integrand at the pole of $\zeta(s)$ at $s = 1$, at the zeros of $\zeta(s)$, and at the additional pole of the integrand at $s = 0$. The main term turns out to be

$$\mathrm{Res}_{s=1} \left\{ -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \right\} = \lim_{s \to 1} (s-1) \left(\frac{1}{s-1} + O(1) \right) \frac{x^s}{s} = x,$$

whereas each nontrivial zero ρ gives the contribution

$$\mathrm{Res}_{s=\rho}\left\{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right\} = -\frac{x^\rho}{\rho}.$$

By the same reasoning, the trivial zeros altogether contribute

$$\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} = -\tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Incorporating the residue at $s=0$, this leads to the *exact* explicit formula

$$\psi(x) = x - \sum_{\rho}\frac{x^\rho}{\rho} - \tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \log(2\pi),$$

which is equivalent to RIEMANN's formula (14). This formula is valid for any $x \notin \mathbb{Z}$. Notice that the right-hand side of this formula is not absolutely convergent. If $\zeta(s)$ would have only finitely many nontrivial zeros, the right-hand side would be a continuous function of x, contradicting the jumps of $\psi(x)$ for prime powers x. Going on it is more convenient to cut the integral in (16) at $t = \pm T$ which leads to the truncated version

$$\psi(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log(xT))^2\right), \tag{17}$$

valid for all values of x. Next we need information on the distribution of the nontrivial zeros. Already the non-vanishing of $\zeta(s)$ on the line $\mathrm{Re}\, s = 1$ yields the asymptotic relations $\psi(x) \sim x$, resp. $\pi(x) \sim \mathrm{li}\,(x)$, which is GAUSS' conjecture (6) and sufficient for many applications. However, more precise asymptotics with a remainder term can be obtained by a zero-free region inside the critical strip. The largest known zero-free region for $\zeta(s)$ was found by VINOGRADOV [95] and KOROBOV [59] (independently) in 1958 who proved

$$\zeta(s) \neq 0 \qquad \text{in} \quad \mathrm{Re}\, s \geq 1 - c(\log(|t|+3))^{-\frac{1}{3}}(\log\log(|t|+3))^{-\frac{2}{3}},$$

where c is some positive absolute constant. In combination with the RIEMANN-VON MANGOLDT formula (12) we can estimate the sum over the nontrivial zeros in (17). Balancing T and x, we obtain the prime number theorem with the sharpest known remainder term: *there exists an absolute positive constant C such that for sufficiently large x*

$$\pi(x) = \mathrm{li}\,(x) + O\left(x\exp\left(-C\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right).$$

By the explicit formula (17) the impact of the RIEMANN hypothesis on the prime number distribution becomes visible. In 1900, VON KOCH [55] showed that for fixed $\theta \in [\frac{1}{2}, 1)$

$$\pi(x) - \mathrm{li}(x) \ll x^{\theta+\varepsilon} \quad \iff \quad \zeta(s) \neq 0 \quad \text{for} \quad \mathrm{Re}\, s > \theta\,; \tag{18}$$

equivalently, one can replace the left-hand side by $\psi(x) - x$; here ε stands for an arbitrary small positive constant. In view of the existence of zeros on the critical line an error term with $\theta < \frac{1}{2}$ is impossible. HARDY [39] proved that infinitely many zeros lie on the critical line. Refining a method of ATLE SELBERG [84], LEVINSON [66] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as HEATH-BROWN [44] and SELBERG (unpublished) discovered, those zeros are all simple. The current record is due to BUI, CONREY & YOUNG [13] who showed, by extending Levinson's method, that more than 41 percent of the zeros are on the critical line and more than 40.5 percent are simple and on the critical line. For further reading on the theory of the RIEMANN zeta-functuion we refer to the classical monograph [91] by TITCHMARSH and the current book [50] by IVIĆ; many historical details about prime numbers can be found in SCHWARZ's survey [83] and NARKIEWICZ's monograph [75].

4 The Ordinates of Zeta Zeros Are Uniformly Distributed Modulo One

Obviously, the trivial zeros are not uniformly distributed modulo one. In 1956 RADEMACHER [79] proved on the contrary the remarkable result that the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one provided that the RIEMANN hypothesis is true; later ELLIOTT [20] remarked that the latter condition can be removed, and (independently) HLAWKA [46] obtained the following unconditional

Theorem 5. *For any real number $\alpha \neq 0$ the sequence $\alpha\gamma$, where γ ranges through the set of positive ordinates of the nontrivial zeros of $\zeta(s)$ in ascending order, is uniformly distributed modulo one. In particular, the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one.*

Proof. We need some deeper results from zeta-function theory. We start with a theorem of LANDAU [62] who proved, for $x > 1$,

$$\sum_{0<\gamma<T} x^{\rho} = -\Lambda(x)\frac{T}{2\pi} + O(\log T), \tag{19}$$

where the summation is over all nontrivial zeros $\rho = \beta + i\gamma$ and $\Lambda(x)$ is the VON MANGOLDT Λ-function, defined by (15); if $x \in (0,1)$ one has to replace $\Lambda(x)$ by $x\Lambda(\frac{1}{x})$ because of the symmetrical distribution of nontrivial zeros. (We shall give a proof of LANDAU's formula in the following section!) Let $x > 1$. In view of (19) and the RIEMANN-VON MANGOLDT-formula (12) it follows that

$$\frac{1}{N(T)} \sum_{0<\gamma<T} x^{\rho} \ll \frac{\log x}{\log T}. \tag{20}$$

To avoid the assumption of the RIEMANN hypothesis we observe that

$$\begin{aligned}|x^{\frac{1}{2}+i\gamma}-x^{\beta+i\gamma}| &\leq \max\{x^{\beta},x^{\frac{1}{2}}\}|\exp((\tfrac{1}{2}-\beta)\log x)-1| \\ &\leq \max\{x^{\beta},x^{\frac{1}{2}}\}\log x\,|\beta-\tfrac{1}{2}|.\end{aligned}$$

Thus,

$$\frac{1}{N(T)}\sum_{0<\gamma<T}|x^{\frac{1}{2}+i\gamma}-x^{\beta+i\gamma}| \leq \frac{\max\{x,x^{\frac{1}{2}}\}|\log x|}{N(T)}\sum_{0<\gamma<T}|\beta-\tfrac{1}{2}|. \tag{21}$$

In the sequel the implicit constants may depend on x. Next we shall use a result of LITTLEWOOD [68], namely

$$\sum_{0<\gamma\leq T}|\beta-\tfrac{1}{2}| \ll T\log\log T. \tag{22}$$

It should be noted that SELBERG [84] improved upon this result in replacing the right hand-side by T by integration of his density theorem

$$N(\sigma,T) \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}\log T \tag{23}$$

for the number $N(\sigma,T)$ of hypothetical zeros $\rho=\beta+i\gamma$ with $0<\gamma\leq T$ and $\beta>\sigma$, i.e.,

$$\sum_{0<\gamma\leq T}|\beta-\tfrac{1}{2}| = \int_{\frac{1}{2}}^{1}N(\sigma,T)\,\mathrm{d}\sigma \ll T.$$

Both estimates indicate that most of the zeta zeros are clustered around the critical line.[10]

Inserting this in (21) and using (12), leads to

$$\frac{1}{N(T)}\sum_{0<\gamma\leq T}(x^{\frac{1}{2}+i\gamma}-x^{\beta+i\gamma}) \ll \frac{\log\log T}{\log T}.$$

Thus, it follows from (20) that also

$$\frac{1}{N(T)}\sum_{0<\gamma\leq T}x^{\frac{1}{2}+i\gamma} \ll \frac{\log\log T}{\log T}.$$

Letting $x=z^m$ with some real number $z>1$ and $m\in\mathbb{Z}$, we deduce

$$\frac{1}{N(T)}\sum_{0<\gamma\leq T}\exp(im\gamma\log z) \ll \frac{\log\log T}{\log T},$$

[10] A different approach to results around the sum of terms $|\beta-\frac{1}{2}|$ is due to KONDRATYUK [57] based on a variant of the CARLEMAN–NEVANLINNA theorem.

which tends to zero as $T \to \infty$. Hence, it follows from WEYL's criterion, Theorem 3, that the sequence of numbers $\alpha\gamma$ with $\alpha = \frac{\log z}{2\pi}$ is uniformly distributed modulo one. •

ELLIOTT's paper [20] is from 1972; it is a transcript of a talk he had given at a meeting on number theory at Oberwolfach in 1968. The main focus of his work, however, was on the frequency of negative values of the LEGENDRE symbol. ELLIOTT's approach is based on the following formula (in slightly different notation)

$$\begin{aligned}\sum_{|\gamma_n|<T} \exp(i\omega\gamma_n) &= 2\mathrm{Re} \int_0^{T-} \exp(i\omega y)\,\mathrm{d}N(y) \\ &= 2\mathrm{Re}\left(i\omega \int_0^{T-} S(y)\exp(i\omega y)\,\mathrm{d}y\right) + O(\log T),\end{aligned}$$

where $S(t) := \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$ is the argument of the zeta-function on the critical line[11] and $\omega \neq 0$ is a fixed real number. Using a conditional asymptotic formula for the second moment of $S(t)$ (under assumption of the RIEMANN hypothesis) due to SELBERG (unpublished), RADEMACHER's theorem follows. This is followed by an added note saying

> *"In a lecture given at the same meeting in Oberwolfach, Professor Selberg indicated that he had improved his result concerning* $\arg\zeta(\frac{1}{2}+it)$ *to give unconditional information concerning the distribution of the values of* $\zeta(s)$ *in regions centred on the line* $\sigma = \frac{1}{2}$*. In particular it is possible with a suitable interpretation to give an unconditional form of Theorem 2."*

In this quotation Theorem 2 is exactly the same statement as in Theorem 5 above. The paper [47] by HLAWKA is from 1975 and does not include a reference to ELLIOTT's paper. HLAWKA's approach is slightly different; his proof is more or less identical to our reasoning above. A last word about the reception of these papers. It seems that ELLIOTT [20] was unaware of RADEMACHER's work [79] since he does not cite his paper. On the contrary, HLAWKA [47] quotes RADEMACHER's paper but not the one by ELLIOTT. In his *Zentralblatt* review HLAWKA wrote that he learned about ELLIOTT's previous result by his colleague BUNDSCHUH.[12] It should be mentioned that HLAWKA [47] also gave a multidimensional analogue of the above theorem. Another proof of the uniform distribution modulo one of the ordinates was given by FUJII [25]; this paper contains as well further related results.

It is a long-standing conjecture that the ordinates of the nontrivial zeros are linearly independent over the rationals. So $\gamma+\gamma'$ should never equal another ordinate of a zeta zero. Of course, one should not expect any algebraic relation for the zeta zeros, hence it is reasonable to expect the converse. INGHAM [49] observed an interesting impact on the distribution of values of the MÖBIUS μ-function. Let $M(x) = \sum_{n\leq x}\mu(n)$, where $\mu(n)$ is defined by

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[11] defined by continuous variation from the principal branch of the logarithm on the real axis

[12] *"Der Referent [Hlawka] wurde von Herrn Bundschuh aufmerksam gemacht, daß auch P.D.T.A. Elliott (...) diese Tatsache bemerkt hat."*

$$\zeta(s)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

the latter identity being valid for $\mathrm{Re}\, s > 1$. It is not difficult to deduce that $\mu(n) = (-1)^r$ if n is squarefree and r denotes the number of prime divisors of n; otherwise $\mu(n) = 0$. INGHAM showed that, if the ordinates of the nontrivial zeros are indeed linearly independent over the rationals, then $\limsup_{x\to\infty} M(x)x^{-\frac{1}{2}} = +\infty$ and $\liminf_{x\to\infty} M(x)x^{-\frac{1}{2}} = -\infty$ which should be compared with (24) below.

Since $\mu(n) \in \{0, \pm 1\}$ one may interpret $M(x)$ as the realization of a one-dimensional symmetric random walk starting at zero. It was DENJOY [15] who argued as follows. Assume that $\{X_n\}$ is a sequence of random variables with distribution $\mathbf{P}(X_n = +1) = \mathbf{P}(X_n = -1) = \frac{1}{2}$. Define

$$S_0 = 0 \qquad \text{and} \qquad S_n = \sum_{j=1}^{n} X_j,$$

then $\{S_n\}$ is a symmetrical random walk in $\mathbb{Z}$ with starting point at 0. By the theorem of MOIVRE-LAPLACE this can be made more precise. It follows that

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| < cn^{\frac{1}{2}}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

Since the right hand-side above tends to 1 as $c \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| \ll n^{\frac{1}{2}+\varepsilon}\right\} = 1$$

for every $\varepsilon > 0$. We observe that this might be regarded as a model for the value-distribution of MÖBIUS μ-function. The law of the iterated logarithm in order to get the strong estimate for a symmetric random walk

$$\lim_{n\to\infty} \mathbf{P}\left\{|S_n| \ll (n \log\log n)^{\frac{1}{2}}\right\} = 1.$$

This suggests for $M(x)$ the upper bound $(x \log\log x)^{\frac{1}{2}}$ which is pretty close to the so-called weak MERTENS hypothesis stating

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 \mathrm{d}x \ll \log X.$$

The latter bound implies the RIEMANN hypothesis and that all zeros are simple. On the contrary, ODLYZKO & TE RIELE [76] disproved the original MERTENS hypothesis [72], i.e., $|M(x)| < x^{\frac{1}{2}}$, by showing

$$\liminf_{x\to\infty} \frac{M(x)}{x^{\frac{1}{2}}} < -1.009 \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{M(x)}{x^{\frac{1}{2}}} > 1.06; \tag{24}$$

for more details see the notes to §14 in Titchmarsh [91].

LANDAU's formula and variations have been used, in particular by FUJII in a series of papers, in order to evaluate discrete moments of the zeta-function or its derivative near or at its zeros, e.g. $\sum_{0<\gamma<T}\zeta'(\frac{1}{2}+i\gamma)$ with very precise error terms under assumption of the RIEMANN hypothesis (see [28]). Furthermore, FUJII [29] investgated the sequence $\gamma+\gamma'$ where both γ and γ' range through the set of positive ordinates of zeta zeros (in ascending order). Assuming the RIEMANN hypothesis, he obtained an asymptotic formula for

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<T}} x^{i(\gamma+\gamma')} \sim \frac{\Lambda(x)^2}{x}\frac{T^2}{8\pi^2} + \frac{x^{iT}}{\log x}\frac{T(\log T)^2}{4\pi^2 i}$$

with an explicit error term. Since the number of terms $\gamma+\gamma'<T$ is asymptotically equal to $\frac{T^2(\log T)^2}{8\pi^2}$, it follows that the sequence of $\gamma+\gamma'$ is uniformly distributed modulo one. This has been used by EGAMI & MATSUMOTO [18] to motivate a related conjecture on distances between different pairs of zero ordinates in order to show that a certain multiple zeta-function has a natural boundary.

5 Questions Around the Distribution of Values of $\zeta(s)$

Bounds for the RIEMANN zeta-function rely heavily on estimates for certain exponential sums. In order to see that consider the DIRICHLET polynomial obtained from the defining series for $\zeta(s)$, i.e.,

$$\sum_{n\le x}\frac{1}{n^s} = 1 + \sum_{1<n\le x}\frac{1}{n^\sigma}\exp(-it\log n),$$

where we have written $s=\sigma+it$. One may hope to estimate $\zeta(\sigma+it)$ by finding a good bound for this DIRICHLET polynomial; of course, we exclude here any treatment of the tail of the series expansion. By partial summation it suffices to consider the latter sum in case of $\sigma=0$. Replacing the logarithm $\log n$ by an appropriate polynomial P of sufficiently high degree, the problem is reduced to an estimation of the exponential sum

$$\sum_{1<n\le x}\frac{1}{n^\sigma}\exp(-itP(n).$$

This type of quantity was already treated by WEYL [103] when he was generalizing BOHL's theorem from the case of linear polynomials to arbitrary polynomials. The works of HARDY & LITTLEWOOD [40, 41] had been following this line of investigation (see [91], Chapter V); exponential sums have found further applications in their approach to the WARING problem by introducing the cricle method. In 1921, WEYL [104] pushed his method further to deal with exponential sums associated with the zeta-function and obtained stronger bounds for $\zeta(1+it)$. Later I.M.

VINOGRADOV gave another, in the case of the zeta-function more powerful method to bound exponential sums which led him to the still best known zero-free region so far (see [91], Chapter V for more details).

Studies on the general distribution of values of the zeta-function started with the research of BOHR and his school. In fact BOHR and his contemporaries were using DIOPHANTINE approximation in order to prove that the zeta-function assumes *large* and *small* values. For the sake of simplicity, let us consider the truncated EULER product

$$\prod_{p\leq x}\left(1-\frac{1}{p^s}\right)^{-1}; \tag{25}$$

by observing $p^s = p^\sigma \exp(it\log p)$ one may use a multi-dimensional version of KRONECKER's approximation theorem in order to find some real number τ such that the values $\frac{1}{2\pi}t\log p$ are close to $\frac{1}{2}$ modulo one for all primes $p \leq x$ which in turn implies that p^{it} is close to -1. This leads to a small value for the truncated EULER product and thereby proves that $\inf|\zeta(s)| = 0$ in the half-plane of absolute convergence although $\zeta(s)$ does not vanish (see [91], Chapter VIII for details). Inside the critical strip the situation is more subtle since (25) does not converge any longer to $\zeta(s)$. Nevertheless, by mean-square approximation this idea can be transported in some way to deduce similar results for $\zeta(s)$ on and around vertical lines $\sigma + i\mathbb{R}$ inside the critical strip. A central role is played by the following extension of KRONECKER's approximation theorem due to WEYL [103]: *Let $a_1,\ldots,a_N$ be real numbers, linearly independent over $\mathbb{Q}$, and let γ be a subregion of the N-dimensional unit cube with* JORDAN *content Γ.*[13] *Then*

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\{\tau\in(0,T)\,:\,(\tau a_1,\ldots,\tau a_N)\in\gamma \bmod 1\} = \Gamma.$$

Moreover, suppose that the curve

$$\{\omega(\tau)\} := (\{\omega_1(\tau)\},\ldots,\{\omega_N(\tau)\}),$$

is uniformly distributed mod 1 *in* $\mathbb{R}^N$ (extending the discrete and one-dimensional definition from §2 in a natural way). *Let $\mathscr{D}$ be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on $\mathscr{D}$. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\{\omega(\tau)\})\mathbf{1}_{\mathscr{D}}(\tau)\,\mathrm{d}\tau = \int_{\mathscr{D}} f(x_1,\ldots,x_N)\,\mathrm{d}x_1\ldots\mathrm{d}x_N$$

uniformly with respect to $f\in\Omega$, where $\mathbf{1}_{\mathscr{D}}(\tau)$ is equal to 1 *if $\omega(\tau)\in\mathscr{D}$* mod 1, *and* 0 *otherwise.* A proof of WEYL's theorem can be found in his paper [101] as well as in KARATSUBA & VORONIN [54]. In [43], §23.6, HARDY & WRIGHT state a

[13] Note that the notion of JORDAN content is more restrictive than the notion of LEBESGUE measure. But, if the JORDAN content exists, then it is also defined in the sense of LEBESGUE and equal to it.

multi-dimensional analogue of KRONECKER's theorem and comment on this result as "*one of those mathematical theorems which assert (...) that what is not impossible will happen sometimes however improbable it may be*. Outside mathematics this is known as 'MURPHY's law'. The unique prime factorization of integers implies the linear independence of the logarithms of the prime numbers over the field of rational numbers. Thus, in some sense, the logarithms of prime numbers behave like random variables and everything that can happen will happen!

Exploiting this idea, and sometimes further methods, namely addition of convex sets, BOHR and his school obtained plenty of remarkable and beautiful results on the value-distribution of the zeta-function, e.g. that the set of values of $\zeta(\sigma+it)$ is dense in $\mathbb{C}$ as t ranges through $\mathbb{R}$ for any fixed $\frac{1}{2} < \sigma \leq 1$ (see [91], Chapter XI). Notice that the problem whether the values taken by the zeta-function on the critical line lie dense in the complex plane is still unsolved.

However, the most spectacular statement in the value-distribution theory of the RIEMANN zeta-function was found by VORONIN [96] in 1975 who discovered the following remarkable approximation property of the zeta-function: *Let* $0 < r < \frac{1}{4}$ *and* $g(s)$ *be a non-vanishing continuous function defined on the disk* $|s| \leq r$, *which is analytic in the interior of the disk. Then, for any* $\varepsilon > 0$, *there exists a real number* $\tau > 0$ *such that*

$$\max_{|s|\leq r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon;$$

moreover, the set of all $\tau \in [0,T]$ *with this porperty has positive lower density with respect to the* LEBESGUE *measure.* This is the so-called universality theorem since it allows the approximation of a huge class of target functions by a single function, namely the RIEMANN zeta-function. Also here a key role in VORONOIN's proof is played by WEYL's refinement of KRONECKER's approximation theorem. Besides VORONIN's original proof there is a probabilistic approach to universality due to BAGCHI, REICH, LAURINČIKAS and further developed by many others (see [65, 88]). In this method the pointwise ergodic theorem due to BIRKHOFF replaces the use of WEYL's uniform distribution theorem in VORONIN's approach.

It was EDMUND LANDAU [63] who started in his invited talk at the occasion of the fifth International Mathematical Congress held at Cambridge in 1912 a new direction in the value distribution theory of the zeta-function. He announces this line of investigation as follows:

> "*Now let me discuss some different investigations about* $\zeta(s)$. *Given an analytic function, the points for which this function is* 0 *are very important; however, of equal interest are those points where the function assumes a given value a. It is easy to prove that* $\zeta(s)$ *takes any value a. But where do the roots of* $\zeta(s) = a$ *lie?*"[14]

Notice that for LANDAU the distribution of the roots of

$$\zeta(s) = a$$

[14] "*Ich komme jetzt zu einigen anderen Untersuchungen über* $\zeta(s)$. *Es sind bei einer analytischen Funktion die Punkte, an denen sie* 0 *ist, zwar sehr wichtig; ebenso interessant sind aber die Punkte, an denen sie einen bestimmten Wert a annimmt. Zu beweisen, dass* $\zeta(s)$ *jeden Wert a annimmt, ist ein leichtes. Wo liegen aber die Wurzeln von* $\zeta(s) = a$*?*"

is for each complex value a equally important. These roots are called a-points and will be denoted by $\rho_a = \beta_a + i\gamma_a$. In the following year LANDAU [10] proved that there is an a-point near any trivial zero $s = -2n$ for any sufficiently large positive integer n, which we shall call trivial. One can show that the trivial a-points are not uniformly distributed modulo one (since they lie too close to the trivial zeros $-2n$, see [67]). All other a-points are said to be nontrivial. For any fixed a, there exist left and right half-planes free of nontrivial a-points (see formula (28, resp. [10]). Moroever, LANDAU [10] obtained an asymptotic formula for the number $N_a(T)$ of nontrivial a-values with imaginary part γ_a satisfying $0 < \gamma_a \leq T$, namely,

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e c_a} + O(\log T), \tag{26}$$

as $T \to \infty$, where $c_a = 1$ if $a \neq 1$, and $c_1 = 2$. Here and in the sequel the a-points are counted according multiplicities and the multiplicity of an a-point $\rho_a = \beta_a + i\gamma_a$ is therefore bounded by $O(\log(3 + |\gamma_a|))$. The paper [10] of BOHR, LANDAU & LITTLEWOOD is of special interest in our context: it was published exactly one hundred years ago, when in Göttingen WEYL was proving his powerful criterion, Theorem 3, and LANDAU was at the same time professor at Göttingen. This very paper consists of three independent chapters, the first belonging essentially to BOHR, the second to LANDAU, and the third to LITTLEWOOD; it had been submitted in November 1913, the same year as WEYL proved his powerful criterion for uniform distribution modulo one.[15] The asymptotic formula (26) extends the RIEMANN–VON MANGOLDT–formula for the number $N(T)$ of nontrivial zeros to arbitrary a-points and shows that the main term is independent of a. Finally, LANDAU [10] proved that almost all a-points are clustered around the critical line provided the RIEMANN hypothesis is true. The latter assumption was removed by LEVINSON [67] who showed that *all but $O(N_a(T)/\log\log T)$ of the a-points $\rho_a = \beta_a + i\gamma_a$ with imaginary part in $\gamma_a \in (T, 2T)$ satisfy*

$$|\beta_a - \tfrac{1}{2}| < \frac{(\log\log T)^2}{\log T}. \tag{27}$$

His reasoning is based on the identity

$$2\pi \sum_{\substack{T<\gamma_a \leq T+U \\ \beta_a > -b}} (\beta_a + b) = \int_T^{T+U} \log|\zeta(-b+it) - a|\,\mathrm{d}t \\ -U\log|1-a| + O(\log T)$$

with some real constant b (as follows from LITTLEWOOD's lemma).

In some literature, density theorems showing that *most* of the zeros of $\zeta(s)$ lie close to the critical line were interpreted as an indicator for the truth of the RIEMANN hypothesis, however, this is only correct if the quantitative difference to the clustering of arbitrary a-points is taken into account (as for example (23) vs.

[15] and we shall make use of both results later on!

LEVINSON's theorem above). We illustrate this observation with a quotation from LEVINSON [67]:

> *"In his recent book (...) Edwards states that the clustering of the zeros of $\zeta(s)$ near $\sigma = 1/2$, first proved by Bohr and Landau (...), is the best existing evidence for the Riemann Hypothesis. Titchmarsh (...) also emphasizes with italics the clustering phenomenon of the zeros of $\zeta(s)$. It will be shown here that for any complex a the roots of $\zeta(s) = a$ cluster at $\sigma = 1/2$ and so, in this sense, the case $a = 0$ is not special. However, (...) it is clear that the clustering for the case $a = 0$ is more pronounced than for $a \neq 0$..."*

The books in question are [17] and [91] by EDWARDS and TITCHMARSH, respectively. It seems that LANDAU's conditional results on the distribution of a-points have been forgotten. As LEVINSON pointed out, a general clustering of a-points around the critical line is true, not only for zeros. However, in the case of zeros the quantity of those zeros which do not lie inside this cluster set are smaller than for any other a-points.

6 Generalizing Landau's Theorem and Applications

LANDAU's formula, resp. theorem (19) has been extended and generalized in different ways. For instance, KACZOROWSKI, LANGUASCO & PERELLI [53] introduced weights in order to obtain an error term of more flexible shape. We aim at an application to the distribution of a-points, hence our generalization is completely different:

Theorem 6. *Let x be a positive real number $\neq 1$. Then, as $T \to \infty$,*

$$\sum_{0<\gamma_a<T} x^{\rho_a} = \left(\alpha(x) - x\Lambda\left(\tfrac{1}{x}\right)\right)\frac{T}{2\pi} + O(T^{\frac{1}{2}+\varepsilon}),$$

where $\alpha(x)$ and $\Lambda(x)$ equal the DIRICHLET *series coefficients in (29) and (30), respectively, if $x = n$ or $x = 1/n$ for some integer $n \geq 2$, and zero otherwise.*

The implicit constant in the error term may depend here and elsewhere on x. The theorem gives an explicit formula with a-points in place of zeros. The case $a = 0$ was first treated by LANDAU [62]; later improvements, resp. generalizations are due to GONEK [34], FUJII [26, 27, 28] (with an improved uniform error estimate), and M.R. & V.K. MURTY [74] (for L-functions from the SELBERG class). For the special case $a = 2$ HILLE [45] proved that the coefficients $f(n)$ of the DIRICHLET series for $(\zeta(s) - 2)^{-1}$ count the number of representations of n as a product of integers strictly greater than one; this allows a simple computation of the arithmetical function $n \mapsto \alpha(n)$ via the convolution $-\log = \alpha * f$. For other values of a the author is not aware of any number-theoretical meaning of α.

In order to generalize LANDAU's formula (19) we shall use some ideas from GARUNKŠTIS & STEUDING [31].

Proof. First we assume $a \neq 1$. Since $\zeta(s) - a$ has a convergent DIRICHLET series representation for sufficiently large $\mathrm{Re}\, s$, it follows that there exists a half-plane

$\mathrm{Re}\, s > B$ which is free of a-points. In order to compute such an abscissa B explicitly, we assume $\sigma := \mathrm{Re}\, s > 1$ and estimate

$$|\zeta(s) - 1| \leq \sum_{n\geq 2} n^{-\sigma} < \int_1^\infty u^{-\sigma}\,\mathrm{d}u = \frac{1}{\sigma - 1}.$$

Thus,

$$\zeta(s) - a \neq 0 \qquad \text{for} \quad \sigma > 1 + \frac{1}{|a-1|}. \tag{28}$$

Consequently, as shown by LANDAU [64], the inverse $(\zeta(s) - a)^{-1}$ has a convergent DIRICHLET series expansion in the same half-plane. After multiplying with the convergent DIRICHLET series for $\zeta'(s)$, we end up with

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n\geq 2} \frac{\alpha(n)}{n^s}; \tag{29}$$

In case of $a = 0$ this equals the logarithmic derivative of the zeta-function

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\geq 2} \frac{\Lambda(n)}{n^s} \tag{30}$$

which plays a central role in LANDAU's proof of Theorem 6 in the special case $a = 0$ as well as in proofs of other explicit formulae in prime number theory. Notice that $\alpha(n) = -\Lambda(n)$ if $a = 0$. Moreover, we observe that both series have no constant term since the series for $\zeta'(s)$ has not. By partial summation it follows that the abscissa of convergence and the abscissa of absolute convergence of an ordinary DIRICHLET series differ by at most one. Hence, the abscissa of absolute convergence of the DIRICHLET series (29) is less than or equal to $B := 2 + |a-1|^{-1}$ (see [42]). In view of (26) for any positive T_0 we can find some $T \in [T_0, T_0 + 1)$ such that the distance between T to the nearest ordinate γ_a of the a-points is bounded by $(\log T)^{-1}$. Moreover, let $b := 1 + (\log T)^{-1}$. Then only finitely many a-points lie to the left of the vertical line $\mathrm{Re}\, s = 1 - b$. Finally, note that the logarithmic derivative of $\zeta(s) - a$ has simple poles at each a-point with residue equal to the order. Hence

$$\sum_{0<\gamma_a<T} x^{\rho_a} = \frac{1}{2\pi i} \int_{\mathsf{R}} x^s \frac{\zeta'(s)}{\zeta(s) - a}\,\mathrm{d}s,$$

where R denotes the counterclockwise oriented rectangle with vertices $B + i, B + iT, 1 - b + iT, 1 - b + i$ and the error term arises from possible contributions of a-points outside R. We rewrite

$$\int_{\mathsf{R}} x^s \frac{\zeta'(s)}{\zeta(s) - a}\,\mathrm{d}s = \left\{ \int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i} + \int_{1-b+i}^{B+i} \right\} x^s \frac{\zeta'(s)}{\zeta(s) - a}\,\mathrm{d}s = \sum_{j=1}^{4} I_j,$$

say. We start with the vertical integral on the right-hand side. Interchanging summation and integration we find

$$I_1 = \sum_{n\geq 2} \alpha(n) \int_{B+i}^{B+iT} \left(\frac{x}{n}\right)^s \mathrm{d}s = i\alpha(x)T + O(1),$$

where $\alpha(x)$ equals the coefficient $\alpha(n)$ in the DIRICHLET series expansion (29) if $x = n$ or $x = 1/n$, and $\alpha(x) = 0$ otherwise (i.e., $x \neq n, 1/n$ for all $2 \leq n \in \mathbb{N}$).

Next we consider the horizontal integrals. Recall the functional equation,

$$\zeta(s) = \Delta(s)\zeta(1-s)\ , \quad \text{where} \quad \Delta(s) = 2(2\pi)^{s-1} \sin\tfrac{\pi s}{2}\Gamma(s).$$

By STIRLING's formula, we find

$$\zeta(\sigma \pm it) \asymp |t|^{\frac{1}{2}-\sigma} |\zeta(1-\sigma \mp it)|,$$

uniformly in σ, as $|t| \to \infty$. This in combination with the PHRAGMÉN-LINDELÖF principle yields the bound

$$\zeta(\sigma + it) \ll t^{\mu(\sigma)+\varepsilon},$$

where

$$\mu(\sigma) \ll \begin{cases} 0 & \text{if} \quad \sigma > 1, \\ \frac{1}{2}(1-\sigma) & \text{if} \quad 0 \leq \sigma \leq 1, \\ \frac{1}{2} - \sigma & \text{if} \quad \sigma < 0. \end{cases}$$

Using the partial fraction decomposition for the logarithmic derivative (as in [31]), we get

$$I_2 = -\left\{\int_{-(\log T)^{-1}}^{1} + \int_{1}^{B}\right\} x^{\sigma+iT} \frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)-a} \mathrm{d}\sigma \ll xT^{\frac{1}{2}+\varepsilon} + x^B T^{\varepsilon}.$$

Next we evaluate the vertical integral on the left-hand side $\mathrm{Re}\, s = 1-b$ of the contour. It is not difficult to show

$$\zeta(\sigma + it) \gg \frac{t^{\frac{1}{2}-\sigma}}{\log t} \tag{31}$$

(otherwise consult Lemma 4 in [31]). Hence, the left-hand side has absolute value larger than $1/2$ for $t \geq t_0$. For such values of t we may expand the logarithmic derivative into a geometric series

$$\frac{\zeta'(s)}{\zeta(s)-a} = \frac{\zeta'}{\zeta}(s) \frac{1}{1-a/\zeta(s)} = \frac{\zeta'}{\zeta}(s)\left(1 + \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k\right).$$

This gives

$$I_3 = O(1) - \int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s)\left(1 + \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k\right) \mathrm{d}s.$$

In view of (31) we find

$$\int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s) \sum_{k\geq 1}\left(\frac{a}{\zeta(s)}\right)^k \mathrm{d}s \ll x^{1-b}T(\log T)\sum_{k\geq 1}\left(\frac{\log T}{T^{\frac{1}{2}}}\right)^k \ll x^{1-b}T^{\frac{1}{2}}(\log T)^2.$$

Using the functional equation, we get in view of (30)

$$\begin{aligned} -\int_{1-b+it_0}^{1-b+iT} x^s \frac{\zeta'}{\zeta}(s)\,\mathrm{d}s &= \int_{1-b+it_0}^{1-b+iT} x^s \left(\frac{\zeta'}{\zeta}(1-s) - \frac{\Delta'}{\Delta}(s)\right)\mathrm{d}s \\ &= -ix^{1-b}\sum_{n\geq 2}\Lambda(n)n^{-b}\int_{t_0}^{T}(xn)^{it}\,\mathrm{d}t + \\ &\quad + ix^{1-b}\int_{t_0}^{T} x^{it}\left(\log\tfrac{t}{2\pi} + O(t^{-1})\right)\mathrm{d}t. \end{aligned}$$

The first term on the right-hand side equals $ix\Lambda(\frac{1}{x})T + O(1)$ whereas the second term can be bounded by $\log T/|\log x|$. Finally, the remaining horizontal integral is independent of T, hence $I_4 \ll 1 + x^B$. Thus we arrive at

$$\sum_{0<\gamma_a<T} x^{\rho_a} = \left\{\alpha(x) - x\Lambda(\tfrac{1}{x})\right\}\frac{T}{2\pi} + O_x(T^{\frac{1}{2}+\varepsilon}).$$

In order to have the asymptotic formula uniform for all T we add an error of size $O(\log T)$. This proves the theorem for $a \neq 1$. If $a = 1$ we consider the function

$$\ell(s) = 2^s(\zeta(s) - 1) = 1 + \sum_{n\geq 3}\left(\frac{2}{n}\right)^s$$

and its logarithmic derivative

$$\frac{\ell'}{\ell}(s) = \log 2 + \frac{\zeta'(s)}{\zeta(s) - 1}.$$

Applying contour integration to this logarithmic derivative yields the asymptotic formula for $a = 1$. •

The above reasoning with $x = 1$ and a more careful treatment of the error term leads to the asymptotic formula (26) for $N_a(T)$.

Next, using our new explicit formula, we shall generalize the result on the uniform distribution modulo one from the zeros to a-points:

Theorem 7. *For any complex number a and any real $\alpha \neq 0$, the sequence of numbers $\alpha\gamma_a$ (with γ_a denoting the ordinates of the a-points) are uniformly distributed modulo one.*

In view of our generalization of LANDAU's theorem the proof of the latter result is straightforward.

Proof. Recall LEVINSON's theorem [67] from the previous section that all but $O(N_a(T)/\log\log T)$ of the a-points $\rho_a = \beta_a + i\gamma_a$ with imaginary part in $\gamma_a \in (T, 2T)$ satisfy (27). More precisely, let $\delta(T) = (\log\log T)^2/\log T$; then LEVINSON showed that the number of a-points $\rho_a = \beta_a + i\gamma_a$ for which $|\beta_a - 1/2| > \delta$ and $T < \gamma_a < 2T$ is bounded by $T\log T/\log\log T$. This yields

$$\sum_{T<\gamma_a\leq 2T} |\beta_a - \tfrac{1}{2}| = \left\{ \sum_{\substack{T<\gamma_a\leq 2T \\ \beta_a - 1/2 > \delta}} + \sum_{\substack{T<\gamma_a\leq 2T \\ |\beta_a - 1/2| \leq \delta}} + \sum_{\substack{T<\gamma_a\leq 2T \\ \beta_a - 1/2 < -\delta}} \right\} |\beta_a - \tfrac{1}{2}|$$
$$\ll \frac{T\log T}{\log\log T} + T(\log\log T)^2.$$

Using this with $2^{-k}T$ in place of T and adding the corresponding estimates over all $k \in \mathbb{N}$, we deduce

$$\sum_{0<\gamma_a\leq T} |\beta_a - \tfrac{1}{2}| \ll \frac{T\log T}{\log\log T} = o(N_a(T)). \tag{32}$$

Since

$$\exp(y) - 1 = \int_0^y \exp(t)\,\mathrm{d}t \ll |y| \max\{1, \exp(y)\},$$

we find, for $x \neq 1$,

$$|x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}| \leq x^{\beta_a} |\exp((\tfrac{1}{2} - \beta_a)\log x) - 1| \leq |\beta_a - \tfrac{1}{2}| \max\{x^{\beta_a}, x^{\frac{1}{2}}\} |\log x|.$$

Hence,

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} |x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}| \leq \frac{X}{N_a(T)} \sum_{0<\gamma_a\leq T} |\beta_a - \tfrac{1}{2}|,$$

where $X = \max\{x^B, 1\}|\log x|$ and B is the upper bound for the real parts of the a-points. In view of (32) we have

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} (x^{\frac{1}{2}+i\gamma_a} - x^{\beta_a+i\gamma_a}) \ll \frac{X}{\log\log T}.$$

Recall Theorem 6,

$$\sum_{0<\gamma_a<T} x^{\beta_a+i\gamma_a} \ll T;$$

here and in the sequel we drop the dependency on x since only the limit as $T \to \infty$ is relevant. Hence, we obtain

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} x^{\frac{1}{2}+i\gamma_a} \ll \frac{1}{\log\log T}.$$

Let $x = z^m$ with some positive real number $z \neq 1$ and $m \in \mathbb{N}$. Then, after dividing the previous formula by $x^{\frac{1}{2}}$, we may deduce

$$\lim_{T\to\infty} \frac{1}{N_a(T)} \sum_{0<\gamma_a\leq T} \exp(im\gamma_a \log z) = 0$$

Now WEYL's criterion, Theorem 3, implies that the sequence of numbers $\frac{1}{2\pi}\gamma_a \log z$ is uniformly distributed modulo one. •

We shall give a new application of Theorem 7. Given sequences of monotonically increasing positive real numbers $\mathfrak{a} = (a_n)_n$ and $\mathfrak{b} = (b_k)_k$, both being uniformly distributed modulo one, AKBARY & M.R. MURTY [1] showed that then also the union of both, namely the sequence $\mathfrak{a} \cup \mathfrak{b} := (a_n, b_k)_{n,k}$ is uniformly distributed modulo one. Here the sequence $(a_n, b_k)_{n,k}$ is ordered according to the absolute value of its elements. The easy proof is as follows. Denote by $N_{\mathfrak{a}}(x)$ the number of elements a_n from $\mathfrak{a}$ satisfying $a_n \leq x$. Then,

$$\begin{aligned} \frac{1}{N_{\mathfrak{a}\cup\mathfrak{b}}(x)} \left| \sum_{a_n,b_k\leq x} \mathrm{e}(m(a_n,b_k)) \right| &= \frac{1}{N_{\mathfrak{a}}(x)+N_{\mathfrak{b}}(x)} \left| \sum_{a_n\leq x} \mathrm{e}(ma_n) + \sum_{b_k\leq x} \mathrm{e}(mb_k) \right| \\ &\leq \frac{1}{N_{\mathfrak{a}}(x)} \left| \sum_{a_n\leq x} \mathrm{e}(ma_n) \right| + \frac{1}{N_{\mathfrak{b}}(x)} \left| \sum_{b_k\leq x} \mathrm{e}(mb_k) \right|. \end{aligned}$$

Hence, WEYL's criterion, Theorem 3, implies the assertion. As a consequence, we may deduce from Theorem 7 the following

Corollary 1. *Let $M \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_M$ be arbitrary positive real numbers and $a_1, \ldots, a_M$ be arbitrary complex numbers. Then the sequence*

$$\cup_{1\leq m\leq M}(\alpha_m \gamma_{a_m}) = \{\alpha_1\gamma_{a_1}, \ldots, \alpha_M\gamma_{a_M}\}$$

is uniformly distributed modulo one. In particular, the ordinates of the zeros of $P(\zeta(s))$ are uniformly distributed modulo one, where P is any non-constant polynomial with complex coefficients.

The application to $P(\zeta(s))$ follows from the factorization $P(\zeta) = \prod_j(\zeta - a_j)$ with certain complex numbers a_j by the fundamental theorem of algebra and an application of the uniform distribution modulo one of the union of the imaginary parts of the a_j-points.

7 Discrepancy and Further Concluding Remarks

We conclude with a few further problems related to applications of uniform distribution modulo one in the context of the RIEMANN zeta-function. Already WEYL noticed that the appearing limits are uniform which has been studied ever since under the notion of *discrepancy*. This topic has important applications, for instance, in billiards where we may ask how soon an aperiodic ray of light will visit a given domain? First results for effective billiards are due to WEYL [102], interesting and surprising results on square billiards have recently been discovered by BECK [2] showing that the typical billiard path is extremely uniform far beyond what one might expect. Also important in this setting are effective versions of the inhomogeneous KRONECKER approximation theorem from the introductory section as, for example, [97]. In the case of the zeros of the RIEMANN zeta-function first estimates for the discrepancy were already given by HLAWKA [47]; using the ERDÖS-TURÁN inequality he proved

$$\sup_{0\leq\alpha\leq 1}\frac{1}{N(T)}\left|\sharp\left\{0<\gamma<T\,:\,\{\gamma\tfrac{1}{2\pi}\log X\}-\alpha N(T)\right\}\right|\ll\frac{\log X}{\log\log T},$$

valid for $X>1$ and all sufficiently large T, where C is an explicit positive constant; the right hand-side can be replaced by $\frac{\log X}{\log T}$ if the RIEMANN hypothesis is assumed. Further results in this direction are due to FUJII in a series of papers [24, 25, 30], AKBARY & M.R. MURTY [1], and FORD, SOUNDARARAJAN, respectively. In ZAHARESCU [22, 23] connections to MONTGOMERY's pair correlation conjecture and the distribution of primes in short intervals are established. In all these investigations the dependency of the error term in LANDAU's explicit formula on x is relevant. Improvements of the explicit formula with an error terms that is uniform in x were given in particular by FUJII [26, 27] and GONEK [33, 34].

The most natural question seems to be *whether the uniform distribution modulo one is a common feature for all arithmetical L-functions?* There is no precise definition of an L-function. M.N. HUXLEY said *"What is a zeta-function (or an L-function)? We know one when we see one."* Therefore it seems natural to consider classes of L-functions. The SELBERG class provides a rather general axiomatic setting for L-functions (see [85] and [88] for its definition); the most simple examples are DIRICHLET L-functions to residue class characters (or L-series as in the above quotation). These L-functions share many patterns with the RIEMANN zeta-function and for those it was already noticed by HLAWKA [46] that the results for the zeta-function carry over without any difficulty. However, the situation for L-functions associated with modular forms is more delicate; HLAWKA finishes his investigations with the following words:

> *"The investigations can be generalized to L-series as well as to Dedekind zeta-functions. (...) It would be interesting to extend this to other zeta-functions as, for example,* $\sum\frac{\tau(n)}{n^s}$*, where* τ *is the well-known Ramanujan function.."*[16]

The RAMANUJAN function $n\mapsto\tau(n)$ provides the FOURIER coefficients of the modular discriminant; the associated L-function is the prototype of a degree two element

[16] *"Die Überlegungen lassen sich auf L-Reihen wie auf die Dedekindsche Zetafunktion übertragen. (...) Interessant wäre es, dies auf andere Zetafunktionen, wie z.B. auf die* $\sum\frac{\tau(n)}{n^s}$ *auszudehnen, wo* τ *die bekannte Ramanujansche Funktion ist."*

of the SELBERG class arising from a modular form. AKBARY & M.R. MURTY [1] proved conditionally uniform distribution modulo one for the non-trivial zeros of L-functions $L(s)$ from a certain class containing the SELBERG class; however, their condition is a conjecture on power moments, resp. an analogue of LEVINSON's bound, namely the so-called *average density hypothesis* claiming that

$$\sum_{\substack{0\leq\gamma<T\\ \beta>\frac{1}{2}}}(\beta-\tfrac{1}{2})=o(N_L(T)), \tag{33}$$

where the nontrivial zeros of $L(s)$ are denoted by $\beta+i\gamma$ and their number up to height T is counted by $N_L(T)$. Such an estimate implies that the zeros are clustered around the critical line. In the case of L-functions to modular forms AKBARY & M.R. MURTY [1] succeeded to prove (33), however, different to HLWAKA's statement, for DEDEKIND zeta-function the uniform distribution modulo one of the zeros has been proved only in the case of ABELian number fields $\mathbb{K}/\mathbb{Q}$ (since in this case the DEDEKIND zeta-function splits into a product of more simple L-functions). Obvious question are whether Condition (33) can be proved in general and what can be done with respect to a-points. A certain progress here is due to JAKHLOUTI et al. [51] who considered an extension of Theorem 7 to L-functions with polynomial EULER products, and GARUNKŠTIS et al. [32] obtained an analogue for certain Selberg zeta-functions. The distribution of a-points of an L-function from the Selberg class has been started already with SELBERG's influential paper [85]; further results in this direction are in particular due to GONEK et al. [35].

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